Solutions

Solution 1. (1) *By a simple change of variable we get that* $(T_1, \dots, T_n)$ *has density:*

$$(t_1, \dots, t_n) \mapsto \lambda^n e^{-\lambda t_n} 1_{\{0<t_1<\dots<t_n\}}. \tag{A.88}$$

(2) *By integrating on the* $(n-1)$*-first variables we get that* T_n *has density*

$$t_n \mapsto \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} 1_{\{t>0\}}, \tag{A.89}$$

that is a $\Gamma(n, \lambda)$ *random variable.*

(3) *Let* $n \in \mathbb{N}_0$ *and* $t \geq 0$*. Remind that* $T_0 = 0$*. We note that* $\mathrm{P}(N_t = n) = \mathrm{P}(T_n \leq t < T_{n+1})$*, which we may compute by using the density of* $(T_1, \dots, T_{n+1})$*.*

(4) *By using the density of* $(T_1, \dots, T_{n+1})$*, one may check that for any measurable function* $f \colon \mathbb{R}_n \to \mathbb{R}_+$*,*

$$\mathrm{E}[f(T_1, \dots, T_n) | N_t = n] = \int_{0 \leq t_1 < \dots < t_n \leq t} f(t_1, \dots, t_n) \frac{n!}{t^n} \mathrm{d}t_1 \dots \mathrm{d}t_n. \tag{A.90}$$

(5) *Since the process* (N_t) *is non-decreasing , we get that* $\mathrm{P}(N_t < \infty, \forall t \geq 0) = \mathrm{P}(N_k < \infty, \forall k \in \mathbb{N})$*. This probability equals one since* N_k *is finite a.s, for all* k *in the countable set of integers..*

Solution 2. *We have for all* $t \geq 0$*:*

$$N_{ct} = \#\{n \geq 1 \colon T_n \leq ct\} = \#\{n \geq 1 \colon T_n/c \leq t\}. \tag{A.91}$$

It is now just a matter of noticing that the increments of the sequence $(T_n/c)_{n \geq 1}$ *are independent exponential random variables with parameter* $c\lambda$*.*

Solution 3. *We have* $\mathrm{P}(\Delta N_t > 0) = \sum_{n \in \mathbb{N}} \mathrm{P}(T_n = t) = 0$ *and* $\mathrm{P}(\Delta N_t = 0, \forall t > 0) = \mathrm{P}(T_1 = +\infty) = 0$*.*

Solution 4. *The characteristic function of* (X_t) *may be easily computed by decomposing on the value of* N_t*. With the same technique we get that*

$$\mathrm{E}(X_t) = \mathrm{E}(Z)\mathrm{E}(N_t) = \lambda t \int_{\mathbb{R}^d} z \nu(\mathrm{d}z), \tag{A.92}$$

which yields

$$\phi_{\bar{X}_t}(u) = \exp\Big(\lambda t \int_{\mathbb{R}^d} (e^{i\langle u, z \rangle} - 1 - i\langle u, z \rangle) \nu(\mathrm{d}z)\Big). \tag{A.93}$$