Solution 5. *We have*

$$\mathcal{N}((0,t]) = \sum_{n\in\mathbb{N}} \delta_{T_n}((0,t]) = \sum_{n\in\mathbb{N}} 1_{(0,t]}(T_n) = N_t. \tag{A.94}$$

We will now check that for all $t > 0$, the restriction of $\mathcal{N}$ to $[0,t]$ is a RPM *with intensity λ times Lebesgue measure (restricted to $[0,t]$). Conditionally on $\{N_t = n\}$ ($n \in \mathbb{N}$), the jump times are distributed like n uniform random variables on $[0,t]$ (see Exercise 1). Hence, conditionally on $\{N_t = n\}$ and for all Borel sets $B_1, \ldots, B_k$ on $[0,t]$ ($k \in \mathbb{N}$), the random vector $(\mathcal{N}(B_1), \ldots, \mathcal{N}(B_k))$ is distributed as a multinomial random variable with parameters n and $(|B_1|/t, \ldots, |B_k|/t)$, where $|\cdot|$ stands for Lebesgue measure. To remove the restriction to $[0,t]$, one may invoke the superposition property (see Proposition 2.3)*

Solution 6. *This follows the same idea as in the solution of the previous exercise. Conditionally on $\{N_t = n\}$, the (unordered) set of points $\{(T_i, Z_i),\ 1 \le i \le n\}$ is distributed as a collection of n i.i.d. random variables with law $\mathcal{U}([0,t]) \otimes \nu$. We may deduce thereof that $\mathcal{N}$ is a* $\mathrm{RPM}(\lambda \mathrm{d}t \otimes \nu)$ *on $E = \mathbb{R}_+ \times \mathbb{R}^d$, where* $\mathrm{d}t$ *is Lebesgue measure on the positive half-line.*

Solution 7. *When m is a finite measure, $\mathcal{N}$ may be written as $\sum_{1\le i\le N} \delta_{X_i}$, where N is $\mathcal{P}(m(E))$ and $(X_i)_{i\in\mathbb{N}}$ is a sequence of independent random variables distributed as $m/m(E)$. Recall that the integral of f w.r.t. Dirac measure δ_x equals $f(x)$. Therefore, we get*

$$\int f \mathrm{d}\mathcal{N} = \sum_{1\le i\le N} f(X_i). \tag{A.95}$$

Solution 8. *Define $\mathcal{N}$ as in Exercice 6. To emphasize the distinct roles of time and space, we write $\mathcal{N}(\mathrm{d}s, \mathrm{d}z)$ instead of $\mathrm{d}\mathcal{N}$ (s for time and z for space). Then, one can check that*

$$X_t = \int_{(0,t]\times\mathbb{R}^d} z \mathcal{N}(\mathrm{d}s, \mathrm{d}z). \tag{A.96}$$

This integral is still well defined when $\nu(\mathbb{R}^d) < \infty$, as there will still be a.s. finitely many points in $(0,t] \times \mathbb{R}^d$. Alternatively, one may apply the usual construction with $\nu/\nu(\mathbb{R}^d)$ as jump distribution and replace λ by $\lambda\nu(\mathbb{R}^d)$.

Solution 9. *Let $\mathcal{N}$ be a* $\mathrm{RPM}(\lambda \mathrm{d}t \otimes \mathrm{d}\nu)$. *By Proposition 2.6, the random variables*

$$X_t = \int_{(0,t]\times\mathbb{R}^d} z \mathcal{N}(\mathrm{d}s, \mathrm{d}z). \tag{A.97}$$

are well-defined and a.s. finite for all $t > 0$. This defines a compound Poisson process with prescribed jump measure.