Solution 15. *See Exercise 12.*

Solution 16. *Let $t > 0$. It is enough to check that for all $n \in \mathbb{N}$, X has a.s. finitely many jumps with modulus larger than $1/n$ on $[0,t]$. This comes from the fact that X has a.s. càdlàg sample paths. For details, we refer to Theorem 2.8.1 and Lemma 2.3.4 in* [1].

Solution 17. *In this case we have that $X_t = \sum_{s\in(0,t]} \Delta X_s = \sum_{s\in(0,t]} |\Delta X_s|$ is finite a.s. hence $\int(1 \wedge |z|)\nu(\mathrm{d}z)$ is finite, by Proposition 4.5.*

Solution 18. *A Poisson counting process is a subordinator. A* CPP *with jump measure supported by $(0,+\infty)$ is also a subordinator.*

Solution 19. *By using stationarity of increments, we get that for all $0 < s < t$,*

$$\mathrm{P}(X_t - X_s \geq 0) = \mathrm{P}(X_{t-s} \geq 0) = 1. \tag{A.99}$$

From this we deduce that

$$\mathrm{P}(X_t - X_s \geq 0,\ \forall s,t \in \mathbb{Q}\colon 0 < s < t) = 1, \tag{A.100}$$

and we conclude by right-continuity of the paths.

Solution 20. *The corresponding Lévy-Itô decomposition writes*

$$X_t = bt + \int_{(0,t]\times(0,+\infty)} z\mathcal{N}(\mathrm{d}s,\mathrm{d}z), \tag{A.101}$$

where $b \geq 0$ and $\mathcal{N}$ is a RPM *on $(0,+\infty)^2$ with intensity measure $\mathrm{d}t \otimes \nu$.*

Solution 21. (1) *By continuity of the Brownian sample paths,*

$$T_a = \inf\{t \geq 0\colon B_t \geq a\}, \tag{A.102}$$

from which we get that $a \mapsto T_a$ is a.s. non-decreasing (and T_a is a stopping time w.r.t. to the filtration generated by B, since $[a,+\infty)$ is a closed set, see [8, Proposition 3.9]*). Let us now prove that it is a Lévy process. First, let us show that sample paths are càdlàg. The existence of left limits are a direct consequence of monotonicity. If T were not right-continuous at $a \geq 0$, there would exist $\varepsilon > 0$ such that $B_t \leq a$ for all $t \in [T_a, T_a+\varepsilon]$, which cannot happen. Indeed, conditionally on $T_a < +\infty$ (which is a.s. satisfied), $(B_{T_a+t} - a)$ is distributed as Brownian motion (strong Markov property) and it is known that the set of return times to the origin of a Brownian motion has 0 as accumulation point. Let us now prove that the process has independent and stationary increments, i.e. for every $0 \leq a \leq b$, the random variable $T_b - T_a$ is independent of $\sigma(T_c,\ 0 \leq c \leq a)$ and is distributed as T_{b-a}. First, note that if $c \leq a$ then $T_c \leq T_a$ and then T_c is $\mathcal{F}_{T_a}$ measurable. By using the strong Markov property and stationary increments of Brownian motion, we get for all $t \geq 0$, a.s (recall that $\mathrm{P}(T_a < \infty) = 1$):*

$$\begin{aligned} \mathrm{P}(T_b - T_a \geq t|\mathcal{F}_{T_a}) &= \mathrm{P}(\sup_{0\leq s\leq t} B_{T_a+s} \leq b|\mathcal{F}_{T_a}) \\ &= \mathrm{P}(\sup_{0\leq s\leq t} B_s \leq b|B_0 = a) \\ &= \mathrm{P}(\sup_{0\leq s\leq t} B_s \leq b-a) \\ &= \mathrm{P}(T_{b-a} \geq t), \end{aligned} \tag{A.103}$$

which completes the proof.

(2) *If the process* (T_a) *had a.s. continuous sample paths then by Corollary 4.1 and Theorem 4.2 it would write* $T_a = ca$ *for some* $c > 0$*, which is absurd.*

(3) *Apply Doob's optional stopping theorem to the martingale* $(\exp(\lambda B_s - \frac{1}{2}\lambda^2 s))_{s\geq 0}$ *and bounded stopping time* $t \wedge T_a$*, then let* $t \to \infty$*.*

(4) *By using the change of variable hinted on, we see that* $\mathcal{L}$ *satisfies* $a\mathcal{L}(u) = -(2u)^{1/2} t\mathcal{L}'(u)$ *for* $u > 0$*, with the initial condition* $\mathcal{L}(0) = 1$*. The unique solution is* $\mathcal{L}(u) = \exp(-a\sqrt{2u})$ *(it is actually enough to check this for* $a = 1$*, why?). Therefore,* $\mathcal{L}(u) = \mathrm{E}(e^{-uT_a})$ *for all* $u \geq 0$*, which completes the proof (Laplace transforms characterize probability distributions on* $[0, \infty)$*).*

Solution 22. *The process* M *has continuous sample paths and is non-decreasing a.s. If it were a subordinator then, by Corollary 4.1 and Theorem 4.2, we would get* $M_t = bt$ *for some* $b \geq 0$ *and all* $t \geq 0$*, which is absurd.*

Solution 23. *The* 2*-stable rotationally invariant Lévy processes are the Brownian motions (with arbitrary variance and no drift).*

Solution 24. *For all* $u \geq 0$*, we have*

$$\int_0^\infty (1 - e^{-uz}) \frac{\mathrm{d}z}{z^{1+\alpha}} = \Gamma(1-\alpha)\frac{u^\alpha}{\alpha}. \tag{A.104}$$

(Write $1 - e^{-uz} = \int_0^z ue^{-uy}\mathrm{d}y$ *and interchange integrals). Therefore,*

$$\mathcal{L}(u) = \exp\Big(- bu - u^\alpha + u\int_0^1 z\nu(\mathrm{d}z)\Big), \tag{A.105}$$

and it is enough to pick $b = \int_0^1 z\nu(\mathrm{d}z) = \frac{\alpha}{(1-\alpha)\Gamma(1-\alpha)}$*.*

Solution 25. *Using the solution to Exercise 24 with* $\alpha = 1/2$*, we see that the jump measure of the process of Brownian ladder times must be*

$$\nu(\mathrm{d}z) = \frac{\mathrm{d}z}{\Gamma(1/2)\sqrt{2z^3}}. \tag{A.106}$$