Solution 26. (1) *Let $G \in \mathcal{H}^2(T)$ such that G is orthogonal to $\mathcal{S}$. It is enough to show that $G = 0$, $\mathrm{d}t \otimes \nu \otimes \mathrm{P}$-a.s (see Lemma A.1). Let $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\nu(A) < \infty$, $s \le t \le T$ and F_s a bounded and $\mathcal{F}_s$-measurable random variable. Consider*

$$F(r,z) = F_s \mathbf{1}_{(s,t]}(r) \mathbf{1}_A(z). \tag{A.107}$$

Then

$$0 = \langle F, G\rangle_{\mathcal{H}^2} = \int_{(s,t]\times A} \mathrm{E}[F_s G(r,z)] \mathrm{d}r\ \nu(\mathrm{d}z). \tag{A.108}$$

Define the following process:

$$X_t^{(A)} := \int_{(0,t]\times A} G(r,z)\mathrm{d}r\nu(\mathrm{d}z), \qquad 0 \le t \le T. \tag{A.109}$$

One may check that it is square-integrable and adapted. Indeed,

- *By Jensen's inequality, $\mathrm{E}[(X_t^{(A)})^2] \le t\nu(A)\|G\|^2_{\mathcal{H}^2(T)} < +\infty$.*
- *Since G is predictable, the mapping*

$$(\omega, z) \in \Omega \times \mathbb{R}^d \mapsto \int_{(0,t]} G(r,z)\mathbf{1}_A(z)\mathrm{d}r \tag{A.110}$$

  is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$-measurable[§]. By Fubini's theorem, we readily obtain that

$$\omega \in \Omega \mapsto \int_{(0,t]\times\mathbb{R}^d} G(r,z)\mathbf{1}_A(z)\mathrm{d}r\nu(\mathrm{d}z) \tag{A.111}$$

  is $\mathcal{F}_t$-measurable.

By (A.108), we get that $\mathrm{E}[F_s(X_t^{(A)} - X_s^{(A)})] = 0$, which yields

$$\mathrm{E}[X_t^{(A)} | \mathcal{F}_s] = X_s^{(A)}. \tag{A.112}$$

Hence, $(X_t^{(A)})$ is an $\mathcal{F}$-martingale. Moreover, one can check that this is a process with finite variations (difference between two non-decreasing processes, see [8, Section 4.1]). Hence, it is a.s. constant (see [8, Theorem 4.8]), so $X_t^{(A)} = X_0^{(A)} = 0$ P-a.s. Let us now deduce that $G = 0$, $\mathrm{d}t \otimes \nu \otimes \mathrm{P}$-a.s, with the help of a monotone class argument. To this end, define

$$\mathcal{C} = \Big\{ C \subseteq [0,T] \times \mathbb{R} \colon \int_C G(r,z)\mathrm{d}r\nu(\mathrm{d}z) = 0 \quad \mathrm{P} - a.s. \Big\}. \tag{A.113}$$

We have proven so far that $\mathcal{C}$ contains the set

$$\mathcal{C}_0 := \Big\{ (s,t] \times A, \qquad 0 \le s \le t \le T, \quad A \in \mathcal{B}(\mathbb{R}) \Big\}, \tag{A.114}$$

that is a π-system generating $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R})$. Furthermore, one can check that $\mathcal{C}$ is a monotone class. By Dynkin's theorem, the property defining $\mathcal{C}$ is therefore valid for all sets in $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R})$. The reader may check that this implies $G = 0, \mathrm{d}t \otimes \nu \otimes \mathrm{P}$-a.s. Hint : use $\{G \ge 0\}$ and $\{G \le 0\}$ as test sets.

[§]Any adapted process with left-continuous (or right-continuous) sample paths is progressive, see [8, Proposition 3.4]

(2) *$F(t_j)$ is bounded and $\widetilde{\mathcal{N}}((t_j,t_{j+1}]\times A_i)$ is square-integrable (we recall that $\nu(A_i)$ is finite) so $F(t_j)\widetilde{\mathcal{N}}((t_j,t_{j+1}]\times A_i)$ is square-integrable for all i and j. A finite sum of square-integrable random variables is square-integrable. Since $F(t_j)$ is $\mathcal{F}_{t_j}$-measurable and $\widetilde{\mathcal{N}}((t_j,t_{j+1}]\times A_i)$ is independent from $\mathcal{F}_{t_j}$, we get*

$$\mathrm{E}\Big[F(t_j)\widetilde{\mathcal{N}}((t_j,t_{j+1}]\times A_i)\Big]=0, \tag{A.115}$$

so $I_T(F)$ is centered.

(3) *The mapping I_T is clearly linear so it remains to prove that $\mathrm{E}(I_T(F)^2)=\|F\|^2_{\mathcal{H}^2(T)}$. For convenience, let us define for every i and j,*

$$I_{i,j}=F(t_j)\widetilde{\mathcal{N}}((t_j,t_{j+1}]\times A_i), \tag{A.116}$$

so that

$$I_T(F)=\sum_{1\le j\le m}\Big(\sum_{1\le i\le n}c_iI_{i,j}\Big). \tag{A.117}$$

Since $F(t_j)$ is $\mathcal{F}_{t_j}$-measurable and $\widetilde{\mathcal{N}}((t_j,t_{j+1}]\times A_i)$ is centered and independent from $\mathcal{F}_{t_j}$, one may check that the random variables $(\sum_i c_iI_{i,j})_{1\le j\le m}$'s are orthogonal. Therefore,

$$\begin{aligned}\mathrm{E}(I_T(F)^2)&=\sum_{1\le j\le m}\mathrm{E}\Big[\Big(\sum_{1\le i\le m}c_iI_{i,j}\Big)^2\Big]\\&=\sum_{1\le j\le m}\mathrm{Var}\Big(\sum_{1\le i\le m}c_iI_{i,j}\Big).\end{aligned} \tag{A.118}$$

Since the $(A_i)_{1\le i\le n}$'s are disjoint, we get from Proposition 2.4 that for every j, the random variables $(I_{i,j})_{1\le i\le n}$'s are independent, hence

$$\begin{aligned}\mathrm{E}[I_T(F)^2]&=\sum_{1\le j\le m}\sum_{1\le i\le m}c_i^2\,\mathrm{Var}(I_{i,j})\\&=\sum_{1\le j\le m}\sum_{1\le i\le n}c_i^2\mathrm{E}[F(t_j)^2\widetilde{\mathcal{N}}((t_j,t_{j+1}]\times A_i)^2]\\&=\sum_{1\le j\le m}\sum_{1\le i\le n}c_i^2\mathrm{E}[F(t_j)^2](t_{j+1}-t_j)\nu(A_i)=\|F\|^2_{\mathcal{H}^2(T)}.\end{aligned} \tag{A.119}$$

(4) *Since $\mathcal{S}$ is dense, the isometry may be uniquely extended to $\mathcal{H}^2(T)$.*

Solution 27. *Pick $b(t)=b$, $\sigma(t)=\sigma$, $H(t,z)=K(t,z)=z$ for all (t,z).*

Solution 28. (1) *We have $N_t=\int_0^t\int_{|z|\ge1}(z)\mathcal{N}(\mathrm{d}s,\mathrm{d}z)$ where $\mathcal{N}$ is a $\mathrm{RPM}(\lambda\mathrm{d}t\otimes\delta_1)$. We may then write $\mathrm{d}N_s:=\int_{|z|\ge1}(z)\mathcal{N}(\mathrm{d}s,\mathrm{d}z)$, that is a RPM on $(0,\infty)$ with intensity $\lambda\mathrm{d}t$ (i.e. a homogeneous Poisson point process with intensity $\lambda>0$). By analogy with compensated Poisson measures we define $\mathrm{d}\widetilde{N}_s$ as the signed measure $\mathrm{d}N_s-\lambda\mathrm{d}s$.*

(2) *We have, with the usual notations,*

$$\int_0^t f(s)\mathrm{d}N_s=\sum_{1\le i\le N_t}f(T_i)\qquad\text{and}\qquad\int_0^t f(s)\mathrm{d}\widetilde{N}_s=\sum_{1\le i\le N_t}f(T_i)-\lambda\int_0^t f(s)\mathrm{d}s. \tag{A.120}$$

(3) $\int_0^t B_s\mathrm{d}N_s=\sum_{1\le i\le N_t}B_{T_i}$.

(4) *The presence of* s^- *instead of* s *is here to ensure predictability of the process* (N_{s^-})*. We have*

$$X_t = \int_0^t N_{s^-}\mathrm{d}N_s = \sum_{1\le i\le N_t} N_{T_i^-} = \sum_{1\le i\le N_t}(i-1) = \frac{N_t(N_t-1)}{2} \tag{A.121}$$

(with the convention that the sum over an empty set is zero, which is consistent with the case $N_t = 0$*). Note that* X_t *is* not *equal to* $\frac{1}{2}N_t^2$*, as we would get from a blind application of standard (non-stochastic) calculus. Let us now look at the compensated version:*

$$\widetilde{X}_t := \int_0^t N_{s^-}\mathrm{d}\widetilde{N}_s = \frac{N_t(N_t-1)}{2} - \lambda\int_0^t N_{s^-}\mathrm{d}s, \tag{A.122}$$

from which we get by a straightforward computation that $\mathrm{E}(\widetilde{X}_t) = 0$*. As one can check, this would no longer hold if we were to replace* N_{s^-} *by* N_s *in the definition of* X_t*.*

(5) *Apply Itô's formula with* $f(t,z) = z^2$ *and* $K(t,z) = 1$*. Only the first term of (III) is present. We get*

$$\mathrm{d}Y_t = [(N_{t^-}+1)^2 - N_{t^-}^2]\mathrm{d}N_t = (2N_{t^-}+1)\mathrm{d}N_t. \tag{A.123}$$

One may check that this is consistent with our previous answer.

Solution 29. *Let us write both formulas (in one dimension) next to each other to highlight the analogy:*

$$\begin{aligned} \mathcal{L}f(x) &= bf'(x) + \tfrac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}}[f(x+z) - f(x) - zf'(x)1_{\{|z|\le1\}}]\nu(\mathrm{d}z),\\ \Psi(u) &= ibu - \frac{1}{2}au^2 + \int_{\mathbb{R}^d}\left(e^{iuz} - 1 - iuz1_{\{|z|\le1\}}\right)\nu(\mathrm{d}z). \end{aligned} \tag{A.124}$$

Derivation corresponds to multiplication by iu *in Fourier mode (hence differentiating twice corresponds to multiplication by* $(iu)^2 = -u^2$*) and shifting by* z *corresponds to multiplication by* e^{iuz} *(addition of a phase).*