Solution 32. (1) *We proceed by backward iteration. The formula clearly holds for $k = n$. Let $1 \leq k < n$ and assume that the formula holds for $k+1$. Using what we know from the one-step model, we get*

$$W_k(X_1, \dots, X_k) = qW_{k+1}(X_1, \dots, X_k, +) + (1-q)W_{k+1}(X_1, \dots, X_k, -), \tag{A.127}$$

with

$$q = \frac{1 - 1/e}{e - 1/e} = \frac{e-1}{e^2-1} = \frac{1}{1+e}. \tag{A.128}$$

We conclude by using the formula at step $k+1$.

(2) *Same idea, with this time $A_0/A_1 = e^{-r}$ and*

$$q = \frac{e^{r+1} - 1}{e^2 - 1}. \tag{A.129}$$

(3) *Check that $\mathrm{E}[e^{X_1}] = e^r$.*

Solution 33. *Let us denote by Q_t the restriction of Q to events in $\mathcal{F}_t$. By the martingale property we get $Q_t(\Omega) = \mathrm{E}_P(e^{Y_t}) = \mathrm{E}_P(e^{Y_0}) = 1$ and the consistency condition:*

$$Q_t(A) = \mathrm{E}_P(e^{Y_t} A) = \mathrm{E}_P(e^{Y_s} A) = Q_s(A), \qquad 0 \leq s \leq t, A \in \mathcal{F}_s. \tag{A.130}$$

We conclude by Kolmogorov's extension theorem.

Solution 34. *By taking logarithm, we have*

$$\mathrm{d}(\log S_t) = \sigma \mathrm{d}B_t + \Big(\mu - \frac{1}{2}\sigma^2\Big)\mathrm{d}t, \tag{A.131}$$

hence

$$\mathrm{d}(\log \widetilde{S}_t) = \Big(\sigma \mathrm{d}B_Q(t) - \frac{1}{2}\sigma^2 \mathrm{d}t\Big) + \Big(\mu - r + \sigma F(t)\Big)\mathrm{d}t. \tag{A.132}$$

The only possible choice is

$$F(t) = \frac{r - \mu}{\sigma}, \qquad \forall t \geq 0, \tag{A.133}$$

in which case, by Itô's formula,

$$\mathrm{d}\widetilde{S}_t = \sigma \widetilde{S}_{t-} \mathrm{d}B_Q(t). \tag{A.134}$$

Solution 35. (1) *For all $t \in [0, T]$, we have*

$$\begin{aligned} W_t &= U_t S_t + V_t A_t \\ &= \gamma_t S_t + Z_t A_t - \gamma_t \widetilde{S}_t A_t \\ &= Z_t A_t \\ &= e^{-r(T-t)} \mathrm{E}_Q(Z | \mathcal{F}_t). \end{aligned} \tag{A.135}$$

(2) *The portfolio is replicating, since $W_T = \mathrm{E}(Z|\mathcal{F}_T) = Z$.*

(3) *By Itô's formula, see Theorem 5.6.4 in* [1].

(4) *The arbitrage-free value of the contingent claim is*

$$W_0 = e^{-rT} \mathrm{E}_Q(Z) \tag{A.136}$$

(compare with (8.15)).

Solution 36. *Using the result of Exercise 35, the arbitrage-free price of the option writes*

$$W_0 = e^{-rT}\mathrm{E}_Q[(S_T - k)_+]. \tag{A.137}$$

Check that

$$S_T = S_0 e^{\sigma B_Q(T) + (r - \frac{1}{2}\sigma^2)T}. \tag{A.138}$$

Using that B_Q is a standard Brownian motion under P_Q, we obtain

$$W_0 = \int_{\mathbb{R}} (S_0 e^x - k e^{-rT})_+ \, e^{-\frac{(x+\frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}} \frac{\mathrm{d}x}{\sigma\sqrt{2\pi T}}. \tag{A.139}$$

Deduce thereof that

$$W_0 = S_0 \Phi\Big(\frac{\log(S_0/k) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\Big) - k e^{-rT} \Phi\Big(\frac{\log(S_0/k) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\Big), \tag{A.140}$$

where Φ is the cumulative distribution function of a standard Gaussian random variable. See [1, Theorem 5.6.4] *and references therein for a complete solution.*

Solution 37. *Let g_{m,v^2} be the density of $\mathcal{N}(m, v^2)$. Then,*

$$\nu(\mathrm{d}z) = \lambda g_{m,v^2}(\mathrm{d}z), \qquad b = \mu + \int_{-1}^{1} z\nu(\mathrm{d}z). \tag{A.141}$$

Solution 38. *Clearly, ν_θ is a non-negative measure. Moreover,*

$$\int_{\mathbb{R}} (1 \wedge |z|^2)\nu_\theta(\mathrm{d}z) \le \int_{\mathbb{R}} \nu_\theta(\mathrm{d}z) = \int_{\mathbb{R}} e^{\theta z} g_{m,v^2}(z)\mathrm{d}z < \infty. \tag{A.142}$$

Solution 39. (1) *By dominated convergence,*

$$f'(\theta) = \int_{\mathbb{R}} x(e^x - 1)e^{\theta x}\nu(\mathrm{d}x) \ge 0, \tag{A.143}$$

so f is non-decreasing.

(2) *Since $\nu((0,\infty)) > 0$ and $\nu((-\infty, 0)) > 0$ we respectively get*

$$\begin{aligned} f'(\theta) &\ge \int_0^\infty x(e^x - 1)\nu(\mathrm{d}x) > 0 \qquad (\theta \ge 0) \\ f'(\theta) &\ge \int_{-\infty}^0 x(e^x - 1)\nu(\mathrm{d}x) > 0 \qquad (\theta \le 0). \end{aligned} \tag{A.144}$$

(3) *From what precedes, f is a bijection from $\mathbb{R}$ to $\mathbb{R}$, so that (8.32) has a unique solution.*