Solutions

Solution 1. (1) *By a simple change of variable we get that* $(T_1, \dots, T_n)$ *has density:*

$$(t_1, \dots, t_n) \mapsto \lambda^n e^{-\lambda t_n} 1_{\{0<t_1<\dots<t_n\}}. \tag{A.86}$$

(2) *By integrating on the* $(n-1)$*-first variables we get that* T_n *has density*

$$t_n \mapsto \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} 1_{\{t>0\}}, \tag{A.87}$$

that is a $\Gamma(n, \lambda)$ *random variable.*

(3) *Let* $n \in \mathbb{N}_0$ *and* $t \geq 0$*. Remind that* $T_0 = 0$*. We note that* $\mathrm{P}(N_t = n) = \mathrm{P}(T_n \leq t < T_{n+1})$*, which we may compute by using the density of* $(T_1, \dots, T_{n+1})$*.*

(4) *By using the density of* $(T_1, \dots, T_{n+1})$*, one may check that for any measurable function* $f\colon \mathbb{R}_n \to \mathbb{R}_+$*,*

$$\mathrm{E}[f(T_1, \dots, T_n)|N_t = n] = \int_{0\leq t_1<\dots<t_n\leq t} f(t_1, \dots, t_n) \frac{n!}{t^n} \mathrm{d}t_1 \dots \mathrm{d}t_n. \tag{A.88}$$

(5) *Since the process* (N_t) *is non-decreasing , we get that* $\mathrm{P}(N_t < \infty, \forall t \geq 0) = \mathrm{P}(N_k < \infty, \forall k \in \mathbb{N})$*. This probability equals one since* N_k *is finite a.s, for all* k *in the countable set of integers..*

Solution 2. *We have for all* $t \geq 0$*:*

$$N_{ct} = \#\{n \geq 1\colon T_n \leq ct\} = \#\{n \geq 1\colon T_n/c \leq t\}. \tag{A.89}$$

It is now just a matter of noticing that the increments of the sequence $(T_n/c)_{n\geq 1}$ *are independent exponential random variables with parameter* $c\lambda$*.*

Solution 3. *We have* $\mathrm{P}(\Delta N_t > 0) = \sum_{n\in\mathbb{N}} \mathrm{P}(T_n = t) = 0$ *and* $\mathrm{P}(\Delta N_t = 0, \forall t > 0) = \mathrm{P}(T_1 = +\infty) = 0$*.*

Solution 4. *The characteristic function of* (X_t) *may be easily computed by decomposing on the value of* N_t*. With the same technique we get that*

$$\mathrm{E}(X_t) = \mathrm{E}(Z)\mathrm{E}(N_t) = \lambda t \int_{\mathbb{R}^d} z\nu(\mathrm{d}z), \tag{A.90}$$

which yields

$$\phi_{\bar{X}_t}(u) = \exp\Big(\lambda t \int_{\mathbb{R}^d} (e^{i\langle u,z\rangle} - 1 - i\langle u, z\rangle)\nu(\mathrm{d}z)\Big). \tag{A.91}$$

Solution 5. *We have*

$$\mathcal{N}((0,t]) = \sum_{n\in\mathbb{N}} \delta_{T_n}((0,t]) = \sum_{n\in\mathbb{N}} \mathbf{1}_{(0,t]}(T_n) = N_t. \tag{A.92}$$

We will now check that for all $t>0$, the restriction of $\mathcal{N}$ to $[0,t]$ is a RPM *with intensity λ times Lebesgue measure (restricted to $[0,t]$). Conditionally on $\{N_t = n\}$ ($n\in\mathbb{N}$), the jump times are distributed like n uniform random variables on $[0,t]$ (see Exercise 1). Hence, conditionally on $\{N_t=n\}$ and for all Borel sets $B_1,\dots,B_k$ on $[0,t]$ ($k\in\mathbb{N}$), the random vector $(\mathcal{N}(B_1),\dots,\mathcal{N}(B_k))$ is distributed as a multinomial random variable with parameters n and $(|B_1|/t,\dots,|B_k|/t)$, where $|\cdot|$ stands for Lebesgue measure. To remove the restriction to $[0,t]$, one may invoke the superposition property (see Proposition 2.3)*

Solution 6. *This follows the same idea as in the solution of the previous exercise. Conditionally on $\{N_t=n\}$, the (unordered) set of points $\{(T_i,Z_i),\ 1\le i\le n\}$ is distributed as a collection of n i.i.d. random variables with law $\mathcal{U}([0,t])\otimes\nu$. We may deduce thereof that $\mathcal{N}$ is a* $\mathrm{RPM}(\lambda \mathrm{d}t\otimes\nu)$ *on $E=\mathbb{R}_+\times\mathbb{R}^d$, where $\mathrm{d}t$ is Lebesgue measure on the positive half-line.*

Solution 7. *When m is a finite measure, $\mathcal{N}$ may be written as $\sum_{1\le i\le N}\delta_{X_i}$, where N is $\mathcal{P}(m(E))$ and $(X_i)_{i\in\mathbb{N}}$ is a sequence of independent random variables distributed as $m/m(E)$. Recall that the integral of f w.r.t. Dirac measure δ_x equals $f(x)$. Therefore, we get*

$$\int f\,\mathrm{d}\mathcal{N} = \sum_{1\le i\le N} f(X_i). \tag{A.93}$$

Solution 8. *Define $\mathcal{N}$ as in Exercice 6. To emphasize the distinct roles of time and space, we write $\mathcal{N}(\mathrm{d}s,\mathrm{d}z)$ instead of $\mathrm{d}\mathcal{N}$ (s for time and z for space). Then, one can check that*

$$X_t = \int_{(0,t]\times\mathbb{R}^d} z\mathcal{N}(\mathrm{d}s,\mathrm{d}z). \tag{A.94}$$

This integral is still well defined when $\nu(\mathbb{R}^d)<\infty$, as there will still be a.s. finitely many points in $(0,t]\times\mathbb{R}^d$. Alternatively, one may apply the usual construction with $\nu/\nu(\mathbb{R}^d)$ as jump distribution and replace λ by $\lambda\nu(\mathbb{R}^d)$.

Solution 9. *Let $\mathcal{N}$ be a* $\mathrm{RPM}(\lambda\mathrm{d}t\otimes\mathrm{d}\nu)$*. By Proposition 2.6, the random variables*

$$X_t = \int_{(0,t]\times\mathbb{R}^d} z\mathcal{N}(\mathrm{d}s,\mathrm{d}z). \tag{A.95}$$

are well-defined and a.s. finite for all $t>0$. This defines a compound Poisson process with prescribed jump measure.

Solution 10. *Let us denote by μ the probability law under consideration and μ_n the law such that $\mu = \mu_n^{*n}$.*

(1) $\mu_n = \mathcal{N}(\frac{m}{n}, \frac{\sigma^2}{n})$;
(2) $\mu_n = \mathcal{P}(\lambda/n)$;
(3) *pick μ_n as the law of a compound Poisson process with intensity λ/n and jump distribution ν, evaluated at time 1;*
(4) *pick $\mu_n = \mathrm{Gamma}(\frac{a}{n}, b)$ (when $k \in \mathbb{N}$ recall that $\mathrm{Gamma}(k, b)$ is the law of the sum of k i.i.d. $\mathcal{E}(b)$ random variables); All one needs to prove is that*

$$\mathrm{Gamma}(a_1, 1) * \mathrm{Gamma}(a_2, 1) = \mathrm{Gamma}(a_1 + a_2, 1), \tag{A.96}$$

(note that b is but a scaling parameter) which can be done via the Laplace transform or the convolution formula.

(5) $\mu_n = \delta_{a/n}$.

Solution 11. *Let X be a $\mathrm{Ber}(p)$ random variable. Suppose that $X = Y_1 + Y_2$ with Y_1 and Y_2 independent and identically distributed. Prove that necessarily $\mathrm{P}(Y_1 = 1/2) = \sqrt{p}$ and $\mathrm{P}(Y_1 = 0) = \sqrt{1-p}$. This is impossible when $p \in (0, 1)$. See Example 9 in* [15] *for a full solution.*

Solution 12. (1) $\Psi(u) = ium - \frac{u^2\sigma^2}{2}$;

(2) $\Psi(u) = \lambda(e^{iu} - 1)$;
(3) $\Psi(u) = \lambda \int (e^{iuz} - 1)\nu(\mathrm{d}z)$;
(4) $\Psi(u) = -a\log(1 - i\frac{u}{b})$, *where* $\log$ *denotes the principal value of the complex logarithm.*
Method 1: *First compute it when $a \in \mathbb{N}$ using the interpretation of $\Gamma(a, b)$ as the sum of independent exponential variables, then extend the formula to $a \in \mathbb{Q} \cap (0, +\infty)$ using infinite divisibility. To extend it to $a > 0$, verify that for all $u \in \mathbb{R}$ the mapping $a > 0 \mapsto \int_0^\infty e^{iut} t^{a-1} e^{-t} \mathrm{d}t$ is continuous (by dominated convergence).*
Method 2: *Let $a > 0$ be fixed and $b = 1$ (the general case $b > 0$ follows by scaling). Using a change of variable, we have for every $\lambda > 0$:*

$$\int_0^{+\infty} \frac{t^{a-1} e^{-\lambda t}}{\Gamma(a)} \mathrm{d}t = \lambda^{-a}. \tag{A.97}$$

Let us now consider these two expressions (in the left and right-hand sides respectively) as functions of the complex variable λ. *Both functions are well-defined and holomorphic (that is $\mathbb{C}$-differentiable) on the open connected set $\{\lambda \in \mathbb{C} \colon \mathrm{Re}(\lambda) > 0\}$ so they coincide on this set, by the Principle of Analytic Continuation* [9, Theorems 1 and 2 in Section 1.3]. *This set includes complex numbers of the form $1 - iu$, $u \in \mathbb{R}$, which allows to conclude.*

(5) $\Psi(u) = iua$.

Solution 13. *For any fixed $u \in \mathbb{R}^d$, the function $z \mapsto e^{i\langle u,z\rangle} - 1 - i\langle u, z\rangle 1_{\{|z| \leq 1\}}$ is bounded (in modulus) by some constant times $1 \wedge |z|^2$. Since ν is a Lévy measure, the integral is well-defined.*

Solution 14. *First of all, the assumption that ν is finite allows us to split in two the integral in (3.3) and proves that c is well-defined. Moreover,*

$$\begin{aligned} \phi_Y(u) &= \exp\Big(-\frac{1}{2}\langle u, Au\rangle\Big), \\ \phi_{\widetilde{Y}}(u) &= \exp\Big(\int (e^{i\langle u,z\rangle}-1)\nu(\mathrm{d}z)\Big) \qquad \textit{(see Exercise 4)}. \end{aligned} \tag{A.98}$$

We conclude by using the independence of Y and $\widetilde{Y}$.

Solution 15. *See Exercise 12.*

Solution 16. *Let $t > 0$. It is enough to check that for all $n \in \mathbb{N}$, X has a.s. finitely many jumps with modulus larger than $1/n$ on $[0,t]$. This comes from the fact that X has a.s. càdlàg sample paths. For details, we refer to Theorem 2.8.1 and Lemma 2.3.4 in* [1].

Solution 17. *In this case we have that $X_t = \sum_{s\in(0,t]} \Delta X_s = \sum_{s\in(0,t]} |\Delta X_s|$ is finite a.s. hence $\int(1 \wedge |z|)\nu(\mathrm{d}z)$ is finite, by Proposition 4.5.*

Solution 18. *A Poisson counting process is a subordinator. A* CPP *with jump measure supported by $(0,+\infty)$ is also a subordinator.*

Solution 19. *By using stationarity of increments, we get that for all $0 < s < t$,*

$$\mathrm{P}(X_t - X_s \geq 0) = \mathrm{P}(X_{t-s} \geq 0) = 1. \tag{A.99}$$

From this we deduce that

$$\mathrm{P}(X_t - X_s \geq 0,\ \forall s,t \in \mathbb{Q}\colon 0 < s < t) = 1, \tag{A.100}$$

and we conclude by right-continuity of the paths.

Solution 20. *The corresponding Lévy-Itô decomposition writes*

$$X_t = bt + \int_{(0,t]\times(0,+\infty)} z\mathcal{N}(\mathrm{d}s, \mathrm{d}z), \tag{A.101}$$

where $b \geq 0$ and $\mathcal{N}$ is a RPM *on $(0,+\infty)^2$ with intensity measure $\mathrm{d}t \otimes \nu$.*

Solution 21. (1) *By continuity of the Brownian sample paths,*

$$T_a = \inf\{t \geq 0\colon B_t \geq a\}, \tag{A.102}$$

from which we get that $a \mapsto T_a$ is a.s. non-decreasing (and T_a is a stopping time w.r.t. to the filtration generated by B, since $[a,+\infty)$ is a closed set, see [8, Proposition 3.9]*). Let us now prove that it is a Lévy process. First, let us show that sample paths are càdlàg. The existence of left limits are a direct consequence of monotonicity. If T were not right-continuous at $a \geq 0$, there would exist $\varepsilon > 0$ such that $B_t \leq a$ for all $t \in [T_a, T_a+\varepsilon]$, which cannot happen. Indeed, conditionally on $T_a < +\infty$ (which is a.s. satisfied), $(B_{T_a+t} - a)$ is distributed as Brownian motion (strong Markov property) and it is known that the set of return times to the origin of a Brownian motion has 0 as accumulation point. Let us now prove that the process has independent and stationary increments, i.e. for every $0 \leq a \leq b$, the random variable $T_b - T_a$ is independent of $\sigma(T_c,\ 0 \leq c \leq a)$ and is distributed as T_{b-a}. First, note that if $c \leq a$ then $T_c \leq T_a$ and then T_c is $\mathcal{F}_{T_a}$ measurable. By using the strong Markov property and stationary increments of Brownian motion, we get for all $t \geq 0$, a.s (recall that $\mathrm{P}(T_a < \infty) = 1$):*

$$\begin{aligned}
\mathrm{P}(T_b - T_a \geq t|\mathcal{F}_{T_a}) &= \mathrm{P}(\sup_{0\leq s\leq t} B_{T_a+s} \leq b|\mathcal{F}_{T_a}) \\
&= \mathrm{P}(\sup_{0\leq s\leq t} B_s \leq b|B_0 = a) \\
&= \mathrm{P}(\sup_{0\leq s\leq t} B_s \leq b - a) \\
&= \mathrm{P}(T_{b-a} \geq t),
\end{aligned} \tag{A.103}$$

which completes the proof.

(2) *If the process* (T_a) *had a.s. continuous sample paths then by Corollary 4.1 and Theorem 4.2 it would write* $T_a = ca$ *for some* $c > 0$*, which is absurd.*

(3) *Apply Doob's optional stopping theorem to the martingale* $(\exp(\lambda B_s - \frac{1}{2}\lambda^2 s))_{s\geq 0}$ *and bounded stopping time* $t \wedge T_a$*, then let* $t \to \infty$*.*

(4) *By using the change of variable hinted on, we see that* $\mathcal{L}$ *satisfies* $a\mathcal{L}(u) = -(2u)^{1/2}t\mathcal{L}'(u)$ *for* $u > 0$*, with the initial condition* $\mathcal{L}(0) = 1$*. The unique solution is* $\mathcal{L}(u) = \exp(-a\sqrt{2u})$ *(it is actually enough to check this for* $a = 1$*, why?). Therefore,* $\mathcal{L}(u) = \mathrm{E}(e^{-uT_a})$ *for all* $u \geq 0$*, which completes the proof (Laplace transforms characterize probability distributions on* $[0,\infty)$*).*

Solution 22. *The process* M *has continuous sample paths and is non-decreasing a.s. If it were a subordinator then, by Corollary 4.1 and Theorem 4.2, we would get* $M_t = bt$ *for some* $b \geq 0$ *and all* $t \geq 0$*, which is absurd.*

Solution 23. *For all* $u \geq 0$*, we have*

$$\int_0^\infty (1 - e^{-uz})\frac{\mathrm{d}z}{z^{1+\alpha}} = \Gamma(1-\alpha)\frac{u^\alpha}{\alpha}. \tag{A.104}$$

(Write $1 - e^{-uz} = \int_0^z ue^{-uy}\mathrm{d}y$ *and interchange integrals). Therefore,*

$$\mathcal{L}(u) = \exp\Big(- bu - u^\alpha + u\int_0^1 z\nu(\mathrm{d}z)\Big), \tag{A.105}$$

and it is enough to pick $b = \int_0^1 z\nu(\mathrm{d}z) = \frac{\alpha}{(1-\alpha)\Gamma(1-\alpha)}$*.*

Solution 24. *Using the solution to Exercise 23 with* $\alpha = 1/2$*, we see that the jump measure of the process of Brownian ladder times must be*

$$\nu(\mathrm{d}z) = \frac{\mathrm{d}z}{\Gamma(1/2)\sqrt{2z^3}}. \tag{A.106}$$

Solution 25. (1) *The random variable Y_j is bounded and $\widetilde{\mathcal{N}}((t_j, t_{j+1}] \times A_i)$ is square-integrable (we recall that $\nu(A_i)$ is finite) so $Y_j\widetilde{\mathcal{N}}((t_j, t_{j+1}] \times A_i)$ is square-integrable for all i and j. A finite sum of square-integrable random variables is square-integrable. Since Y_j is $\mathcal{F}_{t_j}$-measurable and $\widetilde{\mathcal{N}}((t_j, t_{j+1}] \times A_i)$ is independent from $\mathcal{F}_{t_j}$, we get*

$$\mathrm{E}\Big[Y_j\widetilde{\mathcal{N}}((t_j, t_{j+1}] \times A_i)\Big] = 0, \tag{A.107}$$

so $I_T(F)$ is centered.

(2) *The mapping I_T is clearly linear so it remains to prove that $\mathrm{E}(I_T(F)^2) = \|F\|^2_{\mathcal{H}^2(T)}$. For convenience, let us define for every i and j,*

$$I_{i,j} = Y_j\widetilde{\mathcal{N}}((t_j, t_{j+1}] \times A_i), \tag{A.108}$$

so that

$$I_T(F) = \sum_{1\le j\le m} \Big(\sum_{1\le i\le n} c_i I_{i,j}\Big). \tag{A.109}$$

Since Y_j is $\mathcal{F}_{t_j}$-measurable and $\widetilde{\mathcal{N}}((t_j, t_{j+1}] \times A_i)$ is centered and independent from $\mathcal{F}_{t_j}$, one may check that the random variables $(\sum_i c_i I_{i,j})_{1\le j\le m}$'s are orthogonal. Therefore,

$$\begin{aligned}\mathrm{E}(I_T(F)^2) &= \sum_{1\le j\le m} \mathrm{E}\Big[\Big(\sum_{1\le i\le n} c_i I_{i,j}\Big)^2\Big]\\ &= \sum_{1\le j\le m} \mathrm{Var}\Big(\sum_{1\le i\le n} c_i I_{i,j}\Big).\end{aligned} \tag{A.110}$$

Since the $(A_i)_{1\le i\le n}$'s are disjoint, we get from Proposition 2.4 that for every j, the random variables $(I_{i,j})_{1\le i\le n}$'s are independent, hence

$$\begin{aligned}\mathrm{E}[I_T(F)^2] &= \sum_{1\le j\le m}\sum_{1\le i\le m} c_i^2\,\mathrm{Var}(I_{i,j})\\ &= \sum_{1\le j\le m}\sum_{1\le i\le n} c_i^2 \mathrm{E}[Y_j^2\widetilde{\mathcal{N}}((t_j, t_{j+1}] \times A_i)^2]\\ &= \sum_{1\le j\le m}\sum_{1\le i\le n} c_i^2 \mathrm{E}[Y_j^2](t_{j+1} - t_j)\nu(A_i) = \|F\|^2_{\mathcal{H}^2(T)}.\end{aligned} \tag{A.111}$$

(3) *Let $G \in \mathcal{H}^2(T)$ such that G is orthogonal to $\mathcal{S}$. It is enough to show that $G = 0$, $\mathrm{d}t \otimes \nu \otimes \mathrm{P}$-a.s (see Lemma A.1). Let $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\nu(A) < \infty$, $s \le t \le T$ and Y_s a bounded and $\mathcal{F}_s$-measurable random variable. Consider*

$$F(r, z) = Y_s \mathbf{1}_{(s,t]}(r)\mathbf{1}_A(z). \tag{A.112}$$

Then

$$0 = \langle F, G\rangle_{\mathcal{H}^2} = \int_{(s,t]\times A} \mathrm{E}[Y_s G(r, z)]\mathrm{d}r\ \nu(\mathrm{d}z). \tag{A.113}$$

Define the following process:

$$X_t^{(A)} := \int_{(0,t]\times A} G(r, z)\mathrm{d}r\nu(\mathrm{d}z), \qquad 0 \le t \le T. \tag{A.114}$$

One may check that it is square-integrable and adapted. Indeed,

- *By Jensen's inequality, $\mathrm{E}[(X_t^{(A)})^2] \le t\nu(A)\|G\|^2_{\mathcal{H}^2(T)} < +\infty$.*

- *Since G is predictable, the mapping*

$$(\omega, z) \in \Omega \times \mathbb{R}^d \mapsto \int_{(0,t]} G(r,z)\mathbf{1}_A(z)\mathrm{d}r \tag{A.115}$$

is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$-measurable[**]*. By Fubini's theorem, we readily obtain that*

$$\omega \in \Omega \mapsto \int_{(0,t]\times\mathbb{R}^d} G(r,z)\mathbf{1}_A(z)\mathrm{d}r\nu(\mathrm{d}z) \tag{A.116}$$

is $\mathcal{F}_t$-measurable.

By (A.113), we get that $\mathrm{E}[Y_s(X_t^{(A)} - X_s^{(A)})] = 0$*, which yields*

$$\mathrm{E}[X_t^{(A)}|\mathcal{F}_s] = X_s^{(A)}. \tag{A.117}$$

Hence, $(X_t^{(A)})$ is an $\mathcal{F}$-martingale. Moreover, one can check that this is a process with finite variations (difference between two non-decreasing processes, see [8, Section 4.1]*). Hence, it is a.s. constant (see* [8, Theorem 4.8]*), so $X_t^{(A)} = X_0^{(A)} = 0$ P-a.s. Let us now deduce that $G = 0$, $\mathrm{d}t \otimes \nu \otimes \mathrm{P}$-a.s, with the help of a monotone class argument. To this end, define*

$$\mathcal{C} = \Big\{ C \subseteq [0,T] \times \mathbb{R} \colon \int_C G(r,z)\mathrm{d}r\nu(\mathrm{d}z) = 0 \quad \mathrm{P}-a.s. \Big\}. \tag{A.118}$$

We have proven so far that $\mathcal{C}$ contains the set

$$\mathcal{C}_0 := \Big\{ (s,t] \times A, \qquad 0 \le s \le t \le T, \quad A \in \mathcal{B}(\mathbb{R}) \Big\}, \tag{A.119}$$

that is a π-system generating $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R})$. Furthermore, one can check that $\mathcal{C}$ is a monotone class. By Dynkin's theorem, the property defining $\mathcal{C}$ is therefore valid for all sets in $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R})$. The reader may check that this implies $G = 0, \mathrm{d}t \otimes \nu \otimes \mathrm{P}$-a.s. Hint : use $\{G \ge 0\}$ and $\{G \le 0\}$ as test sets.

(4) *Since $\mathcal{S}$ is dense, the isometry may be uniquely extended to $\mathcal{H}^2(T)$.*

Solution 26. *Pick $b(t) = b$, $\sigma(t) = \sigma$, $H(t,z) = K(t,z) = z$ for all (t,z).*

Solution 27. (1) *We have $N_t = \int_0^t \int_{|z|\ge 1} \mathcal{N}(\mathrm{d}s, \mathrm{d}z)$ where $\mathcal{N}$ is a $\mathrm{RPM}(\lambda \mathrm{d}t \otimes \delta_1)$. We may then write $\mathrm{d}N_s := \int_{|z|\ge 1} \mathcal{N}(\mathrm{d}s, \mathrm{d}z)$, that is a RPM on $(0,\infty)$ with intensity $\lambda\mathrm{d}t$ (i.e. a homogeneous Poisson point process with intensity $\lambda > 0$). By analogy with compensated Poisson measures we define $\mathrm{d}\widetilde{N}_s$ as the signed measure $\mathrm{d}N_s - \lambda \mathrm{d}s$.*

(2) *We have, with the usual notations,*

$$\int_0^t f(s)\mathrm{d}N_s = \sum_{1\le i \le N_t} f(T_i) \qquad \textit{and} \qquad \int_0^t f(s)\mathrm{d}\widetilde{N}_s = \sum_{1\le i\le N_t} f(T_i) - \lambda \int_0^t f(s)\mathrm{d}s. \tag{A.120}$$

(3) $\int_0^t B_s \mathrm{d}N_s = \sum_{1\le i\le N_t} B_{T_i}$.

(4) *The presence of s^- instead of s is here to ensure predictability of the process (N_{s^-}). We have*

$$X_t = \int_0^t N_{s^-}\mathrm{d}N_s = \sum_{1\le i\le N_t} N_{T_i^-} = \sum_{1\le i\le N_t} (i-1) = \frac{N_t(N_t-1)}{2} \tag{A.121}$$

[**] Any adapted process with left-continuous (or right-continuous) sample paths is progressive, see [8, Proposition 3.4]

(with the convention that the sum over an empty set is zero, which is consistent with the case $N_t = 0$). Note that X_t is not *equal to $\frac{1}{2}N_t^2$, as we would get from a blind application of standard (non-stochastic) calculus. Let us now look at the compensated version:*

$$\widetilde{X}_t := \int_0^t N_{s^-}\mathrm{d}\widetilde{N}_s = \frac{N_t(N_t-1)}{2} - \lambda\int_0^t N_{s^-}\mathrm{d}s, \tag{A.122}$$

from which we get by a straightforward computation that $\mathrm{E}(\widetilde{X}_t) = 0$. As one can check, this would no longer hold if we were to replace N_{s^-} by N_s in the definition of X_t.

(5) *Apply Itô's formula with $f(t,z) = z^2$ and $K(t,z) = 1$. Only the first term of (III) is present. We get*

$$\mathrm{d}Y_t = [(N_{t^-}+1)^2 - N_{t^-}^2]\mathrm{d}N_t = (2N_{t^-}+1)\mathrm{d}N_t. \tag{A.123}$$

One may check that this is consistent with our previous answer.

Solution 28. *Let us write both formulas (in one dimension) next to each other to highlight the analogy:*

$$\begin{aligned} \mathcal{L}f(x) &= bf'(x) + \tfrac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}} [f(x+z) - f(x) - zf'(x)1_{\{|z|\le 1\}}]\nu(\mathrm{d}z), \\ \Psi(u) &= ibu - \frac{1}{2}au^2 + \int_{\mathbb{R}^d}\left(e^{iuz} - 1 - iuz1_{\{|z|\le 1\}}\right)\nu(\mathrm{d}z). \end{aligned} \tag{A.124}$$

Derivation corresponds to multiplication by iu in Fourier mode (hence differentiating twice corresponds to multiplication by $(iu)^2 = -u^2$) and shifting by z corresponds to multiplication by e^{iuz} (addition of a phase).

Solution 29. *Check Assumption 6.1 with* $b(y) := by$, $\sigma(y) := \sigma^2 y$ *and* $F(y,z) = G(y,z) := yz$.

Solution 30. (1) *By using Remark 7.1, we may write* $S_t = \exp(L_t)$ *with*

$$\begin{aligned} L_t &= bt - \tfrac{1}{2}\sigma^2 t + t\int_{|z|\le 1} (\log(1+z) - z)\nu(\mathrm{d}z) \\ &\quad + \sigma B_t + \int_0^t \int_{|z|>1} \log(1+z)\mathcal{N}(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{|z|\le 1} \log(1+z)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z). \end{aligned} \tag{A.125}$$

The last two integrals are well-defined because $\log(1+z) \sim_0 z$ *and* ν *is a Lévy measure.*

(2) *One may check that* L *is a Lévy process with triplet* $(\bar{b}, \sigma, \bar{\nu})$, *where*

$$\begin{aligned} \bar{b} = b - \frac{1}{2}\sigma^2 &+ \int_{|z|\le 1} (\log(1+z) - z)\nu(\mathrm{d}z) \\ &+ \int_1^{e-1} \log(1+z)\nu(\mathrm{d}z) - \int_{-1}^{1/e-1} \log(1+z)\nu(\mathrm{d}z). \end{aligned} \tag{A.126}$$

and

$$\bar{\nu}(A) = \nu(\{z > -1\colon \log(1+z) \in A\}), \qquad A \in \mathcal{B}(\mathbb{R}). \tag{A.127}$$

The only non-trivial part is to check that $\bar{\nu}$ *is indeed a Lévy measure.*

(3) *We get*

$$\begin{aligned} \mathrm{d}f(L_t) &= f'(L_t)b\mathrm{d}t - \frac{1}{2}\sigma^2 f'(L_t)\mathrm{d}t + \int_{|z|\le 1} (\log(1+z) - z)\nu(\mathrm{d}z) f'(L_t)\mathrm{d}t + \sigma f'(L_t)\mathrm{d}B_t \\ &\quad + \frac{1}{2} f''(L_t)\sigma^2 \mathrm{d}t + \int_{|z|>1} [f(L_{t^-} + \log(1+z)) - f(L_{t^-})]\mathcal{N}(\mathrm{d}t,\mathrm{d}z) \\ &\quad + \int_{|z|\le 1} [f(L_{t^-} + \log(1+z)) - f(L_{t^-})]\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z) \\ &\quad + \int_{|z|\le 1} [f(L_{t^-} + \log(1+z)) - f(L_{t^-}) - \log(1+z) f'(L_{t^-})]\mathrm{d}t\nu(\mathrm{d}z) \end{aligned} \tag{A.128}$$

(4) *Applying it to* $f = \exp$ *and simplifying, we get*

$$\mathrm{d}S_t = S_{t^-}\Big[b\mathrm{d}t + \sigma\mathrm{d}B_t + \int_{|z|>1} z\mathcal{N}(\mathrm{d}t,\mathrm{d}z) + \int_{|z|\le 1} z\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z)\Big] = S_{t^-}\mathrm{d}X_t. \tag{A.129}$$

Solution 31. *Left to reader.*

Solution 32. *Using the same notation as in Exercise 30, we get the (necessary and sufficient) condition:*

$$\bar{b} + \tfrac{1}{2}\sigma^2 + \int \Big(e^z - 1 - z\mathbf{1}_{\{|z|\le 1\}}\Big)\bar{\nu}(\mathrm{d}z) = 0, \tag{A.130}$$

which simplifies as

$$b + \int_{|z|>1} z\nu(\mathrm{d}z) = 0. \tag{A.131}$$

Solution 33. *The case treated in* (7.7) *corresponds to* $\nu_i = \lambda_i \delta_1$, *where* $(i \in \{1,2\})$, *hence* $\phi(z) = \log(\lambda_1/\lambda_2)$ *and we obtain*

(A.132)

$$\nu_1(\mathbb{R}) - \nu_2(\mathbb{R}) = \lambda_1 - \lambda_2, \qquad \sum_{0<s\leq t} \phi(\Delta X_s) = \log(\lambda_1/\lambda_2) \sum_{\substack{0<s\leq t \\ (\Delta X_s \neq 0)}} 1 = \log(\lambda_1/\lambda_2) N_t.$$

Solution 34. *Left to reader.*

Solution 35. (1) *We proceed by backward iteration. The formula clearly holds for $k = n$. Let $1 \leq k < n$ and assume that the formula holds for $k+1$. Using what we know from the one-step model, we get*

$$W_k(X_1, \dots, X_k) = qW_{k+1}(X_1, \dots, X_k, +) + (1-q)W_{k+1}(X_1, \dots, X_k, -), \tag{A.133}$$

with

$$q = \frac{1-1/e}{e-1/e} = \frac{e-1}{e^2-1} = \frac{1}{1+e}. \tag{A.134}$$

We conclude by using the formula at step $k+1$.

(2) *Same idea, with this time $A_0/A_1 = e^{-r}$ and*

$$q = \frac{e^{r+1}-1}{e^2-1}. \tag{A.135}$$

(3) *Check that $\mathrm{E}[e^{X_1}] = e^r$.*

Solution 36. *Let us denote by Q_t the restriction of Q to events in $\mathcal{F}_t$. By the martingale property we get $Q_t(\Omega) = \mathrm{E}_P(e^{Y_t}) = \mathrm{E}_P(e^{Y_0}) = 1$ and the consistency condition:*

$$Q_t(A) = \mathrm{E}_P(e^{Y_t}A) = \mathrm{E}_P(e^{Y_s}A) = Q_s(A), \qquad 0 \leq s \leq t, A \in \mathcal{F}_s. \tag{A.136}$$

We conclude by Kolmogorov's extension theorem.

Solution 37. *By taking logarithm, we have*

$$\mathrm{d}(\log S_t) = \sigma \mathrm{d}B_t + \Big(\mu - \frac{1}{2}\sigma^2\Big)\mathrm{d}t, \tag{A.137}$$

hence

$$\mathrm{d}(\log \widetilde{S}_t) = \Big(\sigma \mathrm{d}B_Q(t) - \frac{1}{2}\sigma^2 \mathrm{d}t\Big) + \Big(\mu - r + \sigma F(t)\Big)\mathrm{d}t. \tag{A.138}$$

The only possible choice is

$$F(t) = \frac{r-\mu}{\sigma}, \qquad \forall t \geq 0, \tag{A.139}$$

in which case, by Itô's formula,

$$\mathrm{d}\widetilde{S}_t = \sigma \widetilde{S}_{t-} \mathrm{d}B_Q(t). \tag{A.140}$$

Solution 38. (1) *For all $t \in [0, T]$, we have*

$$\begin{aligned} W_t &= U_t S_t + V_t A_t \\ &= \gamma_t S_t + Z_t A_t - \gamma_t \widetilde{S}_t A_t \\ &= Z_t A_t \\ &= e^{-r(T-t)} \mathrm{E}_Q(Z|\mathcal{F}_t). \end{aligned} \tag{A.141}$$

(2) *The portfolio is replicating, since $W_T = \mathrm{E}(Z|\mathcal{F}_T) = Z$.*

(3) *By Itô's formula, see Theorem 5.6.4 in* [1].

(4) *The arbitrage-free value of the contingent claim is*

$$W_0 = e^{-rT}\mathrm{E}_Q(Z) \tag{A.142}$$

(compare with (8.15)).

Solution 39. *Using the result of Exercise 38, the arbitrage-free price of the option writes*

$$W_0 = e^{-rT}\mathrm{E}_Q[(S_T - k)_+]. \tag{A.143}$$

Check that

$$S_T = S_0 e^{\sigma B_Q(T)+(r-\frac{1}{2}\sigma^2)T}. \tag{A.144}$$

Using that B_Q is a standard Brownian motion under P_Q, we obtain

$$W_0 = \int_{\mathbb{R}} (S_0 e^x - ke^{-rT})_+ \; e^{-\frac{(x+\frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}} \frac{\mathrm{d}x}{\sigma\sqrt{2\pi T}}. \tag{A.145}$$

Deduce thereof that

$$W_0 = S_0 \Phi\Big(\frac{\log(S_0/k) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}\Big) - ke^{-rT}\Phi\Big(\frac{\log(S_0/k) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\Big), \tag{A.146}$$

where Φ is the cumulative distribution function of a standard Gaussian random variable. See [1, Theorem 5.6.4] *and references therein for a complete solution.*

Solution 40. *Let g_{m,v^2} be the density of $\mathcal{N}(m, v^2)$. Then,*

$$\nu(\mathrm{d}z) = \lambda g_{m,v^2}(\mathrm{d}z), \qquad b = \mu + \int_{-1}^{1} z\nu(\mathrm{d}z). \tag{A.147}$$

Solution 41. *Clearly, ν_θ is a non-negative measure. Moreover,*

$$\int_{\mathbb{R}} (1 \wedge |z|^2)\nu_\theta(\mathrm{d}z) \le \int_{\mathbb{R}} \nu_\theta(\mathrm{d}z) = \int_{\mathbb{R}} e^{\theta z} g_{m,v^2}(z)\mathrm{d}z < \infty. \tag{A.148}$$

Solution 42. (1) *By dominated convergence,*

$$f'(\theta) = \int_{\mathbb{R}} x(e^x - 1)e^{\theta x}\nu(\mathrm{d}x) \ge 0, \tag{A.149}$$

so f is non-decreasing.

(2) *Since $\nu((0,\infty)) > 0$ and $\nu((-\infty,0)) > 0$ we respectively get*

$$\begin{aligned} f'(\theta) &\ge \int_0^\infty x(e^x - 1)\nu(\mathrm{d}x) > 0 \qquad (\theta \ge 0) \\ f'(\theta) &\ge \int_{-\infty}^0 x(e^x - 1)\nu(\mathrm{d}x) > 0 \qquad (\theta \le 0). \end{aligned} \tag{A.150}$$

(3) *From what precedes, f is a bijection from $\mathbb{R}$ to $\mathbb{R}$, so that (8.32) has a unique solution.*