Université Paris-Dauphine, PSL University
Centre de Recherche en Mathématiques de la Décision

Mémoire d'habilitation à diriger des recherches

Spécialité : Mathématiques

présenté par

Julien POISAT

Random Walks, Polymers and Phase Transitions

---

coordonné par :

Mme.	Béatrice	**DE TILIÈRE**	Université Paris-Dauphine

soutenu publiquement le mardi 29 septembre 2020 après avis de :

M.	Kenneth	**ALEXANDER**	University of Southern California
M.	Amine	**ASSELAH**	Université Paris-Est Créteil
M.	Erwin	**BOLTHAUSEN**	Universität Zürich

devant le jury composé de :

M.	Amine	**ASSELAH**	Université Paris-Est Créteil	Rapporteur
M.	Erwin	**BOLTHAUSEN**	Universität Zürich	Rapporteur
M.	Djalil	**CHAFAÏ**	Université Paris-Dauphine	Examinateur
Mme.	Béatrice	**DE TILIÈRE**	Université Paris-Dauphine	Coordinatrice
M.	Giambattista	**GIACOMIN**	Université de Paris	Examinateur
Mme.	Perla	**SOUSI**	University of Cambridge	Examinatrice

Remerciements

Ce document est la synthèse de plusieurs années de travail de recherche. J'aimerais ici remercier chaleureusement toutes les personnes qui m'ont soutenu et aidé dans cette tâche.

Tout d'abord, merci à Béatrice de Tilière d'avoir accepté avec enthousiasme de coordonner ma thèse d'habilitation. Merci aux rapporteurs, Kenneth Alexander, Amine Asselah et Erwin Bolthausen, d'avoir pris le temps de lire et d'évaluer ce manuscrit. Merci Amine pour les nombreux échanges que nous avons eus et et merci de m'avoir proposer de donner un mini-cours sur les polymères chargés lors de notre rencontre à Marseille en 2016 ; ce fut une expérience très enrichissante. Kenneth, I hope we will have the chance to meet again and discuss "in real life", despite the current situation. Erwin, your work is a great source of inspiration and motivation. It is an honor to have you in the committee. Merci à Djalil Chafai, Giambattista Giacomin et Perla Sousi d'avoir accepté de faire partie du jury de soutenance. Djalil, merci pour ton accueil et tes encouragements pendant ces premières années à Dauphine. Giambattista, merci de m'avoir fait découvert le monde des polymères et de la mécanique statistique lors de ton cours de master à l'Université Paris 7. Perla, merci pour les quelques jours passés à Cambridge ; j'en garde un très bon souvenir.

J'en profite pour remercier les collègues avec qui j'ai collaborés pendant ces dernières années. Quentin, nous avons fait beaucoup de chemin ensemble et, pour faire écho à ton propre mémoire, je souhaite moi aussi que nous continuions ainsi. Renato, Dirk and Julián, I have had the pleasure to be your scientific collaborator, your flatmate and your friend all at the same time. You have played a crucial part in making Leiden a second home. Nicolas, merci pour les nombreux séjours nantais et pour tous tes bons conseils. Thank you Francesco, Rongfeng and Nikos. I hope we can work again together in the future. Thank you Dimitris for the pleasant visits in Athens. Thank you Yuki for the Japanese sweets and the fun evenings in Sapporo. I hope to be able to talk Japanese with you some day. Merci François pour ta bonne humeur et ton entrain, mais surtout merci de surveiller ma consommation de biscuits pendant nos séances de travail. Enfin, un grand merci à Nadine Guillotin et à Frank den Hollander. J'ai énormément appris grâce à vous deux. Nadine, merci encore une fois pour tout le temps et l'énergie passés à me former pendant ma thèse de doctorat. Frank, les deux années de postdoc passées à Leiden ont été formidables, sur le plan mathématique comme sur le plan humain.

Merci aussi à tous les collègues dauphinois(es) (et ex-dauphinois) pour la bonne ambiance qui règne au laboratoire, pour les nombreuses discussions à la salle café, à la cantine ou lors des divers pots et picnics (extérieurs et intérieurs). Merci Marie, César et Isabelle pour le soutien administratif.

Enfin, je remercie mes amis et ma famille. Merci pour tous les moments de détente et d'évasion. Merci à ma mère et à mon frère pour leur patience et leur soutien dans les moments de découragement. Merci à ceux qui sont partis prématurément. Merci à mon grand-père, qui a joué un rôle non négligeable dans mon éveil mathématique. Le trésor est bien gardé. Merci Lucile pour ensoleiller mes journées.

Introduction

This *mémoire d'Habilitation à Diriger des Recherches* contains a synthesis of my research work as well as perspectives for future research. A large part of it consists of results that have been already published in journals. These results appear in the form of theorems or propositions* and the corresponding references are encapsulated at the beginning of each chapter. I sketch the proofs of the most important theorems, in order to explain the main ideas behind them. The reader may refer to the original papers for detailed proofs. I also discuss the results and present directions for future research, some of them labelled as conjectures. Some of these research directions are the object of ongoing work, while some others are still open.

The results presented in this *mémoire* mostly deal with polymer models seen from the point of view of Probability Theory and Statistical Mechanics. Polymers are macromolecules that can be found both in nature and in the industry. They are formed by the repetition of a very large number of basic units called *monomers*, which are atoms or groups of atoms. Different kinds of geometries can be achieved via the binding of such monomers but we will here restrict to linear polymers, that are chain-like molecules. To mimick the spatial flexibility of monomer bonds, the idea emerged in the last century to model the configuration of a polymer chain by a random walk path [50, 68]. The simple random walk model is a very popular model in Probability Theory. It is defined as a $\mathbb{Z}^d$-valued random process $(S_n)_{n\geq 0}$ usually started at the origin and so that the increments $(S_n - S_{n-1})_{n\geq 1}$ are independent and uniformly distributed on the $2d$ unit vectors. Hence, a polymer with N monomers shall be represented by a random walk running up to time N. Two types of models arise: in *undirected* models, the position of the n-th monomer in the chain is S_n (d-dimensional model) while in *directed* models, its position is understood as (n, S_n) ($1+d$-dimensional model). The latter model has the advantage of satisfying the exclusion principle, according to which two distinct monomers cannot occupy the same site. Note that we will sometimes also deal with the a priori unphysical dimensions $d > 3$. Another restriction comes from the fact that only *equilibrium* models are considered. The equilibrium states will be defined via the formalism of Statistical Mechanics, which we now briefly introduce. For each $n \in \mathbb{N}$, let H_n be a real-valued function (called Hamiltonian function) depending on the first n steps of the random walk/polymer. This function depends on the particular model we want to study, that is the type of interactions that govern the physics of the molecule (e.g. monomer-monomer or monomer-solvent interactions, external forces). Then, the probability measure describing a polymer of size n, at equilibrium and at inverse temperature $\beta > 0$, can be denoted by $\mathrm{P}_{n,\beta}$ and defined by

$$\frac{\mathrm{dP}_{n,\beta}}{\mathrm{dP}}(S_1, \dots, S_n) = \frac{\exp(-\beta H_n(S_1, \dots, S_n))}{Z_{n,\beta}}, \tag{0.1}$$

where P is the law of the simple random walk and $Z_{n,\beta} = \mathrm{E}[\exp(-\beta H_n(S_1, \dots, S_n))]$ is a normalization factor called *partition function*. One of the main goals is to describe the typical shape of the polymer under the law $\mathrm{P}_{n,\beta}$ in the limit of many monomers ($n \to \infty$), and the first step in undertaking this task is usually to determine the asymptotic behaviour of the sequence of partition functions.

Let us now briefly present below the different problematics that one will find in this document. First of all, one notices that the polymer measure in (0.1) comes as a competition between energy (tendency to minimise the Hamiltonian function) and entropy (randomness). The outcome of this competition may depend on the inverse temperature (or other parameters), leading to different behaviours of the polymer as $n \to \infty$. This drastic change in the limiting polymer measure (or in the free energy of the system as we will see in Chapters 1 and 3) for different values of the parameter is known as a phenomenon of phase transition. What happens at the onset of such transitions is called critical behaviour. Note that we will sometimes break the convention that energy should be minimised by switching the sign of the Hamiltonian, and consider possibly more

*except for Propositions 1.4 and 1.11, which are not contributions from the author.

than one parameter: for instance, the annealed charged polymer model in Chapter 3 (a model for polyelectrolytes) exhibits two phases, depending on both the inverse temperature and the charge bias.

Another important topic is that of disordered systems, of which the first three chapters offer examples. These are systems which contain impurities or a certain degree of inhomogeneity. In the context of polymers, this happens for instance when monomers can exist in various forms inside the same molecule: this is the case for a DNA strand, of which nucleotides can be of four forms, A, T, C or G (Chapter 1) or a polyelectrolyte, of which monomers can have different electrical charges (Chapter 3). Another possible scenario is that of a inhomogeneous solvent (see Chapter 2) and we sometimes say in this case that the polymer evolves in a *random environnement*. This leads to models with two sources of randomness: the first one is that of the random walk itself and the second one is related to the inhomogeneities, also referred as *disorder*. As a consequence, a disordered model usually has two versions: the *quenched* model corresponds to a polymer set in an environment where disorder is frozen, while in the *annealed* model, the polymer and the disorder variables jointly fluctuate. A crucial question is whether disorder affects the critical behavior of a system with a phase transition, and we will come back to this issue of *disorder relevance* in Chapter 1.

This *mémoire* contains four chapters. Each chapter is dedicated to a different model which deals with a specific kind of interaction. Chapters 1 and 2 deal with directed models, while Chapters 3 and 4 deal with undirected models (in Chapter 4 we rather work with the continuous counterpart of the simple random walk, that is Brownian motion). Chapters 1 and 2 both deal with *pinning* yet they are kept separately for a matter of notation and exposition. In Chapter 1, pinning is induced by the presence of a single defect line that attracts or repels the polymer. This model is also adapted to other situations such as DNA denaturation (Poland-Scheraga models). In Chapter 2, however, pinning is induced by the presence of many repulsive lines. As we will see, this model is also connected to other famous models such as the Anderson model and the random walk model among obstacles. Both models in Chapters 1 and 2 are disordered ones: in Chapter 1 disorder sits on the defect line (or equivalently on the monomers) and in Chapter 2, disorder sits on the position of the repulsive lines. Chapter 3 deals with *folding and unfolding* of polymers, a mechanism that is visible in real-life molecules. We will mostly discuss this phenomenon in the context of charged polymers (another class of disordered polymers) for which, in the annealed setup, a phase transition happens between a collapsed phase and an extended phase. Some of our research directions, however, also include the study of folding in other contexts (as in the "Swiss cheese" model and interacting partially directed walks). Finally, Chapter 4 deals with a Poisson collection of Brownian motions, that one may look at a simple model for a system of many non-interacting chains (polymer melt) and the phenomenon at stake there is percolation.

Even if the contents and the structure of what follows are motivated by questions arising from Physics and Biology, we will deal, of course, with the mathematical tools that are used to study such models (see e.g. [53, 72, 73]). The partitions functions will be analyzed using various tools such as coarse graining techniques, rare-stretch strategies, expansion techniques and more generally, tools from the Large Deviations Theory [51, 52, 65]. Some results from Ergodic Theory are used in Section 1.3.1. Some other tools are used to study disordered models, such as change of measure arguments, second moment methods and, when it comes to *correlated disorder*, decoupling techniques. We will also rely on tools from Renewal Theory [8] (Chapter 1) and Random Walk Theory [96, 123] (such as confinement estimates and moment generating functions of hitting times in Chapter 2 or representation of local times in Chapter 3). The notion of Newtonian capacity of a set plays a crucial role in Chapter 4 but also in one of our perspectives on folding (Section 3.6.1).

Notation

- $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$;
- $u_n \sim v_n$ means $u_n = [1 + o(1)]v_n$ as $n \to \infty$;
- we use the symbol $:=$ to define quantities;
- $\varepsilon \downarrow 0$ (resp. $\varepsilon \uparrow 0$) means that ε converges to 0 from above (resp. from below);
- $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, $(a)_+ = \max(a, 0)$;
- if $Z_n = \mathrm{E}[\exp(H_n)]$ is a partition function and A is an event then $Z_n(A) = \mathrm{E}[\exp(H_n)1_A]$;
- $M^\intercal$ is the transpose matrix of M;
- we use the symbols $\#$ or $|\cdot|$ to denote the cardinality of a set;
- unless stated otherwise, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$;
- if U is an open set of $\mathbb{R}^d$ and $k \in \mathbb{N} \cup \{\infty\}$ then $\mathcal{C}^k(U)$ is the set of functions that are k-times continuously differentiable in U;
- $\mathrm{cap}(A)$ is the Newtonian capacity of a set A (in $\mathbb{Z}^d$ or $\mathbb{R}^d$, depending on the context);
- $\mathcal{B}(x, r)$ is the open Euclidean ball centered at $x \in \mathbb{R}^d$ with radius r;
- we abbreviate "independent and identically distributed" by IID or i.i.d;
- $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$.

Chapter 1

Pinning of a polymer on a defect line

This chapter is based on [111–113] and joint works with Q. Berger, F. Caravenna, R. Sun and N. Zygouras [19,23], D. Cheliotis and Y. Chino [44] and F. den Hollander [54].

The pinning model provides a general mathematical framework to study the localization phase transition that occurs in various physical contexts such as the wetting transition of interfaces [56,69], DNA denaturation (Poland-Scheraga models) [115], localization of flux lines in superconducting vortex arrays [105] and localization of a polymer along a defect line. For an account on pinning models, we refer to the two monographs [72,73] and references therein, as well as Chapters 7 and 11 in [53]. This chapter is organized as follows: in Section 1.1 we recall some well-established facts about the homogeneous version of the pinning model and the corresponding phase transition; in Section 1.2 we investigate the effect of disorder in the interaction and present a summary of the results that have been obtained on this topic. We then consider the case of correlated disorder in Section 1.3. We finally close the chapter with a discussion and some perspectives in Section 1.4.

1.1 The homogeneous pinning model

Polymer chain and contact points. The polymer chain is modeled by a Markov chain $S = (S_n)_{n\geq 0}$ on $\mathbb{Z}$ with $S_0 = 0$, the probability distribution and expectation of which are denoted by P and E. Here, one should think of (n, S_n) as the position of the n-th monomer along the polymer chain while the line $\mathbb{N} \times \{0\}$ corresponds to a defect line. We denote by $\tau := (\tau_n)_{n\geq 0}$ the sequence of random times at which S returns to 0, namely $\tau_0 = 0$ and $\tau_{n+1} = \inf\{k > \tau_n \colon S_k = 0\}$. By the Markov property of S, the process τ is a *renewal process*. With a slight abuse of notation, we shall also write τ for the random subset of integers $\{0, \tau_1, \tau_2, \ldots\}$ which, in the polymer interpretation, are the positions (along the polymer chain) of the monomers that touch the defect line. We assume throughout the chapter that the inter-arrival law of τ satisfies a power-law decay:

$$K(n) := \mathrm{P}(\tau_1 = n) = L(n)\, n^{-(1+\alpha)}, \qquad \alpha \geq 0, \quad n \in \mathbb{N}, \tag{1.1}$$

where $L \colon (0,\infty) \to (0,\infty)$ is a slowly varying function whose support is aperiodic, that is, $\gcd\{n \geq 1 \colon L(n) > 0\} = 1$. The aperiodicity assumption is made for convenience and does not hide anything deep. We also assume that τ is recurrent (non-terminating) that is $\mathrm{P}(\tau_1 < \infty) = \sum_{n\geq 1} K(n) = 1$ (otherwise it is said to be transient). Finally, we denote by δ_n the Bernoulli random variable that is 1 if the n-th monomer along the polymer chain touches the defect line and 0 otherwise, which we

may write

$$\delta_n = 1_{\{n\in\tau\}}. \tag{1.2}$$

Example 1.1. *One of the most natural example is when S is the simple random walk on $\mathbb{Z}$, which satisfies (1.1) for $n \in 2\mathbb{N}$, with $\alpha = 1/2$ and $L(\cdot)$ converging to a constant. The fact that the support of K has period 2 in this case can be easily dealt with.*

Remark 1.2. *In this framework the fundamental object is the renewal process τ, and* there is no need to refer to the Markov chain S. *However, let us mention that, for any $\alpha > 0$ and any slowly varying function L, a nearest-neighbour Markov chain S on $\mathbb{Z}$ with Bessel-like drift can be constructed, that satisfies assumption (1.1) asymptotically, that is $K(n) \sim L(n)/n^{1+\alpha}$ as $n \to \infty$, cf. [2].*

We start with the homogeneous version of the model, which is completely solvable and treated in detail in [72]. Let us denote the number of contact points between the n first monomers and the defect line by

$$H_n = \sum_{1\le k\le n} \delta_k = \#\{1 \le k \le n\colon k \in \tau\}. \tag{1.3}$$

The polymer measure with length $n \in \mathbb{N}$ and pinning strength $h \in \mathbb{R}$ is defined by

$$\frac{\mathrm{dP}_{n,h}}{\mathrm{dP}} = \frac{1}{Z_{n,h}} \exp(hH_n), \tag{1.4}$$

where $Z_{n,h} = \mathrm{E}[\exp(hH_n)]$ is a normalizing constant called *partition function*. The measure above corresponds to a polymer chain of which the n first monomers interact with the defect line. The interaction is attractive if $h > 0$, neutral if $h = 0$ and repulsive if $h < 0$. It is sometimes more convenient to work with the *constrained* (as opposed to *free*) version of the measure, that is

$$\frac{\mathrm{dP}^c_{n,h}}{\mathrm{dP}} = \frac{1}{Z_{n,h}} \exp(hH_n)\delta_n, \tag{1.5}$$

as the sequence of constrained partition functions $Z^c_{n,h} = \mathrm{E}[\exp(hH_n)\delta_n]$ is super-multiplicative. However, both versions lead to the same value of the limiting free energy [72, Chapter 1, Remark 1.2] which is defined as

$$\mathrm{F}(h) = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,h} = \lim_{n\to\infty} \frac{1}{n} \log Z^c_{n,h}. \tag{1.6}$$

By restricting the expectation E to the event $\{\tau_1 > n\}$ in the definition of the partition function, and observing that $\log \mathrm{P}(\tau_1 > n) = O(\log n)$ by (1.1), it follows that $\mathrm{F}(h) \ge 0$. It is actually known that $\mathrm{F}(h) = 0$ if $h \le 0$, otherwise

$$\mathrm{F}(h) = \inf\Big\{\mathrm{F} \ge 0\colon \sum_{n\ge1} e^{-\mathrm{F}n} K(n) \le e^{-h}\Big\}. \tag{1.7}$$

The phase transition can be explained in terms of non-analyticity of the free energy at the *critical point* $h_c = 0$ and in terms of the limiting density of contact points:

$$\lim_{n\to\infty} \mathrm{E}_{n,h}\Big(\frac{H_n}{n}\Big) = \begin{cases} 0 & (h < 0) \\ \mathrm{F}'(h) > 0 & (h > 0). \end{cases} \tag{1.8}$$

What happens at criticality depends on the exponent α. It is known that

$$\mathrm{F}(h) = h^{\max(1,1/\alpha)} \hat{L}(1/h), \tag{1.9}$$

where $\hat{L}$ is a slowly varying function related to L, see Theorem 2.1 in [72] for a precise statement. A simple case to keep in mind is when L converges to a constant and so does $\hat{L}$. The exponent $\nu_{\mathrm{hom}} = \max(1, 1/\alpha)$ is called the (homogeneous) *critical exponent* and characterizes the order of the phase transition.

Remark 1.3. *There is no loss of generality by assuming that τ is recurrent, as one can observe by shifting the pinning strength by* $\log \mathrm{P}(\tau_1 < \infty)$.

1.2 The random pinning model

An important challenge in statistical mechanics is to understand the effect of quenched impurities or inhomogeneities in the interaction on the mechanism of phase transition. Such impurities may drastically change the properties of a physical system and alter the nature of its phase transitions, leading to new phenomena. In the pinning model, disorder is attached to the defect line, which can either attract or repel the polymer. This leads to the random (or disordered) version of the pinning model.

The disorder sequence. The disorder is modeled by a sequence $\omega := (\omega_n)_{n\geq 1}$ of i.i.d. real random variables which are sampled independently of τ. The probability distribution and corresponding expectation for ω will be denoted respectively by $\mathbb{P}$ and $\mathbb{E}$. We assume that

$$M(t) := \mathbb{E}[e^{t\omega_1}] < \infty \quad \forall t \in \mathbb{R}\,, \qquad \mathbb{E}[\omega_1] = 0\,, \qquad \mathbb{E}[\omega_1^2] = 1\,. \tag{1.10}$$

The requirement that ω_1 is centered and of unit variance is just a matter of normalization and does not hide anything deep. The log-moment generating function is denoted by

$$\Lambda(t) := \log M(t) \sim \tfrac{1}{2}t^2 \quad \text{as } t \to 0\,. \tag{1.11}$$

The standard normal distribution provides an example of such a probability distribution, with $\Lambda(t) = \frac{1}{2}t^2$ for $t \in \mathbb{R}$. The i.i.d. assumption will be removed in Section 1.3.

The disordered polymer measure. Let us define a disordered analogue of (1.4):

$$\frac{\mathrm{dP}^{\omega}_{n,\beta,h}}{\mathrm{dP}} = \frac{1}{Z^{\omega}_{n,\beta,h}} \exp\Big\{\sum_{k=1}^{n}(h+\beta\omega_k)\delta_k\Big\}, \quad n \in \mathbb{N}, \quad \beta \geq 0, \quad h \in \mathbb{R}, \tag{1.12}$$

where

$$Z^{\omega}_{n,\beta,h} = \mathrm{E}\Big\{\exp\Big(\sum_{k=1}^{n}(h+\beta\omega_k)\delta_k\Big)\Big\} \tag{1.13}$$

is the *quenched* partition function, h is the (average) pinning strength and β is the disorder strength (or coupling constant).

Free energy. Many statistical properties of the model can be captured through the (quenched) *free energy*, which is defined by

$$\mathrm{F}(\beta,h) := \lim_{n\to\infty}\frac{1}{n}\log Z^{\omega}_{n,\beta,h} = \lim_{n\to\infty}\frac{1}{n}\,\mathbb{E}\log Z^{\omega}_{n,\beta,h}\,, \tag{1.14}$$

where the first limit exists $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$, and remains unchanged if we replace the partition function by its constrained counterpart (see [72, Chapter 4]). As in the homogeneous case, we may split the space of parameters into a *localized phase* $\mathcal{L} = \{(\beta,h) : \mathrm{F}(\beta,h) > 0\}$ and a *delocalized phase* $\mathcal{D} = \{(\beta,h) : \mathrm{F}(\beta,h) = 0\}$. These two phases are separated by the (quenched) critical curve

$$h_c(\beta) := \sup\{h\colon\ \mathrm{F}(\beta,h) = 0\}, \tag{1.15}$$

where the (quenched) localization-delocalization transition occurs. We may describe these two phases in terms of the contact fraction $\partial_h \mathrm{F}(\beta,h) = \lim_{n\to\infty}(1/n)\mathrm{E}_{n,\beta,h}(|\tau \cap \{1,\dots,n\}|)$, which is positive inside the localized phase and zero inside the delocalized phase. A simple application of Jensen's inequality leads to the observation that $h_c(\beta) \leq 0$ and with extra work one can prove that actually $h_c(\beta) \in (-\infty, 0)$ if $\beta > 0$ [4]. Apart from the critical point, the other main feature of the phase transition is the (quenched) critical exponent, which is defined by

$$\nu_q(\beta) = \lim_{h\downarrow h_c(\beta)}\frac{\log \mathrm{F}(\beta,h)}{\log(h-h_c(\beta))}, \tag{1.16}$$

when the limit exists, and indicates the smoothness (or order) of the phase transition.

1.2.1 Annealed versus quenched

The effect of disorder is best seen through comparison of the quenched model with its *annealed* counterpart. In particular, the annealed partition function and the annealed free energy are defined by

$$Z^{\mathrm{a}}_{n,\beta,h} := \mathbb{E}(Z^{\omega}_{n,\beta,h}) = \mathbb{E}\mathrm{E}\left[\exp\Big(\sum_{k=1}^{n}(h+\beta\omega_k)\delta_k\Big)\right] \tag{1.17}$$

and

$$\mathrm{F}_{\mathrm{a}}(\beta,h) := \lim_{n\to\infty}\frac{1}{n}\log Z^{\mathrm{a}}_{n,\beta,h}. \tag{1.18}$$

The critical features of this model are the annealed critical curve and exponent:

$$h^{\mathrm{a}}_c(\beta) = \inf\{h\colon \mathrm{F}_{\mathrm{a}}(\beta,h)>0\}, \quad \nu_{\mathrm{a}}(\beta) = \lim_{h\downarrow h^{\mathrm{a}}_c(\beta)}\frac{\log \mathrm{F}_{\mathrm{a}}(\beta,h)}{\log(h-h^{\mathrm{a}}_c(\beta))}, \tag{1.19}$$

when the limit exists. By Jensen's inequality, for all $h\in\mathbb{R}$ and $\beta\geq 0$,

$$\mathrm{F}(\beta,h)\leq \mathrm{F}_{\mathrm{a}}(\beta,h), \tag{1.20}$$

which gives

$$h_c(\beta)\geq h^{\mathrm{a}}_c(\beta). \tag{1.21}$$

It turns out that in the case of i.i.d. disorder the annealed model is exactly solvable since it coincides with the homogeneous model with parameter $h+\Lambda(\beta)$, hence

$$h^{\mathrm{a}}_c(\beta) = -\Lambda(\beta)\sim -\tfrac{1}{2}\beta^2, \qquad \beta\downarrow 0 \tag{1.22}$$

and

$$\nu_{\mathrm{a}}(\beta) = \nu_{\mathrm{hom}} = \max(1,1/\alpha), \qquad \beta\geq 0. \tag{1.23}$$

1.2.2 Disorder relevance and the Harris criterion

Disorder is usually said to be relevant if quenched and annealed critical features differ. Hence, from what precedes, disorder may be relevant with respect to critical points, if $h_c(\beta)-h^{\mathrm{a}}_c(\beta)>0$, or relevant with respect to critical exponents, if $\nu_{\mathrm{a}}(\beta)\neq\nu_q(\beta)$. There has been a lot of activity around this question and several approaches have been used: direct estimates such as fractional moment and second moment estimates [4, 5, 19, 22, 31, 55, 74, 76], martingale theory [92], variational techniques [45] and more recently chaos expansion of the partition function [35–37]. This question is now fully settled.

Critical point shift. It was first noted in [75, Proposition 6.1] that if $h_c(\beta)-h^{\mathrm{a}}_c(\beta)>0$ for some value of β then the same holds for larger values of β. Hence, the question is to know whether there is equality of critical points for small values of β or if there is a positive critical point shift for all $\beta>0$. Finally, the criterion for disorder relevance in the sense of critical point shift reads [22]:

$$[\forall\beta>0, \quad h_c(\beta)>h^{\mathrm{a}}_c(\beta)] \iff \sum_{n\geq 0}\mathrm{P}(n\in\tau)^2=\infty. \tag{1.24}$$

The sum above actually coincides with the expected number of points in the intersection set of two independent copies of τ. By the Renewal Theorem, $\{\mathrm{P}(n\in\tau)\}$ converges to $\frac{1}{\mu}:=\frac{1}{\mathrm{E}[\tau_1]}$ as $n\to\infty$, with the improvement that

$$\mathrm{P}(n\in\tau)\sim L(n)^{-1}n^{\alpha-1} \tag{1.25}$$

when $\alpha\in(0,1)$. Hence, disorder is always relevant when $\alpha>1/2$ and irrelevant for small values of β when $\alpha<1/2$, which comes in agreement with a criterion found in the physics literature and called

the Harris criterion [82]. This criterion gave however no prediction for the marginal case $\alpha = 1/2$, which was the subject of controversies. The answer for this special case, which is actually essential for Example 1.1, depends on the slowly varying function and is given by (1.24) and (1.25).

Critical exponents. A key result for investigating disorder relevance with respect to critical exponents is the following:

Proposition 1.4 (Smoothing inequality)**.** *There exists a constant $\varepsilon_0 > 0$ and a continuous map $(\beta, \delta) \mapsto A_{\beta,\delta}$ from $(0, \varepsilon_0) \times (-\varepsilon_0, \varepsilon_0)$ to $(0, \infty)$, depending only on the disorder distribution and such that $\lim_{(\beta,\delta)\to(0,0)} A_{\beta,\delta} = 1$, with the following properties: for every $0 < \beta < \varepsilon_0$ and $|u| < \beta\varepsilon_0$*

$$0 \le \mathrm{F}(\beta, h_c(\beta) + u) \le \frac{1+\alpha}{2} A_{\beta,\frac{u}{\beta}} \frac{u^2}{\beta^2}; \tag{1.26}$$

The smoothing inequality was first proved in [76], without the precision on the constant and under more restrictive assumptions on the disorder. In the case of Gaussian disorder, it appears in [72] with the right constant $(1+\alpha)/2$, cf. Theorem 5.6 and Remark 5.7 therein. The general statement we use here is proved in [31]. This inequality implies that in all cases $\nu_q(\beta) \ge 2$ when $\beta > 0$, therefore $\nu_q(\beta) > \nu_{\mathrm{a}}(\beta)$ as soon as $\alpha > 1/2$. Hence disorder tends to smoothen the phase transition. In the other direction, it was proved in [1, 6, 45, 77, 92, 133] that when the intersection of two copies of τ is transient (when $\alpha < 1/2$ for instance) then $\nu_q(\beta) = \nu_{\mathrm{a}}(\beta)$ at least for small values of β.

Finally, let us point out that even if the issue of disorder relevance is settled for the pinning model, it is still an open problem to determine the precise order of the quenched phase transition in the relevant regime.

1.2.3 The weak-coupling limit of the quenched critical curve

For the random pinning model with $\alpha > 1$, rough upper and lower bounds of the order β^2 were known for the critical curve $h_c(\beta)$, cf. [5, 55]. In [19, Theorem 1.4], we sharpened these earlier results by establishing the following:

Theorem 1.5. *If $\mu := \mathrm{E}[\tau_1] < \infty$ * then we have*

$$\lim_{\beta\downarrow 0} \frac{h_c(\beta) - h_c^{\mathrm{a}}(\beta)}{\beta^2} = \frac{1}{2\mu} \frac{\alpha}{1+\alpha}. \tag{1.27}$$

Thus, the asymptotic behavior of the critical curve of the random pinning model with $\alpha > 1$ is universal, in the sense that it depends only on the exponent α and on the mean μ of the underlying renewal process, and not on the finer details of the renewal process or the disorder distribution. We stress that the asymptotic behaviour of (1.27) cannot be derived by a naive expansion of the partition function and interchange of $n \to \infty$ and $\beta \to 0$ limits.

Proof strategy for the lower bound. The proof of the lower bound on $h_c(\cdot)$ is based on a refinement of the "fractional moment and coarse graining" method, which was developed in this context in [55, 74], see also [73, Chapters 6 and 7]. The fractional moment method consists in showing that for some $\gamma \in (0, 1)$,

$$\liminf_{n\to\infty} \mathbb{E}[(Z^{\omega}_{n,\beta,h})^{\gamma}] < \infty. \tag{1.28}$$

If the previous relation holds then $\mathrm{F}(\beta, h) = 0$, hence $h \le h_c(\beta)$. The standard application of this moment method makes use of a change of measure, which via the use of Hölder's inequality gives rise to an energy-entropy balance. The change of measure is done by defining the tilted measures:

$$\frac{\mathrm{d}\mathbb{P}_{\delta,k}}{\mathrm{d}\mathbb{P}} = \prod_{1\le i\le k} \exp(\delta\omega_i - \Lambda(\delta)), \qquad \delta \in \mathbb{R}. \tag{1.29}$$

*The condition in [19, Theorem 1.4] is $\alpha > 1$ but it was slightly improved to $\mathrm{E}[\tau_1] < \infty$ in [23].

By Hölder's inequality,

$$\mathbb{E}[(Z^{\omega}_{n,\beta,h})^{\gamma}] \le \mathbb{E}_{\delta,n}[Z^{\omega}_{n,\beta,h}]^{\gamma}\mathbb{E}_{\delta,n}\Big[\Big(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_{\delta,n}}\Big)^{\frac{1}{1-\gamma}}\Big]^{1-\gamma}. \tag{1.30}$$

The first factor in the r.h.s. is an *energy factor* while the second term is an *entropy factor*. In [55,74], the entropy factor can be bounded by an arbitrary constant, while the energy factor can be made arbitrarily small. Our first refinement requires optimizing this energy-entropy balance, which is crucial in obtaining the precise constant. To be more precise, we set $h_\beta = h^{\mathrm{a}}_c(\beta) + c\beta^2$, $n_\beta = \lceil t/\beta^2 \rceil$ ($t > 0$) and $\delta = -a\beta$ ($a \in \mathbb{R}$). By evaluating (1.30) for this specific choice of parameters, letting $\beta \downarrow 0$ and optimising over a (that is $a = (1-\gamma)/\mu$) we obtain:

$$\limsup_{\beta\downarrow 0} \mathbb{E}[(Z^{\omega}_{n_\beta,\beta,h_\beta})^{\gamma}] \le \frac{1}{\mu^{\gamma}} \exp\Big[\frac{\gamma}{\mu}\Big(c - \frac{1-\gamma}{2\mu}\Big)t\Big]. \tag{1.31}$$

The coarse graining step enables us to go from the estimate above to (1.28) by glueing several blocks of length n_β. This step is quite technical so we do not detail it here (see the original paper [19] for the full proof) but the outcome is that (1.28) holds as soon as the r.h.s. above is small and, since t can be chosen arbitrarily large, this means that any $c < (1-\gamma)/(2\mu)$ is a valid choice. The condition that $\gamma > 1/(\alpha+1)$ for the coarse graining step to work finally gives the desired lower bound. Note that we need a refinement in the coarse graining procedure too. In the standard application, the polymer only needs to place a positive fraction of monomers at the interface in each visited coarse-grained block, while in our case, we need to ensure that this positive fraction is in fact close to 1. For this step, $\alpha > 1$ plays a crucial role.

Proof strategy for the upper bound. The idea to prove the upper bound is to couple the smoothing inequality in Proposition 1.4 with a rough linear (but quantitative) lower bound on the free energy. More precisely, we prove that for every $c \in \mathbb{R}$,

$$\liminf_{\beta\downarrow 0} \frac{\mathrm{F}(\beta, h^{\mathrm{a}}_c(\beta) + c\beta^2)}{\beta^2} \ge \frac{1}{\mu}\Big[c - \frac{1}{2\mu}\Big]. \tag{1.32}$$

This is obtained by defining for $q \in \mathbb{N}$,

$$H^{(q)}_{\ell,\omega} = \log \mathrm{E}\Big[\exp\Big(\sum_{1\le k\le \ell} (\beta\omega_k + h)\delta_k\Big)|\tau_q = \ell\Big] \tag{1.33}$$

and applying Jensen's inequality:

$$\mathrm{F}(\beta, h) \ge \frac{1}{q\mu}\mathbb{E}\mathrm{E}(H^{(q)}_{\tau_q,\omega}). \tag{1.34}$$

Then, we set $h = h^{\mathrm{a}}_c(\beta) + c\beta^2$, perform a Taylor expansion as $\beta \downarrow 0$ and later send q to infinity. Remarkably, enforcing the compatibility of the inequality in (1.32) with the corresponding smoothing inequality (1.26) leads to the sharp upper bound on the critical curve. What actually lies behind this compatibility condition is a rare stretch strategy, which we now sketch. We start by decomposing $\mathbb{N} = \cup_{i=1}^{\infty} B_i$ into blocks of (large) length M and we search for such blocks where the sample average of the disorder is about $a\beta$, that is $M^{-1}\sum_{n\in B_\cdot} \omega_n \simeq a\beta$, where a is a fixed parameter. The probability of a block to have a sample average of that order is roughly $\exp(-a^2\beta^2 M/2)$ and the reciprocal of this probability will give the number of blocks that will separate the atypical ones. Once these "atypical blocks" have been identified, we let the polymer jump from the end point of one such block to the start point of the following. In view of (1.1), the cost for this is roughly $\exp(-(1+\alpha)a^2\beta^2 M/2)$. Once at the beginning of an atypical block, the contribution to the free energy of the polymer is roughly, by (1.32) (recall that $h = h^{\mathrm{a}}_c(\beta) + c\beta$ and we have shifted the mean of the disorder to $a\beta$):

$$\frac{\beta^2}{\mu}\Big[c + a - \frac{1}{2\mu}\Big]M. \tag{1.35}$$

The energy-entropy balance gives the following lower bound for the free energy:

$$e^{-a^2\beta^2 M/2}\, M\left[\left(c+a-\frac{1}{2\mu}\right)-\frac{1+\alpha}{2}a^2\right]\beta^2.$$

Finally, optimizing over a, the term in square brackets becomes $\left[c-\frac{\alpha}{1+\alpha}\frac{1}{2\mu}\right]$, which leads to the sharp upper bound on the critical curve. Let us note that rare stretch strategies have been employed extensively in the study of pinning (but also copolymer) models, cf. [72, Sections 6.3 and 5.4] and [73, Section 5.1] for instance.

Remark 1.6. *The case $\alpha \in (\frac{1}{2}, 1)$ has been answered by Caravenna, Toninelli and Torri [37]. There exists a constant $m_\alpha \in (0,\infty)$ and a slowly varying function $\widetilde{L}_\alpha$ (explicitly determined by L and α) such that*

$$\lim_{\beta\downarrow 0}\frac{h_c(\beta)-h_c^{\mathrm{a}}(\beta)}{\widetilde{L}_\alpha(1/\beta)\beta^{\frac{2\alpha}{2\alpha-1}}}=m_\alpha. \tag{1.36}$$

Moreover, m_α is the critical parameter of the continuum disordered pinning model which was built in [35, 36]. However, the continuum limit does not seem to give any information on the limit in (1.27). Indeed, it is shown in [35] that when $\alpha > 1$, the partition function under weak coupling converges, in the continuum limit, to the exponential of a Brownian motion with drift, which depends on μ but not on $\alpha > 1$.

1.3 The random pinning model with correlated disorder

The study of pinning models in correlated disorder is more recent [17, 18, 21, 23, 113]. The motivation is twofold. First, it is quite natural from a mathematical perspective to try and understand what remains of the properties of the model (in particular, the validity of the criterion for disorder relevance) when ones removes the assumption of independence in the disorder sequence. Also, from the point of view of physical and biological motivations, it appears that inhomogeneities may present (more or less strong) correlations: let us mention for instance the correlations in the sequence of nucleotides, which play the role of the disorder sequence in DNA denaturation [86]. The question of the influence of correlated disorder on critical phenomena was actually considered in the context of long-range correlated percolation and the Harris criterion extended to the so-called Weinrib-Halperin criterion [140,141], which suggests that the criterion for disorder relevance should be modified only if the correlations are strong enough.

Our first observation is that the existence of the quenched free energy is still guaranteed under rather mild assumptions on the disorder sequence, namely if ω is a stationary and ergodic sequence of integrable random variables [72, Theorem 4.6]. However, the annealed model is in general not trivial and this is a reason to first restrict to some particular correlated disorder distributions, such as

- correlated Gaussian disorder [17, 23, 111–113];
- random environments with large attractive regions of sub-exponential decay, also referred to as *infinite disorder* [18, 21];
- disorder built on renewal sequences [3, 21, 44].

1.3.1 Correlated Gaussian disorder

The random pinning model with Gaussian correlated disorder has been studied in [17, 23, 111–113] and is still partially open. We assume here that $\omega = (\omega_n)_{n\geq 0}$ is a Gaussian stationary sequence,

the law of which is denoted by $\mathbb{P}$ and we recall that ω_n is the (random) charge on the n-th site of the defect line (or interface). Its correlation function is

$$\rho_n := \mathbb{E}[\omega_0 \omega_n], \qquad n \in \mathbb{Z}, \tag{1.37}$$

with $\rho_{-n} = \rho_n$ as a consequence of stationarity. We still assume that $\mathbb{E}[\omega_0] = 0$ and $\rho_0 = \mathbb{E}[\omega_0^2] = 1$, which is just a matter of normalization. For notational convenience, we also write $\Upsilon := (\rho_{ij})_{i,j \geq 0}$ the covariance matrix, where $\rho_{ij} := \mathbb{E}[\omega_i \omega_j] = \rho_{|j-i|}$.

Example 1.7. *A valid choice for the correlation structure is $\rho_n = (1+n)^{-a}$ for all $n \geq 0$, with $a > 0$ a fixed constant, since it is convex, cf. [116].*

In the Gaussian case, the two-point correlation function is enough to describe the whole correlation structure and to compute exponential moments. This allows us to get an explicit annealed model:

$$Z^{\mathrm{a}}_{n,\beta,h} = \mathrm{E}\Big[\exp\Big(h \sum_{1 \leq k \leq n} \delta_k + \frac{\beta^2}{2} \sum_{1 \leq k,\ell \leq n} \rho_{k\ell} \delta_k \delta_\ell \Big) \Big]. \tag{1.38}$$

Let us observe that when correlations are non-negative and not summable the annealed model is degenerate, in the sense that $\mathrm{F}_{\mathrm{a}}(\beta, h) = +\infty$ for all (β, h), and the quenched free energy is always positive. We refer to [17, 18] for an explanation of this so-called *infinite disorder* phenomenon. Henceforth we shall make the following assumption:

Assumption 1.8. *We assume that correlations are* summable, *that is $\sum_{n \in \mathbb{Z}} |\rho_n| < +\infty$, and we define the constant $\Upsilon_\infty := \sum_{n \in \mathbb{Z}} \rho_n$. This means that Υ is a bounded operator. We also make the additional technical assumption that Υ is invertible.*

Note that Assumption 1.8 implies that $\lim_{n \to \infty} \rho_n = 0$, which entails ergodicity of ω, see [49, Chapter 14, Section 2, Theorem 2]. For the choice $\rho_n = (1+n)^{-a}$, Assumption 1.8 corresponds to having $a > 1$.

Even so, the annealed model is not trivial and it is still not completely solved. A spectral characterization of the annealed critical curve is given in [113], but there is no explicit formula as in the i.i.d. case. Nonetheless, we are able to derive the small coupling asymptotic of the annealed critical curve [†]:

Proposition 1.9 (cf. Theorem 2.3 in [113])**.** *The following limit holds:*

$$\lim_{\beta \downarrow 0} \frac{h^{\mathrm{a}}_c(\beta)}{\beta^2} = -\frac{1}{2} \mathbf{C}_\rho, \quad \text{with} \quad \mathbf{C}_\rho := \sum_{n \in \mathbb{Z}} \rho_n \mathrm{P}(|n| \in \tau). \tag{1.39}$$

In [23] we identified the small coupling asymptotic of the quenched critical point when $\mathrm{E}[\tau_1] < +\infty$, in analogy with Theorem 1.5:

Theorem 1.10. *If $\mu = \mathrm{E}[\tau_1] < +\infty$ then we have*

$$\lim_{\beta \downarrow 0} \frac{h_c(\beta) - h^{\mathrm{a}}_c(\beta)}{\beta^2} = \frac{\Upsilon_\infty}{2\mu} \frac{\alpha}{1+\alpha}. \tag{1.40}$$

Note that the asymptotics given in Theorem 1.10 and that of Theorem 1.5 only differ through the multiplicative constant Υ_∞. In particular, one recovers Theorem 1.5 in the IID case, where $\Upsilon_\infty = 1$. Theorem 1.10 proves disorder relevance in terms of critical points if $\mu < +\infty$ (in particular if $\alpha > 1$), under Assumption 1.8.

Let us now turn to critical exponents. The smoothing inequality in Proposition 1.4 has been extended to the Gaussian correlated case:

[†] In [113, Theorem 2.3] we assume that correlations are doubly-summable, but a look at the proof reveals that summable correlations are enough.

Proposition 1.11 (cf. Theorem 2.3 in [17]). *For every $\beta > 0$ and $u \geq 0$,*

$$0 \leq \mathrm{F}(\beta, h_c(\beta) + u) \leq \left(\frac{1+\alpha}{2\Upsilon_\infty}\right) \frac{u^2}{\beta^2}. \tag{1.41}$$

As for the annealed critical exponent, it is known to coincide with the homogeneous one under the stronger assumption that the correlations are *doubly summable*:

Theorem 1.12 (cf. Theorem 3.1 in [113]). *If $\sum_{n\in\mathbb{N}} n|\rho_n|$ is finite then for all $\beta \geq 0$, there exists $C = C(\beta) \in (0,\infty)$ such that for $u \geq 0$,*

$$(1/C)\,\mathrm{F}(0,u) \leq \mathrm{F}_{\mathrm{a}}(\beta, h_c^{\mathrm{a}}(\beta) + u) \leq C\,\mathrm{F}(0,u). \tag{1.42}$$

The proof uses a general theorem on countable Markov shifts with a potential of summable variations. Let us describe in a few lines the core of the method developed in [113]. First, we define a potential function on $\mathbb{N}^{\mathbb{N}}$ by

$$\phi_\beta(t_1, t_2, \ldots) = \log K(t_1) + \tfrac{1}{2}\beta^2\Big(1 + 2\sum_{k\geq 1} \rho_{t_1+\ldots+t_k}\Big). \tag{1.43}$$

We also consider the following operator, which acts on bounded functions f defined on $\mathbb{N}^{\mathbb{N}}$ and which is closely linked to the annealed partition function:

$$(\mathcal{L}_\beta f)(t_1, t_2, \ldots) = \sum_{n\in\mathbb{N}} \exp(\phi_\beta(n, t_1, t_2, \ldots)) f(n, t_1, t_2, \ldots). \tag{1.44}$$

It turns out that when the correlations are *doubly summable* the potential function has *summable variations* and $\mathcal{L}_\beta$ has nice spectral properties. In particular, one can define $\lambda(\beta)$, the analogue of the Perron-Frobenius eigenvalue of $\mathcal{L}_\beta$, and we obtain

$$h_c^{\mathrm{a}}(\beta) = -\log \lambda(\beta). \tag{1.45}$$

Such a characterization also holds for the annealed free energy. To this end, we define

$$\phi_{\beta,\mathrm{F}}(t_1, t_2, \ldots) = \phi_\beta(t_1, t_2, \ldots) - \mathrm{F}t_1, \tag{1.46}$$

and $\mathcal{L}_{\beta,\mathrm{F}}$ (resp. $\lambda(\beta, \mathrm{F})$) the associated operator (resp. principal eigenvalue). We prove that

$$\mathrm{F}^{\mathrm{a}}(\beta, h) = \inf\{\mathrm{F} \geq 0 \colon\ -\log \lambda(\beta, \mathrm{F}) = h\}, \qquad h \geq h_c^{\mathrm{a}}(\beta). \tag{1.47}$$

Then, Theorem 1.12 follows from a general theorem for phase transitions of dynamical systems. We point out that the bounds in (1.42) can be upgraded to sharp asymptotics under the stronger assumption that the correlations decrease exponentially fast, see [113, Theorem 2.2]. Theorem 1.12 was also independently proved in [17, Theorem 2.2] with more basic arguments (quasi-renewal property) but for small values of β.

Finally, one may deduce from (1.42) and (1.41) that disorder is relevant (in terms of critical exponents) for $\alpha > 1/2$ and $\sum_{n\in\mathbb{N}} n|\rho_n| < \infty$, which corresponds to having $a > 2$ in Example 1.7.

Remark 1.13. *The condition that Υ is invertible is actually only necessary for Proposition 1.11 and Theorem 1.10. A simple case when Υ is invertible is when $1 = \rho_0 > 2\sum_{n\in\mathbb{N}} |\rho_n|$: it is then diagonally dominant. More generally, one has to consider the Laurent series associated to the Toeplitz matrix Υ, namely $f(\lambda) = 1 + 2\sum_{n\in\mathbb{N}} \rho_n \cos(\lambda n)$ (we used that $\rho_0 = 1$, and that $\rho_{-n} = \rho_n$). Then, the fundamental eigenvalue distribution theorem of Szegö [80, Chapter 5] tells that the Toeplitz operator Υ is invertible if and only if $\min_{\lambda\in[0,2\pi]} f(\lambda) > 0$. For example, if $\rho_0 = 1, \rho_1 = 1/2$ and $\rho_n = 0$ for $n \geq 2$, then Assumption 1.8 is not verified: indeed, one then has that $f(\lambda) = 1 + \cos(\lambda)$, and its minimum is 0 so that the operator Υ is not invertible.*

One of the main difficulty in the correlated setup is that the partition functions associated to disjoint subsets of the defect line are no longer independent. To tackle this, we proved and used the following Gaussian decoupling inequality:

Lemma 1.14 (cf. Lemma B.1 in [23]). *Let $\mathbb{P}$ be the law of a centered Gaussian sequence $\omega = (\omega_n)_{n\in\mathbb{Z}}$, with a correlation matrix $\Upsilon = (\rho_{ij})_{i,j\in\mathbb{Z}}$. If $\mathcal{I}$ and $\mathcal{J}$ are two disjoint subsets of $\mathbb{Z}$, we define*

$$C(\mathcal{I},\mathcal{J}) = \sum_{i\in\mathcal{I},\, j\in\mathcal{J}} |\, \rho_{ij}\,| \,. \tag{1.48}$$

Let $f : \mathbb{R}^{\mathcal{I}} \mapsto \mathbb{R}$ and $g : \mathbb{R}^{\mathcal{J}} \mapsto \mathbb{R}$ be two functions which are $\mathcal{C}^2$ and such that, for some constant $c \in (0,\infty)$,

$$\begin{aligned} \forall i \in \mathcal{I},\, j \in \mathcal{J}, \qquad & |\, \partial_{\omega_i} f\,| \le c\, f, \qquad |\, \partial_{\omega_j} g\,| \le c\, g\,; \\ \forall i, i' \in \mathcal{I},\, j, j' \in \mathcal{J}, \qquad & |\, \partial^2_{\omega_i,\omega_{i'}} f\,| \le c^2\, f, \qquad |\, \partial^2_{\omega_j,\omega_{j'}} g\,| \le c^2\, g\,. \end{aligned} \tag{1.49}$$

Denoting by $\omega_{\mathcal{I}}$ the vector $(\omega_n)_{n\in\mathcal{I}}$, for any subset $\mathcal{I}$ of $\mathbb{Z}$, we have

$$\mathbb{E}[f(\omega_{\mathcal{I}})g(\omega_{\mathcal{J}})] \le e^{c^2 C(\mathcal{I},\mathcal{J})}\mathbb{E}[f(\omega_{\mathcal{I}})]\mathbb{E}[g(\omega_{\mathcal{J}})]. \tag{1.50}$$

The basic idea behind Lemma 1.14 is interpolation. By using two independent copies of ω, we first build a centered Gaussian sequence $\widetilde{\omega}$ such that

$$\mathbb{E}(\widetilde{\omega}_i\widetilde{\omega}_j) = \rho_{ij}(1_{\{i,j\in\mathcal{I}\}} + 1_{\{i,j\in\mathcal{J}\}}), \qquad i,j \in \mathcal{I}\cup\mathcal{J}. \tag{1.51}$$

We then define the interpolating sequence

$$\omega_n(t) = \sqrt{t}\widetilde{\omega}_n + \sqrt{1-t}\omega_n \quad \text{and} \quad \phi(t) = \mathbb{E}[f(\omega_{\mathcal{I}}(t))g(\omega_{\mathcal{J}}(t))], \qquad t \in [0,1], \tag{1.52}$$

so that the expectation in the l.h.s. of (1.50) coincides with $\phi(0)$ whereas the product of the expectations in the r.h.s. coincides with $\phi(1)$. The derivative of ϕ may be evaluated via Gaussian intergration by part and it then boils down to proving the differential inequality $\phi'(t) \ge -C\phi(t)$ for some positive constant C and applying Gronwall's lemma, see [23, Lemma B.1] for details. Note that interpolation techniques have been already successfully applied by Toninelli [133] in the context of random pinning with i.i.d. disorder.

1.3.2 Renewal disorder

This section is based on a joint work with D. Cheliotis and Y. Chino [44]. The disorder sequence we consider here is built on another renewal sequence $\hat{\tau}$ which is independent of τ and starts at the origin. Its law will be denoted by $\hat{\mathrm{P}}$. To be more precise, we assume that if the n-th monomer is on the interface then it is given a reward equal to $\beta + h$ if $n \in \hat{\tau}$ and h otherwise. This corresponds to the following binary correlated disorder sequence:

$$\omega_n = \hat{\delta}_n := 1_{\{n\in\hat{\tau}\}}, \qquad n \in \mathbb{N}_0. \tag{1.53}$$

From now on, the inter-arrival laws of τ and $\hat{\tau}$ satisfy

$$K(n) := \mathrm{P}(\tau_1 = n) \sim c_K\, n^{-(1+\alpha)}, \quad \hat{K}(n) := \hat{\mathrm{P}}(\hat{\tau}_1 = n) = c_{\hat{K}}\, n^{-(1+\hat{\alpha})}, \quad n \in \mathbb{N} \tag{1.54}$$

with $\alpha, \hat{\alpha} > 0$, and

$$\mu := \mathrm{E}(\tau_1), \qquad \hat{\mu} := \hat{\mathrm{E}}(\hat{\tau}_1), \tag{1.55}$$

which may be finite or infinite. Note that these definitions ensure aperiodicity for both renewal processes. In principle, the constants c_K and $c_{\hat{K}}$ may also be replaced by slowly varying functions, which would allow to include the special case $\alpha \in \{0,1\}$ in the discussion, but we refrain from

doing so for the sake of simplicity. Also, we write an equality in the definition of $\hat{K}(n)$ to ensure log-convexity. This technical condition is only needed to prove Theorem 1.22, which we actually believe to hold when the equality sign is replaced by the equivalent sign in the definition of $\hat{K}$ in (1.54). The definitions of the basic quantities such as partition functions and free energies are the same as in the previous sections, except that $\mathbb{P}$ and $\mathbb{E}$ are replaced by $\hat{\mathrm{P}}$ and $\hat{\mathrm{E}}$.

A first dichotomy arises:

- If $\hat{\alpha} < 1$ then the quantity in front of β in the Hamiltonian, which equals $|\tau \cap \hat{\tau} \cap \{1, \dots, n\}|$, is at most $|\hat{\tau} \cap \{1, \dots n\}|$. The latter is of order $n^{\hat{\alpha}} = o(n)$, therefore disorder has no effect on the quenched free energy, which reduces to the homogeneous free energy. However the annealed model is non-trivial.

- If $\hat{\alpha} > 1$ then (i) we may replace $\hat{\mathrm{P}}$ by its stationary version, denoted by $\hat{\mathrm{P}}_s$, under which the distribution of the increments $(\hat{\tau}_{n+1} - \hat{\tau}_n)_{n \in \mathbb{N}_0}$ is the same as in $\hat{\mathrm{P}}$, whereas that of $\hat{\tau}_0$ becomes $\{\hat{\mathrm{P}}(\hat{\tau}_1 > n)/\hat{\mu}\}_{n \in \mathbb{N}_0}$, see e.g. [8, Chapter V, Corollary 3.6] (Again, this does not affect the free energy, see Propositions 1.15 and 1.21) ; (ii) the correlations of the disorder sequence have a power-law decay with exponent $\hat{\alpha} - 1$, since for $n > m$,

$$\begin{aligned} \rho_{m,n} := \mathrm{Cov}_{\hat{\mathrm{P}}_s}(\hat{\delta}_m, \hat{\delta}_n) &= \hat{\mathrm{E}}_s(\hat{\delta}_m \hat{\delta}_n) - \hat{\mathrm{E}}_s(\hat{\delta}_m)\hat{\mathrm{E}}_s(\hat{\delta}_n) \\ &= \hat{\mathrm{P}}_s(m \in \hat{\tau})\Big(\hat{\mathrm{P}}_s(n \in \hat{\tau} \mid m \in \hat{\tau}) - \hat{\mathrm{P}}_s(n \in \hat{\tau})\Big) \\ &= \frac{1}{\hat{\mu}}\Big(\hat{\mathrm{P}}(n - m \in \hat{\tau}) - \frac{1}{\hat{\mu}}\Big) \\ &\sim c(n-m)^{1-\hat{\alpha}}, \qquad \text{as } n - m \to \infty, \end{aligned} \tag{1.56}$$

  for some positive constant c. The latter can be deduced from the Renewal Theorem and the following renewal convergence estimate [70, Lemma 4]

$$\hat{\mathrm{P}}(n \in \hat{\tau}) - \frac{1}{\hat{\mu}} \sim \frac{c_{\hat{K}}}{\hat{\alpha}(\hat{\alpha}-1)\hat{\mu}^2} \frac{1}{n^{\hat{\alpha}-1}}, \qquad n \to \infty. \tag{1.57}$$

This choice of disorder distribution and correlation structure is motivated by the following:

- By tuning the value of the parameter exponent $\hat{\alpha}$, one finds a whole spectrum of correlation exponents ranging from non-summable correlations to summable correlations, according to whether the sum $\sum_{n \geq 0} |\rho_{0,n}|$ is infinite or finite. According to (1.56), correlations are summable when $\hat{\alpha} > 2$ and non-summable when $\hat{\alpha} < 2$.

- Our disorder sequence is bounded, therefore the annealed free energy is always finite, in contrast to the case of Gaussian variables with non-summable correlations [17].

- The probability of observing a long sequence of ones decays exponentially in the length, which rules out the infinite disorder regime discussed in [18].

- The renewal structure of the disorder sequence makes the study of the annealed model and decoupling inequalities more tractable.

The rest of the section is organized as follows: we first introduce the intersection set, which serves as preliminary for the annealed results that come next, and we end with two theorems on the quenched model. We refer the reader to the last section of this chapter for further discussion and perspectives, including one conjecture.

The intersection set

Before going any further, we need to introduce some quantities. The intersection set of τ and $\hat{\tau}$, which we denote by

$$\widetilde{\tau} = \tau \cap \hat{\tau}, \tag{1.58}$$

will play a fundamental role in the sequel. Let us notice that it is itself a renewal starting at $\widetilde{\tau}_0 = 0$. We denote its law by $\widetilde{\mathrm{P}}$ and write

$$\widetilde{\delta}_n := 1_{\{n\in\widetilde{\tau}\}} = 1_{\{n\in\tau\}} 1_{\{n\in\hat{\tau}\}} = \delta_n \hat{\delta}_n, \qquad n \in \mathbb{N}_0. \tag{1.59}$$

For $h \le 0$, denote by P_h the probability of the renewal process with $\tau_0 = 0$ and inter-arrival law

$$K_h(n) = e^h K(n), \quad n \in \mathbb{N}, \qquad K_h(\infty) = 1 - e^h. \tag{1.60}$$

We denote the corresponding expectation by E_h. If $h < 0$ then τ is transient under P_h. The expected number of points in the renewal set $\widetilde{\tau}$ (including 0) under the law $\mathrm{P}_h \otimes \hat{\mathrm{P}}$ is denoted by

$$\mathcal{I}(h) := \mathrm{E}_h \hat{\mathrm{E}}(|\widetilde{\tau}|) \in [1, \infty]. \tag{1.61}$$

Note that

$$\begin{aligned} \mathcal{I}(h) &= \sum_{n\in\mathbb{N}_0} \mathrm{P}_h(n \in \tau)\hat{\mathrm{P}}(n \in \hat{\tau}) = \sum_{n,k\in\mathbb{N}_0} e^{hk}\mathrm{P}(\tau_k = n)\hat{\mathrm{P}}(n \in \hat{\tau}) \\ &= \sum_{k\in\mathbb{N}_0} e^{hk}(\mathrm{P}\otimes\hat{\mathrm{P}})(\tau_k \in \hat{\tau}). \end{aligned} \tag{1.62}$$

The function $\mathcal{I}$ is finite and infinitely differentiable on $(-\infty, 0)$. It is also continuous on $(-\infty, 0]$, increasing and strictly convex. Its range is $[1, \mathcal{I}(0)]$ with $\mathcal{I}(0) = \mathrm{E}\hat{\mathrm{E}}(|\tau \cap \hat{\tau}|)$, which may be finite or infinite. It follows from standard renewal theory (see [8, Proposition 2.4, Chapter 1]) that

$$p(h) := (\mathrm{P}_h \otimes \hat{\mathrm{P}})(\widetilde{\tau}_1 < \infty) = 1 - \mathcal{I}(h)^{-1}. \tag{1.63}$$

Results on the annealed model

We begin with the existence of the annealed free energy.

Proposition 1.15. *For all $\beta \ge 0$ and $h \in \mathbb{R}$, the annealed free energy*

$$\mathrm{F}_{\mathrm{a}}(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{a}}_{n,\beta,h} \tag{1.64}$$

exists and it is finite and non-negative. The result still holds, without changing the value of the free energy, when $\hat{\mu} < \infty$ and $\hat{\mathrm{P}}$ is replaced by $\hat{\mathrm{P}}_s$.

The following basic properties of the annealed free energy are standard: the function $(\beta, h) \mapsto \mathrm{F}_{\mathrm{a}}(\beta, h)$ is convex, continuous and non-decreasing in both variables. Our next result provides an expression for the annealed critical curve involving the function $\mathcal{I}$.

Proposition 1.16. *Let $\beta_0 = -\log p(0)$. The annealed critical curve is*

$$h^{\mathrm{a}}_c(\beta) = \begin{cases} \mathcal{I}^{-1}\left(\frac{1}{1-e^{-\beta}}\right) & \text{if } \beta > \beta_0, \\ 0 & \text{if } 0 \le \beta \le \beta_0. \end{cases} \tag{1.65}$$

Remark 1.17. *From (1.63) we have that $\beta_0 = -\log(1 - \{\mathrm{E}\hat{\mathrm{E}}(|\widetilde{\tau}|)\}^{-1})$ is non-negative. Therefore, using the Renewal Theorem, we see that*

$$\beta_0 \begin{cases} > 0 & \text{if } \alpha + \hat{\alpha} < 1, \\ = 0 & \text{if } \alpha + \hat{\alpha} > 1. \end{cases} \tag{1.66}$$

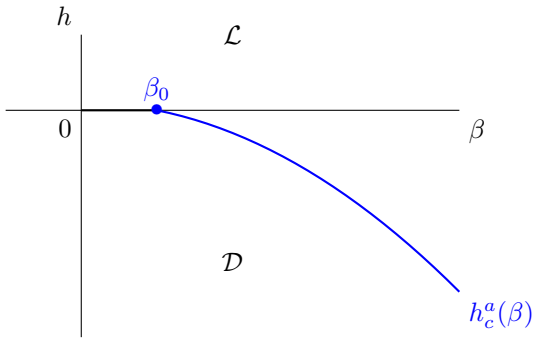

Figure 1.1: Shape of the annealed critical curve (in blue). The critical point $\beta_0 = -\log p(0)$ and the slope at β_0 might be positive or equal to zero, depending on the values of α and $\hat{\alpha}$, see Remark 1.17 and Theorem 1.19.

By the properties of $\mathcal{I}$ we get that $\beta \mapsto h_c^{\mathrm{a}}(\beta)$ is infinitely differentiable in $[0,\infty)\backslash\{-\log p(0)\}$ and has negative derivative in $(-\log p(0),\infty)$. Moreover, $\beta \mapsto h_c^{\mathrm{a}}(\beta)$ is concave because $(\beta,h) \mapsto \mathrm{F}^{\mathrm{a}}(\beta,h)$ is convex, see Figure 1.1. The next two results provide the scaling behaviour of the annealed critical curve close to β_0.

Theorem 1.18. *Suppose $\alpha+\hat{\alpha}>1$ (then $\beta_0=0$). There exists $c_{\mathrm{a}}>0$ such that*

$$h_c^{\mathrm{a}}(\beta) = -\frac{\beta}{\hat{\mu}} - c_{\mathrm{a}}\beta^{\gamma_{\mathrm{a}}}[1+o(1)], \qquad \text{as } \beta \downarrow 0, \tag{1.67}$$

where

$$\gamma_{\mathrm{a}} = \begin{cases} 1+\left\lceil \frac{\hat{\alpha}-1}{\alpha\wedge 1}\wedge 1\right\rceil. & \text{if } \hat{\alpha}>1 \text{ and } \hat{\alpha}\neq 1+\alpha\wedge 1 \\ \frac{\alpha\wedge 1}{\hat{\alpha}-1+\alpha\wedge 1} & \text{if } \hat{\alpha}<1. \end{cases} \tag{1.68}$$

If $\hat{\alpha}=1+\alpha\wedge 1$, we get instead

$$h_c^{\mathrm{a}}(\beta) = -\frac{\beta}{\hat{\mu}} - c_{\mathrm{a}}\beta^2|\log\beta|[1+o(1)]. \tag{1.69}$$

The first term $-\beta/\hat{\mu}$ in the expansion simply accounts for the fact that our disorder sequence is not centered and that by the Renewal Theorem, $\lim_{n\to\infty}\hat{\mathrm{E}}(\hat{\delta}_n) = 1/\hat{\mu}$. Note that by Jensen's inequality, $h_c^{\mathrm{a}}(\beta) \le -\beta/\hat{\mu}$, and this already gives that $c_{\mathrm{a}} \ge 0$ in Theorem 1.18. If $\hat{\alpha} > 1+\alpha\wedge 1$, then $\gamma_{\mathrm{a}}=2$, as in the i.i.d. case, but if $\hat{\alpha}<1+\alpha\wedge 1$, there is an anomalous scaling of the annealed critical curve. Moreover, if $\hat{\alpha}<1$ then $\hat{\mu}=\infty$ so the term $\beta/\hat{\mu}$ disappears and $\gamma_{\mathrm{a}}>1$ gives the first order term.

Theorem 1.19. *Suppose $\alpha+\hat{\alpha}<1$ (then $\beta_0>0$). As $\beta\downarrow\beta_0$, there is a constant $c\in(-\infty,0)$ such that*

$$h_c^{\mathrm{a}}(\beta) \sim \frac{c(\beta-\beta_0)^{\gamma_{\mathrm{a}}}}{1+|\log(\beta-\beta_0)|1_{\{1-\hat{\alpha}=2\alpha\}}}, \quad \text{where} \quad \gamma_{\mathrm{a}} = 1\vee\frac{\alpha}{1-\alpha-\hat{\alpha}}. \tag{1.70}$$

Our next result is about the order of the annealed phase transition.

Proposition 1.20 (The annealed critical exponent)**.** *Suppose $\hat{\alpha}>0$. Let $\beta>0$. There exists a constant $C=C(\beta)\in(0,\infty)$ such that*

$$(1/C)u^{\nu_{\mathrm{a}}(\beta)} \le \mathrm{F}_{\mathrm{a}}(\beta, h_c^{\mathrm{a}}(\beta)+u) \le Cu^{\nu_{\mathrm{a}}(\beta)} \tag{1.71}$$

for all $0 < u \le 1$, *with*

$$\nu_{\mathrm{a}}(\beta) := \begin{cases} \frac{1}{\alpha_{\mathrm{eff}}} \vee 1 & \text{if } \beta > \beta_0, \\ \frac{1}{\alpha} \vee 1 & \text{if } 0 \le \beta \le \beta_0, \end{cases} \tag{1.72}$$

where $\alpha_{\mathrm{eff}} := \alpha + (1 - \hat{\alpha})_+$.

Therefore, the annealed critical exponent remains unchanged compared to the homogeneous case if $\hat{\alpha} > 1$, but is changed for large values of β when $\hat{\alpha} < 1$ and $\alpha < 1$.

Results on the quenched model

We start with the existence of the quenched free energy.

Proposition 1.21. *For* $\beta > 0$ *and* $h \in \mathbb{R}$ *the sequence* $\{(1/n) \log Z^{\omega}_{n,\beta,h}\}_{n\in\mathbb{N}}$ *converges* $\hat{\mathrm{P}}$*-a.s. and in* $L^1(\hat{\mathrm{P}})$ *to a non-negative constant* $\mathrm{F}(\beta, h)$ *called the quenched free energy. Moreover, if* $\hat{\mu} = \infty$ *then* $\mathrm{F}(\beta, h) = \mathrm{F}(0, h)$*, and if* $\hat{\mu} < \infty$ *then the convergence still holds* $\hat{\mathrm{P}}_s$*-a.s. and in* $L_1(\hat{\mathrm{P}}_s)$ *(without changing the value of the free energy).*

We are able to prove the following smoothing inequality:

Theorem 1.22. *Let* $\hat{\alpha} > 1$ *and* $\beta > 0$*. There exists a constant* $C = C(\beta) \in (0, \infty)$ *such that for* $0 < u \le 1$,

$$\mathrm{F}(\beta, h_c(\beta) + u) \le C u^{2 \wedge \hat{\alpha}} (1 + |\log u| 1_{\{\hat{\alpha}=2\}}). \tag{1.73}$$

The exponent $2 \wedge \hat{\alpha}$ in the theorem above is not expected to be optimal, but in view of Proposition 1.20, this already tells us that disorder is relevant w.r.t. critical exponents if $\hat{\alpha} > 2$ and $\alpha > 1/2$, or if $\hat{\alpha} \in (1, 2)$ and $\hat{\alpha} > 1/\alpha$. This result extends the smoothing inequality in Proposition 1.4.

We also prove the following result on disorder irrelevance:

Theorem 1.23. *If* $\hat{\alpha} > 2$ *and* $\alpha < 1/2$ *then disorder is irrelevant for* β *small enough, meaning that* $h_c(\beta) = h^{\mathrm{a}}_c(\beta)$ *and*

$$\lim_{h \downarrow h^{\mathrm{a}}_c(\beta)} \frac{\log \mathrm{F}(\beta, h)}{\log(h - h^{\mathrm{a}}_c(\beta))} = \frac{1}{\alpha}. \tag{1.74}$$

To the best of our knowledge, such a result on disorder irrelevance (in both critical points and exponents) has not yet been proved for other instances of correlated disorder, e.g. Gaussian disorder with summable correlations.

Sketch of the proof of Theorem 1.23. We follow the standard second moment method, which consists in proving that the second moment of the quenched partition function at the annealed critical point is bounded in the system size. However, the evaluation of such moment is not as direct as in the i.i.d. case and new arguments are needed. First, we consider a slightly more convenient partition function, that is (recall (1.60))

$$\bar{Z}_{n,\beta,h} = \mathrm{E}_h\Big[\exp\Big(\beta \sum_{1 \le k \le n} \delta_k \hat{\delta}_k \Big) \Big], \qquad h \le 0. \tag{1.75}$$

Hence our goal is now to prove that

$$\sup_{n \ge 1} \hat{\mathrm{E}}[\bar{Z}^2_{n,\beta,h^{\mathrm{a}}_c(\beta)}] < \infty. \tag{1.76}$$

The next step is to set $z = z(\beta) = e^{\beta} - 1$ and perform a Meyer expansion on the quenched partition function, which gives:

$$\hat{\mathrm{E}}[\bar{Z}^2_{n,\beta,h}] = \sum_{I,J \subseteq [n]} z^{|I|+|J|} \mathrm{P}_h(I \subseteq \tau) \mathrm{P}_h(J \subseteq \tau) \hat{\mathrm{P}}_h(I \cup J \subseteq \hat{\tau}). \tag{1.77}$$

Then, we provide a decoupling inequality for the last term of the r.h.s., that is a control on

$$\Big|\frac{\hat{\mathrm{P}}_h(I\cup J\subseteq\hat\tau)}{\hat{\mathrm{P}}_h(I\subseteq\hat\tau)\hat{\mathrm{P}}_h(J\subseteq\hat\tau)}-1\Big|. \tag{1.78}$$

At this stage we use that $\hat\alpha>2$ to prove the existence of a constant $C>1$ such that the second moment is bounded from above by:

$$\begin{aligned}&\sum_{I,J\subseteq[n]} z^{|I|+|J|}C^{|I\cap J|}[\mathrm{P}_h(I\subseteq\tau)\hat{\mathrm{P}}_h(I\subseteq\hat\tau)][\mathrm{P}_h(J\subseteq\tau)\hat{\mathrm{P}}_h(J\subseteq\hat\tau)]\\ &=\sum_{I,J\subseteq[n]} z^{|I|+|J|}C^{|I\cap J|}[(\mathrm{P}_h\otimes\hat{\mathrm{P}}_h)(I\subseteq\tau\cap\hat\tau)][(\mathrm{P}_h\otimes\hat{\mathrm{P}}_h)(J\subseteq\tau\cap\hat\tau)]\end{aligned} \tag{1.79}$$

(for ease of exposition we simplify this part of the argument but details can be found in [44]). We now almost have a product. The next step is to write that

$$|I\cap J|=|I|+|J|-|I\cup J|, \tag{1.80}$$

and notice that

$$\Big(\frac{1}{C}\Big)^{|I\cup J|}=\mathrm{E}_X\Big[\prod_{i\in I}X_i\prod_{j\in J}X_j\Big], \tag{1.81}$$

where X is a sequence of i.i.d. Bernoulli random variables with parameter $1/C$. Inserting (1.81) into (1.79), we finally get a product. By rearranging the expression that we obtain and reintegrating over X, we get as an upper bound on the second moment:

$$\widetilde{\mathrm{E}}^{\otimes 2}_{\beta,h}\Big[\exp\Big(\bar\beta^2\sum_{1\le k\le n}\widetilde\delta_k\widetilde\delta'_k\Big)\Big], \tag{1.82}$$

where (i) $\bar\beta$ is an explicit inverse temperature which is equivalent to $\sqrt{C}\beta$ as $\beta\downarrow 0$ (the constant C is the same as in (1.79)) and (ii) $\widetilde\tau$ and $\widetilde\tau'$ are two independent copies of a renewal process which, under $\widetilde{\mathrm{P}}_{\beta,h}$, starts at 0 and satisfies $\widetilde{\mathrm{P}}_{\beta,h}(\widetilde\tau_1=n)=e^\beta(\mathrm{P}_h\otimes\hat{\mathrm{P}})(\widetilde\tau_1=n)$ for $n\ge 1$ (recall (1.58)). This renewal process is transient when $h<h_c^{\mathrm{a}}(\beta)$ and recurrent when $h=h_c^{\mathrm{a}}(\beta)$. Remarkably, the expression in (1.82) is very similar to what we would obtain with i.i.d disorder and one can (almost) conclude from there as in the i.i.d. case. ■

1.4 Discussion and perspectives

(A) The value of the quenched critical exponent in the regime of relevant disorder is still unknown and seems to be a very challenging question, even in the case of i.i.d. disorder. Derrida and Retaux [57] introduced a simplified version of the hierarchical pinning model, which is formulated in terms of a recursive model. For this model they conjectured an infinite order phase transition in the relevant disorder regime, with a free energy vanishing like $\exp(-\mathrm{cst}/\sqrt{h-h_c(\beta)})$ as $h\downarrow h_c(\beta)$, and argued in favor of a similar singularity for the pinning model. This conjecture has been answered recently in [47] for the recursive model (in a slightly weaker form) and in [83] for an exactly solvable continuous version.

(B) The critical slope of the copolymer model. We also obtained in [19, Theorem 1.4] the weak-coupling limit of the quenched critical curve (the critical slope) for the copolymer model, in the case of i.i.d. disorder and $\alpha>1$. The partition function of the copolymer writes

$$Z^{\omega}_{n,\lambda,h}=\mathrm{E}\Big[\exp\Big(-2\lambda\sum_{1\le k\le n}(\omega_k+h)\Delta_k\Big)\Big], \tag{1.83}$$

where $(\Delta_k)_{k\geq 1}$ is a sequence of Bernoulli random variables which indicates whether the k-th monomer is below or above the interface. It is defined by $\Delta_k = \varepsilon_i$, where i is the unique positive integer such that $k \in (\tau_{i-1}, \tau_i]$ and $(\varepsilon_i)_{i\geq 1}$ is a sequence of i.i.d Bernoulli random variables with parameter $\frac{1}{2}$ which indicates whether an excursion of the polymer is below or above the interface. This model corresponds to a polymer in presence of two solvents separated by a flat interface, when the monomers may have different solvent affinities. Our result, which is the analogue of Theorem 1.5, writes (with notations similar to the pinning model):

$$\lim_{\lambda\downarrow 0} \frac{h_c^{\mathrm{a}}(\lambda) - h_c(\lambda)}{\lambda} = \frac{1}{2}\frac{\alpha}{1+\alpha}. \tag{1.84}$$

This answers a conjecture of Bolthausen, den Hollander and Opoku [29], who obtained the corresponding lower bound. The value of the critical slope has been extensively investigated and remains an open question in the case $\alpha \in (0,1)$, where bounds have been obtained and the value is known to be universal and to coincide with the corresponding slope in the *continuum polymer model*, see [28, 29, 33]. In the case of correlated Gaussian disorder and $\alpha > 1$, we proved in [23, Theorem 1.9] that (1.84) becomes

$$h_c^{\mathrm{a}}(\lambda) = \Upsilon_\infty \lambda, \qquad \lim_{\lambda\downarrow 0} \frac{h_c(\lambda)}{\lambda} = \max\Big\{\frac{\Upsilon_\infty}{1+\alpha}, \frac{1}{2}\frac{\Upsilon_\infty}{1+\alpha} + \frac{1}{2}\mathbf{C}_\rho^{\mathrm{cop}}\Big\}, \tag{1.85}$$

(recall Assumption 1.8) where

$$\mathbf{C}_\rho^{\mathrm{cop}} = \mathrm{E}\Big[\frac{1}{\mu} \sum_{1\leq n,m\leq \tau_1} \rho_{nm}\Big]. \tag{1.86}$$

Interestingly, the first term in the maximum corresponds to the Monthus bound [103] whereas the second term is the natural extension of the value found in the i.i.d. case.

(C) The Large Deviations approach. Cheliotis and den Hollander [45] attacked the problem of pinning with disorder from the point of view of Large Deviations Theory. Their approach relies on a quenched Large Deviations Principle (LDP) for words cut in a random sequence by a renewal process, see Birkner, Greven and den Hollander [25]. This LDP was extended in [54] to words cut in a correlated sequence, which could, in principle, be applied to pinning models with correlated disorder.

(D) Pinning model with Gaussian correlated disorder. Many parts are still missing regarding the issue of disorder relevance in this context. Let us stick, for simplicity, to the case of summable correlations of the form $\rho_k = (1+k)^{-a}$ $(a > 1)$ and sum up what we have obtained so far. When $a > 2$, correlations are doubly summable, $\nu_{\mathrm{a}}(\beta) = \nu_{\mathrm{hom}}$ and by the smoothing inequality, disorder is relevant w.r.t. to critical exponents as soon as $\alpha > \frac{1}{2}$, since then $\nu_q(\beta) \geq 2 > \nu_{\mathrm{a}}(\beta)$. We have also proved that disorder is relevant w.r.t. to critical points as soon as $\alpha > 1$, for all $a > 1$. According to the Weinrib-Halperin criterion, $\alpha = \frac{1}{2}$ should remain the borderline between irrelevance and relevance, hence what is still missing to confirm this prediction is a proof of irrelevance when $\alpha < \frac{1}{2}$ and relevance when $\alpha \in (\frac{1}{2}, 1)$ and $a \in (1,2)$. Regarding the annealed critical exponents, previous work on the hierarchical version of the model [24] suggests that $\nu_{\mathrm{a}}(\beta) = \nu_{\mathrm{hom}}$ as soon as $a > 2\min(\alpha, 1)$ and $\nu_{\mathrm{a}}(\beta) > \nu_{\mathrm{hom}}$ when $a < 2\min(\alpha, 1)$.

(E) Pinning model with renewal disorder. We finally collect here some remarks about the model presented in Section 1.3.2:
1. According to the Weinrib-Halperin criterion [141], which aims to generalize the Harris criterion, disorder should be relevant if $\nu < \frac{2}{\xi\wedge 1}$ (at least for small disorder) and irrelevant if $\nu > \frac{2}{\xi\wedge 1}$, where ν is the critical exponent of the pure (homogeneous) system and ξ is the correlation exponent of the environment. The application of this criterion to pinning models was introduced and discussed in [17, 24]. In our case, $\nu = (1/\alpha)\vee 1$, $\xi = \hat{\alpha} - 1$ (assuming that $\hat{\alpha} > 1$) and the

Harris criterion should not be changed if $\xi > 1$, i.e. $\hat{\alpha} > 2$, which is confirmed by Theorem 1.22 and Theorem 1.23. If $\hat{\alpha} \in (1,2)$, the criterion predicts that disorder is relevant (resp. irrelevant) if $\alpha > \frac{\hat{\alpha}-1}{2}$ (resp. $\alpha < \frac{\hat{\alpha}-1}{2}$). However, there is no clear evidence that this criterion gives the right prediction out of the Gaussian regime and it has actually been disproved in several examples [17,18].

2. The recent work of Caravenna, Sun and Zygouras [35, 36] has opened a new perspective on the issue of disorder relevance. Their approach consists in computing weak-coupling limit of the quenched partition functions, with randomness surviving in the limit. More precisely, they determine sequences of parameters in the Hamiltonian (the coupling constants h_n, β_n in our case) that converge to zero as the system size goes to infinity and such that the properly rescaled quenched partition function converges in distribution to a random limit, which is obtained in the form of a Wiener chaos expansion. In several instances, including the one of the pinning model in i.i.d. environment, it was shown that these conditions coincide with those of disorder relevance. Applying this approach to our model leads us to the following conjecture:

Conjecture 1.24. *Disorder is relevant for all $\beta > 0$ (in the sense of critical point shift) if*

$$\alpha > 1 - \frac{1}{\hat{\alpha} \wedge 2}, \tag{1.87}$$

in which case

$$\limsup_{\beta \downarrow 0} \frac{\log(h_c(\beta) - h_c^{\mathrm{a}}(\beta))}{\log \beta} = \frac{(\alpha \wedge 1)(\hat{\alpha} \wedge 2)}{1 - (\hat{\alpha} \wedge 2)(1 - (\alpha \wedge 1))}. \tag{1.88}$$

The reason for the term $1/\hat{\alpha}$ in place of the usual $1/2$ when $\hat{\alpha} \in (1,2)$ is that the partial sums of our disorder sequence are in the domain of attraction of an $\hat{\alpha}$-stable law rather than the standard normal distribution. More precisely, we have:

$$\frac{1}{n^{1/\hat{\alpha}}} \sum_{k=1}^{n} (\hat{\delta}_k - 1/\hat{\mu}) \longrightarrow \hat{\alpha}\text{-stable law}, \quad \text{as } n \to \infty, \quad \hat{\alpha} \in (1,2). \tag{1.89}$$

Therefore we expect that white noise is replaced by a Lévy noise in the weak-coupling limit of the quenched partition function. Note that (1.87) and (1.88) coincide with the case of i.i.d. disorder when $\hat{\alpha} > 2$, that is the summable correlation scenario. Finally, another fact that supports this conjecture is that the chaos expansion approach gives the right prediction for a pinning model in i.i.d. γ-stable environment ($1 < \gamma < 2$), which has been studied recently by Lacoin and Sohier [93]. There, it was proved that disorder is relevant (resp. irrelevant) if $\alpha > 1 - 1/\gamma$ (resp. $\alpha < 1 - 1/\gamma$), which is to be compared to our conjecture.

3. The picture that has emerged for the moment regarding disorder relevance for this model can be summed up in the following exponent diagram, see Figure 1.2.

- The blue area is where we have proved relevance for small β.
- In the region $\hat{\alpha} < 1$ we have relevance in the blue area because the quenched critical curve is trivially 0 while the annealed one is strictly negative.
- In the blue region with $\hat{\alpha} > 1$, we have relevance due to smoothing, see Theorem 1.22. We do not know yet whether the critical points differ but we conjecture that they do so (see Conjecture 1.24 above).
- In the yellow triangle we have irrelevance because there also the annealed critical curve is 0 for small β and the critical exponents agree.
- In the yellow part with $\hat{\alpha} > 2$, we have irrelevance due to Theorem 1.23.

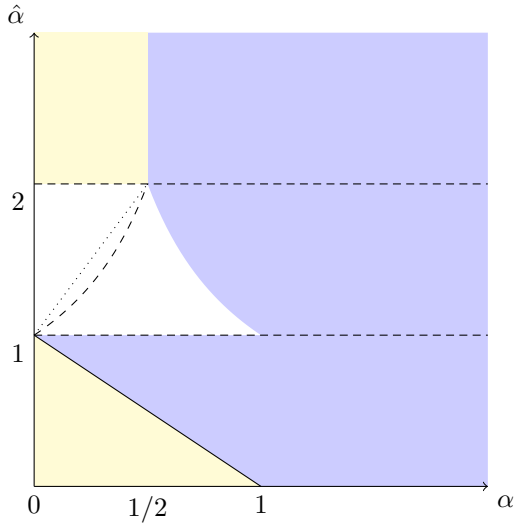

Figure 1.2: Disorder relevance/irrelevance in the exponent diagram.

- The dashed line marks the border of relevance/irrelevance according to the chaos expansion heuristics when $\alpha \in (0,1)$ and $\hat{\alpha} \in (1,2)$, see Conjecture 1.24.
- The dotted line marks the border of relevance/irrelevance according to the Harris-Weinrib-Helperin criterion when $\alpha \in (0,1)$ and $\hat{\alpha} \in (1,2)$, see Item 2 above.

4. Finally, let us mention the recent work of Alexander and Berger [3] who also consider a pinning model with disorder built out of a renewal sequence. Even if they may look similar, the model studied in [3] and the one considered here are actually different in spirit. Indeed, in [3] all the interactions up to the n-th renewal point of the disorder renewal (denoted here by $\hat{\tau}_n$) are taken into account, and the only parameter is the inverse temperature β (no pinning strength h). As a consequence, the results obtained therein are also quite different as for instance, the critical line deciding disorder relevance is at $\alpha + \hat{\alpha} = 1$. However, we do not exclude that the two models are related. For instance, the line $\alpha + \hat{\alpha} = 1$ also appears in Remark 1.17 above and, incidentally, in Proposition 1.20 (see also Figure 1.2).

Chapter 2

Pinning of a polymer with multiple interfaces

This chapter is based on a joint work with F. Simenhaus [114].

We consider $S = (S_n)_{n\in\mathbb{N}_0}$ a simple random walk on $\mathbb{Z}$. We recall that the increments $(S_n - S_{n-1})_{n\in\mathbb{N}}$ are independent and identically distributed (i.i.d.) random variables which are uniformly distributed on $\{-1, 1\}$, and we shall write P_x for the law of the walk started at $S_0 = x$, for $x \in \mathbb{Z}$, with the notational simplification $\mathrm{P}_0 = \mathrm{P}$. As in Chapter 1, we consider a directed polymer model, meaning that the n-th monomer of the polymer chain is located at site $(n, S_n) \in \mathbb{N} \times \mathbb{Z}$. The focus of the previous chapter was on the interaction between the polymer chain and a single defect line or interface. In this chapter, we shall investigate what happens when the polymer is in presence of several (actually an infinite number of) interfaces.

Let us draw the reader's attention on notation for a moment. For simplicity we mostly stick to the notation of the original paper [114], which means that the letter τ, which was used in Chapter 1 to denote the set of contacts points of the polymer with the interface, is now used to denote the set of heights where the interfaces are located. Hence, in order to avoid confusion we chose to treat the case of many interfaces in a different chapter, even though both Chapter 1 and the present chapter deal with the pinning effect.

This chapter is organized as follows: Section 2.1 presents a general overview of the problem while Section 2.2 is dedicated to the model studied in [114], where a particular random set of interfaces is considered. We discuss some perspectives in Section 2.3.

2.1 An overview of the problem

2.1.1 A $(1+1)$-directed polymer with multiple interfaces

Let us assume that the polymer chain is in presence of a countable number of interfaces, the heights of which are denoted by $\tau = (\tau_n)_{n\in\mathbb{N}_0}$. In other words, the n-th interface is the (discrete) horizontal line $\mathbb{N} \times \{\tau_n\}$. We further assume that the interfaces are all *equally repulsive*. Hence, the partition function of our system writes

$$Z_{n,\beta} = \mathrm{E}[\exp(-\beta\mathcal{H}_n)], \qquad \mathcal{H}_n = \sum_{k=1}^{n} 1_{\{S_k\in\tau\}}, \tag{2.1}$$

where $\mathcal{H}_n$ is the Hamiltonian and $\beta > 0$ is the inverse temperature, or repulsion strength. Whenever the polymer touches one of the interfaces it is penalized by a factor $e^{-\beta}$. The associated polymer measure writes

$$\frac{\mathrm{dP}_{n,\beta}}{\mathrm{dP}} = \frac{1}{Z_{n,\beta}} \exp\Big(-\beta \sum_{k=1}^{n} 1_{\{S_k \in \tau\}} \Big). \tag{2.2}$$

By removing the deterministic and linear term n to the Hamiltonian, we see that the situation is the same as that of neutral interfaces with an attractive region in-between them. Therefore, it is not a complete surprise that the presence of many *repulsive* interfaces may eventually lead to *pinning* (hence the name of the chapter), but we will see that the final answer to this question depends heavily on the distribution of τ. We refer to Chapter 1 and references within for an overview of the pinning phenomenon in the case of a single interface.

The set of monomers that touch the interfaces (contact points) shall be denoted by θ, that is (we assume w.l.o.g. that $0 \in \tau$):

$$\theta_0 = 0, \quad \theta_{n+1} = \inf\{k > \theta_n \colon S_k \in \tau\}, \quad n \in \mathbb{N}_0, \tag{2.3}$$

see Figure 2.1.

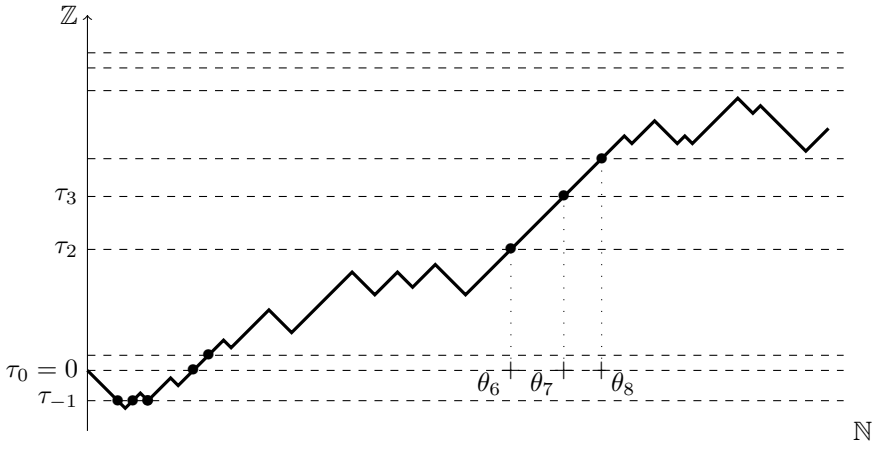

Figure 2.1: Example of a polymer among repulsive interfaces. The horizontal dashed lines correspond to the interfaces, the thick one to the polymer and the dots to the contact points.

2.1.2 A warm-up: the case of periodic interfaces

Let us consider for a moment the case of equally spaced interfaces, that is $\tau = t\mathbb{Z}$, where $t \in \mathbb{N}$ is fixed. This problem is equivalent to a polymer in a slit, that is a random walk wrapped on the torus $\mathbb{Z}/t\mathbb{Z}$ with repulsion at 0.

If we set $\beta = \infty$ this means that contacts with the interfaces are forbidden and the behaviour of the partition function in the limit of large n is then given by the classical *small-ball* probability:

$$Z_{n,\infty} = \mathrm{P}_1(S_k \in (0,t),\ 0 < k < n) = \exp(-[\varphi_\infty(t) + o(1)]n), \qquad n \to \infty \tag{2.4}$$

where

$$\varphi_\infty(t) = \frac{\pi^2}{2t^2} + \Theta\Big(\frac{1}{t^4}\Big), \qquad t \to \infty. \tag{2.5}$$

The first term in the expansion above corresponds to the first eigenvalue of (minus) the continuous Laplacian $(-\Delta)$ on the interval $(0,t)$ with Dirichlet boundary conditions.

The case $\beta \in (0,\infty)$ was considered for instance in [30,99,107] and taken up by Caravenna and Pétrélis [34], who refined the asymptotic expansion of the free energy by proving that

$$Z_{n,\beta} = \exp(-[\varphi_\beta(t) + o(1)]n), \qquad n \to \infty, \tag{2.6}$$

with

$$\varphi_\beta(t) = \frac{\pi^2}{2t^2}\Big[1 - \frac{4}{e^\beta - 1}\frac{1}{t} + o\Big(\frac{1}{t}\Big)\Big], \qquad t \to \infty. \tag{2.7}$$

Caravenna and Pétrélis [34] actually consider the case when the size of the slits t is allowed to depend on n, say $t = t_n = n^a$ where $a \in (0,1)$. The second order term in the expansion (2.7) indicates that the typical time spent in-between interfaces is of order t_n^3. Therefore, there should be a transition at $t_n^3 = n$, that is $a = 1/3$, and this is exactly what Caravenna and Pétrélis [34] prove. Let us summarize what they obtain:

- If $a < 1/3$, the polymer sees the interfaces a number of times of order $n/t_n^3 = n^{1-3a}$. In addition, the random walk seen at the subsequent times $(\theta_i)_{i\geq 1}$ should be diffusive (as the interfaces are periodically distributed, there is no favorite direction) hence S_n is of order $n^a\sqrt{n^{1-3a}} = n^{\frac{1-a}{2}}$.
- If $1/3 < a < 1/2$, the walk essentially sees one interface, that is S_n is of order n^a, and in this case it is actually proved that the polymer localizes into one slit after a $O(1)$ number of steps.
- If $a > 1/2$ then the polymer does not see the interfaces (except the one at the origin) and the polymer is essentially diffusive.

We refer to [34, Theorem 1.1] for precise statements.

In Section 2.2 we shall consider an environment of interfaces where the sizes of the slits are *fixed but random*. This direction of research was actually suggested in the monograph [53, Chap. 7]. We will see however that the size of the relevant slits does depend on the size of the polymer n, which explains why Caravenna and Pétrélis' work [34] is relevant to us.

2.1.3 Another viewpoint: a random walk among obstacles

Another standard formulation of the problem we consider in this chapter is that of a random walk surviving in a field of obstacles, which actually makes sense in any dimension $d \geq 1$. We now see n as a time parameter and S_n as the position of a random walker at time n. The set τ is interpreted as the set of positions of some obstacles and θ is the set of consecutive times when the walker meets any of those obstacles. Each time the walker meets an obstacle, it is killed with probability $1 - e^{-\beta}$ or survives with probability $e^{-\beta}$, independently from the past. The obstacles are called *soft* when $\beta \in (0,\infty)$, that is when there is a positive chance of survival, and *hard* when $\beta = \infty$, that is when any obstacle instantaneously kills the random walk. For a precise mathematical definition, we first enlarge the probability space so as to include a $\mathbb{N}$-valued geometric random variable $\mathcal{N}$ with success parameter $1 - e^{-\beta}$. This random variable plays the role of a clock that kills the walker after the latter meets the set of obstacles $\mathcal{N}$ times. In other terms, the killing time of the walker, denoted by σ, is defined by

$$\sigma = \theta_{\mathcal{N}}. \tag{2.8}$$

Note that our probability law now depends on the parameter β. We shall write P_x^β when we want to stress this dependence or omit the superscript when no confusion is possible. Again we may write P^β instead of P_0^β. We also point out that σ implicitly depends on τ through θ. By integrating out

$\mathcal{N}$ we see that the partition function in (2.1) coincides with the probability that the random walk survives in the field of obstacles τ at least up to time n, that is

$$Z_{n,\beta} = \mathrm{P}^{\beta}(\sigma > n), \qquad n \in \mathbb{N}. \tag{2.9}$$

We stress again that $Z_{n,\beta}$ is a function of the environment of obstacles $(\tau_n)_{n\geq 0}$.

Let us give here a brief account of what is known for the random walk among random obstacles, the two main (related) points of interest being the asymptotic behaviour of the survival probability and the path properties of the random walk conditioned to survive, in the limit of large n. For simplicity, we restrict to the case of random *Bernoulli obstacles*, which is at the same time the most natural and the most studied case. This corresponds to defining, independently from the random walk, a sequence of i.i.d. Bernoulli random variables $\omega = (\omega_i)_{i\in\mathbb{Z}}$ with parameter $p \in (0,1)$ and setting

$$\tau = \{i \in \mathbb{Z} \colon \omega_i = 1\}. \tag{2.10}$$

We shall use $\mathbb{P}$ and $\mathbb{E}$ for probability and expectation with respect to these Bernoulli random variables. One may then consider the *quenched* survival probability (the positions of the obstacles are frozen) or the *annealed* survival probability (the positions of the obstacles are averaged out).

The case of annealed and hard ($\beta = \infty$) Bernoulli obstacles corresponds to the random walk penalized by the cardinality of its range, that is

$$\mathbb{E}(Z_{n,\infty}) = \mathrm{E}[(1-p)^{|R_n|}], \qquad \text{where} \quad R_n := \{S_1, \dots, S_n\}. \tag{2.11}$$

Donsker and Varadhan [63] first proved that

$$\mathbb{E}(Z_{n,\infty}) = \exp\Big(- [c_{\mathrm{a}} + o(1)] n^{\frac{d}{d+2}} \Big), \qquad n \to \infty, \tag{2.12}$$

where $c_{\mathrm{a}} = c_{\mathrm{a}}(p,d)$ is an explicit constant obtained as the minimiser of a variational problem. Sznitman [124, 130] introduced the *method of enlargement of obstacles* (MEO) and obtained thereby a direct analogue of (2.12) in the context of a Brownian motion among Poisson obtacles (BMPO). Antal [7] again derives (2.12) for the random walk in random obstacles (RWRO) in discrete space and continuous-time, this time via the MEO. Let us now turn to path properties. Bolthausen [27] gave a localization result for the continuous-time RWRO in $d = 2$. In short, he proved that the random walk folds into a Euclidian ball with a radius $\rho_n \approx n^{1/4}$ (the precise constant is determined by the variational problem that is behind the value of c_{a}) and conjectures that the same should hold in all dimensions, with $\rho_n \approx n^{1/(d+2)}$. In parallel, Sznitman [126] proved an analogous result for $d = 2$ in the context of BMPO. The case $d \geq 3$ was taken up in the same context (BMPO) by Povel [118] via the MEO. As pointed out in this reference, localization results for $d = 1$ should follow from estimates (0.7) and Remark 1.4 in Sznitman [125]. Povel also claims that it is possible to extend Sznitman's arguments in [125] to derive a weak convergence result for the BMPO conditioned on survival. The one-dimensional version of weak convergence for BMPO was obtained by Schmock [122]. More recently, Berestycki and Cerf [16] proved Bolthausen's conjecture for the discrete RWRO in $d \geq 3$. In an independent and almost simultaneous work, Ding, Fukushima, Sun and Xu [60] proved the same kind of localization results, with an upper bound on the boundary of the range (see also [58] for a biased random walk version).

In the case of quenched Bernoulli obstacles, we have

$$Z_{n,\beta} = \exp\Big(- [c + o(1)] \frac{n}{(\log n)^{2/d}} \Big), \qquad n \to \infty \tag{2.13}$$

where the $o(1)$ holds $\mathbb{P}$-as and $c = c(p,d)$ is an explicit constant obtained as the minimiser of a variational problem. This was proved in the BMPO setting by Sznitman [127], see also [129] and

Fukushima [71]. Antal [7] claims that (2.13) follows from his work, where he uses the MEO in a discrete setting. Sznitman [128,130] proved a confinement property for the BMPO (with path localization in $d = 1$). These results have been recently improved by Ding and Xu [61,62] who proved localization of the random walk in a single Euclidian ball of volume asymptotically equivalent to $d \log_{1/p} n$, in the discrete setting, see also the even more recent improvements by Ding, Fukushima, Sun and Xu [59].

Let us end this section with a reference to the classical monograph by Sznitman [130] on random motions in random media. Chapter 6 therein is of particular interest to us as it highlights the link between random motions in random media and directed polymers in the presence of columnar defects, and presents the concept of pinning effect of quenched path measures.

2.1.4 (Yet) another viewpoint: the parabolic Anderson model

Our model actually fits into a broader class of models, for which the partition function writes

$$Z_n^V = \sum_{x\in\mathbb{Z}^d} Z_{n,x}^V, \qquad Z_{n,x}^V := \mathrm{E}\Big[\exp\Big(\sum_{1\le k\le n} V(S_k)\Big)1_{\{S_n=x\}}\Big], \tag{2.14}$$

where $V = (V(x))_{x\in\mathbb{Z}^d}$ is a potential, and of course (2.1) is simply Z_n^V with $V(x) = -\beta 1_{\{x\in\tau\}}$. The partition function constrained on the endpoint satisfies the equation

$$Z_{n+1,x}^V - Z_{n,x}^V = e^{V(x)}(\Delta Z_n^V)_x + (e^{V(x)} - 1)Z_{n,x}^V, \tag{2.15}$$

where Δ is the discrete Laplacian. This equation actually corresponds to a discrete-time analogue of the parabolic Anderson model (PAM) with potential V, that is the equation

$$\partial_t u = \Delta u + Vu, \qquad u\colon (t,x) \in [0,\infty)\times\mathbb{Z}^d \mapsto \mathbb{R}. \tag{2.16}$$

There is a rich literature and intense activity around the PAM. We refer to König [90] for a recent monograph dedicated to this topic. Note that the potential is usually chosen as a sequence of random variables that are independent in the space parameter and constant in time. In contrast, the set of interfaces we shall consider in Section 2.2 leads to a potential with long-range spatial correlations, that is one of the research direction suggested in [90, Section 7.2]. Such models have also been considered by physicists. Let us mention for instance the Aubry-André model [15] in which the potential, defined by an irrational rotation on the torus, is ergodic but not even mixing. Interestingly, this model is known for exhibiting a phase transition even in dimension one. For a review of potentials in discrete and continuous space, we refer to [90, Section 1.5].

2.2 Renewal interfaces

In what follows we allow ourselves to use the terminology of the polymer point of view or that of the random walk in random obstacles (RWRO). Hence, the trajectory of the random walk is the same as the configuration of the polymer and the positions of the obstacles are the heights of the interfaces. From now on, for convenience, we consider random walks that do not visit the negative half-line $\mathbb{Z}^- = -\mathbb{N}_0$. This extra condition allows us to consider interfaces that are indexed by $\mathbb{N}_0$ instead of $\mathbb{Z}$ and does not hide anything deep nor change the main idea of our result. Hence the partition function now writes

$$Z_{n,\beta} = \mathrm{E}[\exp(-\beta\mathcal{H}_n)1_{\{S_k>0,\,0<k\le n\}}], \qquad \mathcal{H}_n = \sum_{k=1}^{n} 1_{\{S_k\in\tau\}}. \tag{2.17}$$

This simply amounts to putting an *impenetrable wall* on the negative half-plane. In the language of RWRO, this means that we slightly change the definition of the killing time to

$$\sigma = \theta_{\mathcal{N}} \wedge H_{\mathbb{Z}^-} \qquad \text{and} \qquad Z_{n,\beta} = \mathrm{P}^{\beta}(\sigma > n), \tag{2.18}$$

where $H_{\mathbb{Z}^-} = \inf\{n \geq 1 \colon S_n \leq 0\}$ is the hitting time of the negative half-line.

2.2.1 Definition of the model

We now put a probability measure $\mathbb{P}$ on the heights of the interfaces that is different from the Bernoulli distribution presented in Section 2.1.3. Let us denote by $T_k = \tau_k - \tau_{k-1}$, for $k \in \mathbb{N}$, the increments of τ, that is the size of the intervals between two consecutive interfaces, which we call *gaps* (see Figure 2.2). We assume that, under $\mathbb{P}$, τ is a discrete renewal process, meaning that the (T_k)'s are i.i.d. $\mathbb{N}$-valued random variables. We further assume that $\tau_0 = 0$ and that the increments have a power-tail distribution:

$$\mathbb{P}(T_1 = n) \sim c_\tau \; n^{-(1+\gamma)}, \qquad \gamma > 0, \quad n \to \infty, \tag{2.19}$$

where c_τ is a positive constant.

Remark 2.1. *When $\mathbb{P}$ is the Bernoulli distribution of Section 2.1.3, τ is also a renewal process, with the notable difference that the increment distribution is geometric: $\mathbb{P}(T_1 = n) = p(1-p)^{n-1}$ for $n \geq 1$. We will see that going from an exponential tail to a polynomial tail induces some drastic changes to the asymptotic behaviour of the partition function.*

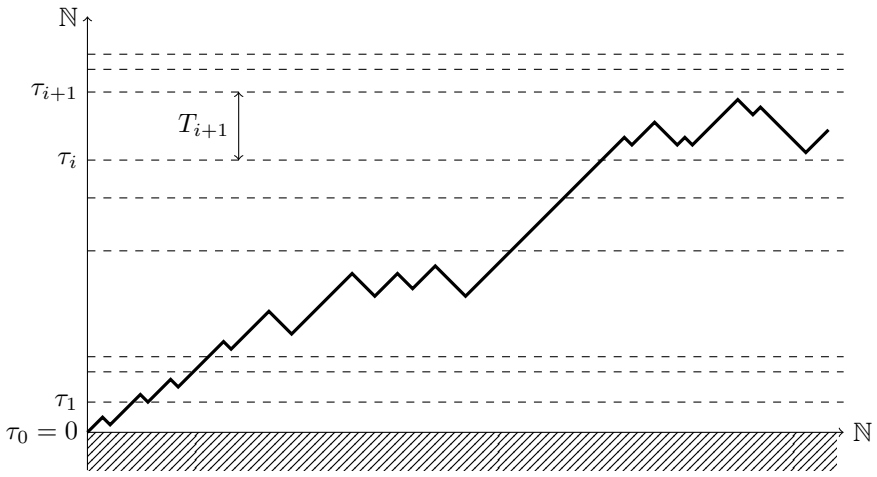

Figure 2.2: Example of a polymer among repulsive interfaces. The dashed lines correspond to the interfaces, the thick one to the polymer and the shaded area to the hard wall.

2.2.2 The quenched Lyapunov exponent

Before we can state our main theorem, we need to introduce the *quenched Lyapunov exponent*. To this end let us define, for $x \in \mathbb{Z}$, the hitting time of x,

$$H_x = \inf\{n \geq 1 \colon S_n = x\}, \tag{2.20}$$

and for $\ell \in \mathbb{N}$, the random variable (with respect to τ)

$$\lambda(\ell, \beta) = -\frac{1}{\ell} \log \mathrm{E}\Big(\exp\Big(-\beta \sum_{1 \leq k \leq H_{\tau_\ell}} 1_{\{S_k \in \tau\}} \Big) 1_{\{S_k > 0,\ 0 < k \leq n\}} \Big). \tag{2.21}$$

In the language of RWRO, the expectation above is the probability that the walk restricted to the positive half-line survives at least until it reaches the ℓ-th obstacle:

$$\lambda(\ell,\beta) = -\frac{1}{\ell}\log \mathrm{P}^{\beta}(H_{\tau_\ell} < \sigma). \tag{2.22}$$

Proposition 2.2. *For all $\beta > 0$ there exists a positive constant $\lambda(\beta) = \lambda(\beta,\mathbb{P})$ such that, $\mathbb{P}$-a.s. and in $L_1(\mathbb{P})$,*

$$\lim_{\ell\to\infty} \lambda(\ell,\beta) = \lambda(\beta), \tag{2.23}$$

with

$$0 \le \lambda(\beta) - \beta \le \mathbb{E}(\log T_1) + \log 2. \tag{2.24}$$

Moreover, the function $\beta \mapsto \lambda(\beta)$ is concave and continuous on $\mathbb{R}_+^*$.

Note that $\log T_1$ is integrable because of our assumption in (2.19). The limit $\lambda(\beta)$ is called the *quenched Lyapunov exponent* and corresponds to the asymptotic cost per new interface or obstacle visited. The existence of this limit is obtained via a standard application of Kingman's sub-additive ergodic theorem (see Theorem 7.4.1 in [64]). We refer to the original paper [114] for a complete proof.

2.2.3 Main result: limiting free energy and variational problem

We are now ready to state our main result. From now on we set

$$N = N(n) = n^{\frac{\gamma}{\gamma+2}}, \qquad \mathrm{F}_n(\beta) = -\frac{1}{N}\log Z_{n,\beta}, \qquad n \ge 1. \tag{2.25}$$

Theorem 2.3. *The sequence of random variables $(\mathrm{F}_n(\beta))_{n\in\mathbb{N}}$ converges in $\mathbb{P}$-distribution to the random variable*

$$\mathrm{F}(\beta) := \inf_{(x,y)\in\Pi}\Big\{\lambda(\beta)x + \frac{\pi^2}{2y^2}\Big\}, \tag{2.26}$$

where Π is a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}_*^+$ with intensity $\mathrm{d}x \otimes c_\tau\gamma\ y^{-(1+\gamma)}\mathrm{d}y$.

In particular, the limit is universal as it only only depends on c_τ and γ. With a slight abuse of notation we have written $(x,y) \in \Pi$ to mean that (x,y) is in the support of the point measure Π. As a slight but not substantial improvement of this result, we also establish in [114] a functional version of this theorem.

2.2.4 Sketch of the proof

In this section we try to convey the main ideas behind the proof of Theorem 2.3.

Choice of scaling.

First of all let us explain, at a heuristic level, the choice of the normalization in (2.25). The argument we use is usually referred to as a *Flory argument*. We assume that at a large time n the walk has visited at most N obstacles and has remained confined in the largest visible gap, and we find the value of N with the best energy-entropy balance. By basic extreme-value theory, the size of the largest visible gap is of order $N^{1/\gamma}$, and by a standard small-ball estimate (see (2.4)-(2.5) above) the entropic cost of being confined in that gap during time n is of order $nN^{-2/\gamma}$. Also, we learn from Proposition 2.2 that the cost of crossing N obstacles is of order $\lambda(\beta)N$. By balancing out these two costs, one finds the optimal choice $N = n^{\frac{\gamma}{\gamma+2}}$.

Limiting variational problem.

Let us now explain how the Poisson point process arises. To this end we define

$$\begin{aligned} (X_{i,N}, Y_{i,N}) &:= \Big(\frac{i}{N}, \frac{T_{i+1}}{N^{1/\gamma}}\Big) \qquad \text{for all } i \geq 1, \\ \text{and} \qquad \Pi_N &= \sum_{i=1}^{\infty} \delta_{(X_{i,N}, Y_{i,N})}, \end{aligned} \tag{2.27}$$

where δ_x is the Dirac mass at x. We observe that Π_N is a (random) Radon point measure on $E = \mathbb{R}^+ \times \mathbb{R}^+_*$ and we shall denote by $M_p(E)$ the space of such measures. It turns out that Π_N converges weakly (for the topology of vague convergence) to the Poisson point process Π introduced in Theorem 2.3. This follows from [121, Proposition 3.21 p. 154] and the fact that for all $y > 0$, the sequence $(n\mathbb{P}(T_1 > yn^{1/\gamma}))_{n\geq 1}$ converges to $c_\tau y^{-\gamma}$ when n goes to infinity. Let us now fix $i \in \mathbb{N}$ and consider the following two-step strategy:

(i) the polymer reaches the i-th interface;

(ii) the polymer stays confined in the interval (τ_i, τ_{i+1}) for the remaining time, which we assume to be $n - o(n)$.

By Proposition 2.2, the cost of (i) is roughly $\lambda(\beta)i$ provided i is large, and by (2.4)-(2.5), the cost of (ii) is roughly $(\pi^2 n)/(2T_{i+1}^2)$. Hence, since $N = n^{\frac{\gamma}{\gamma+2}}$, the total contribution of this strategy to the free energy is

$$\lambda(\beta)i + \frac{\pi^2 n}{2T_{i+1}^2} = \psi_\beta(X_{i,N}, Y_{i,N})N, \qquad \text{where} \quad \psi_\beta(x,y) = \lambda(\beta)x + \frac{\pi^2}{2y^2}. \tag{2.28}$$

Therefore, if we assume that this strategy is the relevant one, the free energy $\mathrm{F}_n(\beta)$ should be well approximated by

$$\min_{(x,y)\in\Pi_N} \psi_\beta(x,y), \tag{2.29}$$

which should hopefully converge, in the limit of large n (or N) to

$$\min_{(x,y)\in\Pi} \psi_\beta(x,y), \tag{2.30}$$

that is the right-hand side in (2.26). Unfortunately, the map

$$\mathcal{E} \in M_p(E) \mapsto \min_{(x,y)\in\mathcal{E}} \psi_\beta(x,y) \tag{2.31}$$

appears not to be vaguely continuous and for this reason we have to go through a number of approximations by first restricting the minimum to a compact set K. We refer to [114] for details.

Lower bound on the partition function.

One can almost directly derive a rigorous lower bound for the partition function out of this heuristic. The only thing we have neglected in the first part of the strategy (see item (i) above) is the time it takes for the random walk to reach the gap (τ_i, τ_{i+1}). However, as the confinement probability

$$\mathrm{P}_{\tau_i}(S_k \in (\tau_i, \tau_{i+1}), 0 < k \leq j) \tag{2.32}$$

decreases with j, we get by applying the Markov property at time H_{τ_i}:

$$Z_{n,\beta} \geq \mathrm{P}(H_{\tau_i} \leq n \wedge \sigma)\mathrm{P}_{\tau_i}(S_k \in (\tau_i, \tau_{i+1}), 0 < k \leq n). \tag{2.33}$$

The second probability is dealt with a sharper version of (2.4). It now remains to show that the constraint $\{H_{\tau_i} \le n\}$ may be removed in the first probability, in order to make the quantity $\lambda(i,\beta) \approx \lambda(\beta) i$ appear. This is done by proving that for typical interfaces and for the values of i considered, we have

$$\mathrm{P}(H_{\tau_i} \le n | H_{\tau_i} \le \sigma) \ge \exp(-o(N)). \tag{2.34}$$

Upper bound on the partition function.

Let us sketch here the proof of the upper bound, which is more challenging, as it is often the case for this kind of models. The difficulty is to prove that there is no better strategy for the polymer than the one previously described. In words, we prove that once the optimal gap has been selected (see (2.29)) there is no contribution on the free energy from the *near-optimal* gaps, at least at first order. The basic idea is a decomposition of the environment of interfaces according to *record gaps*. The records are defined as follows:

$$i(0) = 0, \qquad i(k) = \inf\{i > i(k-1) \colon T_{i+1} > T_{i(k-1)+1}\}, \qquad k \in \mathbb{N}, \tag{2.35}$$

and

$$T_k^* = T_{i(k)+1}, \qquad k \in \mathbb{N}_0. \tag{2.36}$$

Hence, $i(k)$ is the label of the k-th record and T_k^* is the corresponding gap size. Letting $H_k^* = H_{\tau_{i(k)}}$ be the hitting time of the k-th record gap, we decompose the partition function according to the furthest record gap reached by the random walk:

$$Z_{n,\beta} = \sum_{k \ge 0} Z_{n,\beta}^{(k)}, \qquad \text{where} \quad Z_{n,\beta}^{(k)} := Z_{n,\beta}(H_k^* \le n < H_{k+1}^*). \tag{2.37}$$

Thanks to our lower bound, we may already dismiss the records that are too far from the origin (they are too costly to reach) or too close from the origin (they are typically thin hence the entropic cost of confinement is high). This writes, for $\varepsilon > 0$ arbitrarily small and with $\mathbb{P}$-probability close to one,

$$Z_{n,\beta} \le 2 \sum_{k \colon \varepsilon N \le i(k) \le (1/\varepsilon) N} Z_{n,\beta}^{(k)}, \tag{2.38}$$

and we observe that the value of i that optimises (2.29) corresponds to one of those $i(k)$'s with large $\mathbb{P}$-probability, provided ε is small enough. Since we can control the number of records in the window $[\varepsilon N, (1/\varepsilon)N]$, the next step is to give an upper bound on each $Z_n^{(k)}$. The bound we aim for can be roughly written as

$$Z_{n,\beta}^{(k)} \lesssim \exp\Big\{ - \Big[\lambda(\beta) i(k) + \frac{\pi^2 n}{2(T_k^*)^2} \Big] \Big\}, \tag{2.39}$$

up to some prefactor which is $\exp(o(N))$. The first term in the sum corresponds to the cost of reaching the k-th record (Step (i)) while the second term corresponds to the confinement part (Step (ii)). In the remaining part of the proof, we fix k such that $\varepsilon N \le i(k) \le (1/\varepsilon)N$ and we let

$$\phi = \frac{\pi^2}{2(T_k^*)^2}, \tag{2.40}$$

which is the cost of confinement in the k-th record gap $(\tau_{i(k)}, \tau_{i(k)+1})$, per unit of time. We observe that, according to the upper bound we aim for, the cost ϕ is felt on the interval $[0, n]$ (up to some negligible part), which somehow indicates that the walk spends almost all its time confined in the record gap (even though at this point we are not trying yet to obtain path statements). This fact is not completely obvious as one could imagine, for instance, that it might be entropically favorable for the walk to linger in some gaps the sizes of which are relatively close to the record gap size.

However, we prove that this is not the case, or at least, that this is not felt at the level of the free energy.

Let us explain now how we control $Z_{n,\beta}^{(k)} \exp(\phi n)$. By decomposing the partition function according to the time it takes for the polymer to reach the record gap, we see that it suffices to control two parts:

$$\begin{aligned} \text{(I)} &= \mathrm{P}^{\beta}(\sigma > m, H_k^* = m) \exp(\phi m), \\ \text{(II)} &= \mathrm{P}_{\tau_{i(k)}}(\sigma \wedge H_{k+1}^* > n - m) \exp(\phi(n-m)). \end{aligned} \tag{2.41}$$

Control on (I). We first use a Chebychev bound:

$$\text{(I)} \leq \mathrm{E}^{\beta}\Big[\exp(\phi H_k^*) 1_{\{H_k^* < \sigma\}} \Big]. \tag{2.42}$$

To estimate this (truncated) moment generating function, we use an auxiliary process $X = (X_n)_{n \geq 0}$ that is uniquely defined by the relation (recall (2.3))

$$\tau_{X_n} = S_{\theta_n}, \qquad n \in \mathbb{N}_0. \tag{2.43}$$

In words, this proces indicates the labels of the interfaces that are visited by the random walk, in chronological order. By letting ζ_0 and ζ_k^* be the respective hitting times of 0 and $i(k)$ by X, and by conditioning on X, we get

$$\mathrm{E}^{\beta}\Big[\exp(\phi H_k^*) 1_{\{H_k^* < \sigma\}} \Big] = \mathrm{E}^{\beta}\Big[\prod_{1 \leq i \leq \zeta_k^*} \mathrm{E}(\exp(\phi(\theta_i - \theta_{i-1})) | X_{i-1}, X_i) 1_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \Big]. \tag{2.44}$$

To rewrite this expression in a nicer way, we define the (truncated) moment generating functions

$$Q_{ij}(\phi) := \mathrm{E}_{\tau_i}(e^{\phi \theta_1} 1_{\{X_1 = j\}}) = \mathrm{E}_{\tau_i}(e^{\phi \theta_1} 1_{\{S_{\theta_1} = \tau_j\}}), \qquad i, j \in \mathbb{N}_0, \tag{2.45}$$

which are actually zero unless $|i - j| \leq 1$. By integrating out the geometric random variable $\mathcal{N}$, we obtain

$$\mathrm{E}^{\beta}\Big[\exp(\phi H_k^*) 1_{\{H_k^* < \sigma\}} \Big] = \mathrm{E}\Big[\prod_{1 \leq i \leq \zeta_k^*} e^{-\beta} \frac{Q_{X_{i-1}, X_i}(\phi)}{Q_{X_{i-1}, X_i}(0)} 1_{\{\zeta_k^* < \zeta_0\}} \Big]. \tag{2.46}$$

The next step is to control the ratio

$$R_i := \frac{Q_{X_{i-1}, X_i}(\phi)}{Q_{X_{i-1}, X_i}(0)}. \tag{2.47}$$

Explicit formulas are available for the moment generating functions in (2.45), by which we get a rather precise control on the R_i's. It turns out that the control on the ratio depends on whether the gap that lies between $\tau_{X_{i-1}}$ and τ_{X_i} is *small* or *large* (if $X_{i-1} = X_i$ we consider the largest gap adjacent to τ_{X_i}). Let us clarify this dichotomy: first, we fix a threshold $\alpha \in (0, 1)$ and claim that a gap of size T is *large* if $T \geq \alpha T_k^*$, else it is *small*. By tuning α appropriately we find that

$$R_i \leq \begin{cases} \exp(\varepsilon) & \text{if the gap between } \tau_{X_{i-1}} \text{ and } \tau_{X_i} \text{ is small,} \\ f_k & \text{if the gap between } \tau_{X_{i-1}} \text{ and } \tau_{X_i} \text{ is large,} \end{cases} \tag{2.48}$$

where f_k is a positive quantity that depends on the ratio between the first and second record gaps on the portion of space considered:

$$f_k = f(T_{k-1}^*/T_k^*), \tag{2.49}$$

with $f(x)$ diverging as $x \to 1$. Again, we see that the large gaps might be problematic. Let us set

$$\mathcal{I} := \{1 \leq i \leq \zeta_k^* \colon \text{the gap between } \tau_{X_{i-1}} \text{ and } \tau_{X_i} \text{ is large}\}. \tag{2.50}$$

From what precedes, we get as an upper bound

$$(\mathrm{I}) \le \mathrm{E}^{\beta-\varepsilon}(f_k^{|\mathcal{I}|}1_{\{\zeta_k^*<\zeta_0\wedge\mathcal{N}\}}), \tag{2.51}$$

that is, each visit to an interface is penalized by $\exp(-[\beta-\varepsilon])$ while each crossing of a large gap gets a (possibly large) reward f_k. However, each crossing of a large gap also has a probabilistic cost and the last key step is to prove that, in some sense and with large $\mathbb{P}$-probability, this cost is not compensated by the reward f_k. We refer to the original paper [114] for details on this part of the argument. This means, in short, that large gaps are crossed essentially once, so that

$$(\mathrm{I}) \lesssim f_k^L \mathrm{P}(\zeta_k^* < \zeta_0 \wedge \mathcal{N}), \qquad L := \#\{1 \le i \le i(k) \colon T_i \ge \alpha T_k^*\}. \tag{2.52}$$

We finally prove that the (random) prefactor f_k^L is smaller than $\exp(\sqrt{N})$ with large $\mathbb{P}$-probability and use Proposition 2.2 to get

$$(\mathrm{I}) \lesssim \exp(-\lambda(\beta)i(k) + o(N)). \tag{2.53}$$

Control on (II). The probability in (II) corresponds to a confinement of the polymer inbetween levels 0 and $\tau_{i(k+1)}$. We prove that this probability is dominated by that of confinement in the largest gap available, that is the k-th record, which has size T_k^*. This can be done in two ways. The first way is to extend the results in (2.6)-(2.7) (for an environment with only one gap size) to an environment with a periodic pattern of gap sizes, via the tools of Markov renewal theory. The second way is to use a coupling argument and prove that the polymer is more penalized by the interfaces in $(0, \tau_{i(k+1)})$ than by interfaces equally spaced by T_k^*. We obtain thereby that $(\mathrm{II}) \le \exp(o(N))$, which, in view of (2.53), completes the proof.

2.2.5 Discussion

We collect here a number of comments and remarks about our result.

1. We have not considered the annealed partition function, which decreases at most polynomially fast since τ might leave $(0, n]$ free from interfaces:

$$\mathbb{E}(Z_{n,\beta}) \ge \tfrac{1}{2}\mathrm{P}_1(H_0 \ge n)\mathbb{P}(\tau_1 > n) \sim (\mathrm{cst})n^{-(\gamma+\frac{1}{2})}, \tag{2.54}$$

as $n \to \infty$, and we end up with a behaviour that is completely different from that of the quenched partition function.

2. Note that we cannot hope for better than weak convergence. Indeed, if F_n would converge to F almost-surely, then F would be measurable with respect to the tail σ-algebra of the family $(T_i)_{i\in\mathbb{N}}$. As the increments are independent, the latter is trivial due to the $0-1$ law, and F would be deterministic.

3. In the case $\gamma \le 1$, the variational formula in (2.26) admits an alternative representation in terms of a subordinator, which reads

$$\mathrm{F}(\beta) = \inf_{t\ge 0}\Big\{\lambda(\beta)t + \frac{\pi^2}{2(\Delta\mathcal{S}_t)^2}\Big\}, \tag{2.55}$$

where $(\mathcal{S}_t)_{t\ge0}$ is a γ-stable subordinator and $\Delta\mathcal{S}_t = \mathcal{S}_t - \mathcal{S}_t^- = \mathcal{S}_t - \lim_{u\to t^-}\mathcal{S}_u$.

4. We can compute explicitly the tail distribution function of the limiting law in Theorem 2.3:

$$\mathbb{P}(\mathrm{F}(\beta) \ge u) = \exp\Big(-\frac{c_\tau}{\lambda(\beta)\pi^\gamma(\gamma+2)}(2u)^{1+\frac{\gamma}{2}}\Big), \qquad u \ge 0. \tag{2.56}$$

See [114] for details.

5. The case $\gamma = 0$ is left open. In this case, a gap distribution of the form (2.19) is no longer appropriate and one should instead assume that $\mathbb{P}(T_1 = n) \sim L(n)/n$, where L is a non-negative slowly varying function such that $\sum L(n)/n$ is finite. Complications may arise at two levels : (i) the normalization of $\max_{1\le i\le n} T_i$, that we use to guess the value of N, and (ii) the integrability of $\log T_1$, that we use in Proposition 2.2. For instance, if $L(n) = (\log n)^{-2}$ then $\mathbb{E}(\log T_1) = \infty$ and $\max_{1\le i\le n} T_i$ has a completely different type of renormalization since, as one can readily show, $(1/\sqrt{n}) \log \max_{1\le i\le n} T_i$ converges to a non-trivial probability law with cumulative distribution function $x \mapsto \exp(-x^{-2})1_{\{x>0\}}$, as $n \to \infty$.

6. We state without proof an alternative expression for $\lambda(\beta)$ based on ergodic theory considerations. To this end, let $\widetilde{\tau}$ be an independent copy of τ, as defined in Section 2.2. Suppose that the random walk is now free to visit $\mathbb{Z}^-$ but is killed by the set $-\widetilde{\tau}$ (note the minus sign), with the same probability $1 - \exp(-\beta)$, and denote by $\widetilde{\sigma}$ the corresponding killing time. Then,

$$\lambda(\beta) = -\mathbb{E}\widetilde{\mathbb{E}} \log \mathrm{P}^\beta(H_{\tau_1} < \widetilde{\sigma}). \tag{2.57}$$

Assuming this last equality, we could readily prove using the dominated convergence theorem that λ is also continuous at 0.

7. Equation (2.24) does not give much information about the behaviour of $\lambda(\beta)$ at 0, which remains an open question. We expect however that $\beta = o(\lambda(\beta))$ as $\beta \to 0$ and we now explain why. To this end, recall (2.57) and the related notations above. By integrating over $\mathcal{N}$ and differentiating in β we obtain

$$\lim_{\beta\to 0} \lambda'(\beta) = \widetilde{\mathbb{E}}\mathbb{E}\mathrm{E}\Big(\sum_{k=1}^{H_{\tau_1}} 1_{\{S_k\in-\widetilde{\tau}\}}\Big), \tag{2.58}$$

that we expect to be infinite. Indeed, by first restricting the walk to make its first step to the left and then using the symmetry of the random walk,

$$\widetilde{\mathbb{E}}\mathbb{E}\mathrm{E}\Big(\sum_{k=1}^{H_{\tau_1}} 1_{\{S_k\in-\widetilde{\tau}\}}\Big) \ge \frac{1}{2}\widetilde{\mathbb{E}}\mathrm{E}_{-1}\Big(\sum_{k=1}^{H_0} 1_{\{S_k\in-\widetilde{\tau}\}}\Big) = \frac{1}{2}\widetilde{\mathbb{E}}\mathrm{E}_1\Big(\sum_{k=1}^{H_0} 1_{\{S_k\in\widetilde{\tau}\}}\Big). \tag{2.59}$$

We now interchange integrals and use the Renewal Theorem to obtain, at least for $\gamma \neq 1$,

$$\lim_{\beta\to 0} \lambda'(\beta) \ge \frac{1}{2}\mathrm{E}_1\Big(\sum_{k=1}^{H_0} \widetilde{\mathbb{P}}(S_k \in \widetilde{\tau})\Big) \ge \frac{C}{2}\mathrm{E}_1\Big(\sum_{k=1}^{H_0} (1+S_k)^{(\gamma-1)\wedge 0}\Big). \tag{2.60}$$

Since, by Ray-Knight's theorem, the mean number of visits to $x \in \mathbb{N}_0$ between time 1 and H_0 equals 1 under P_1, we get

$$\lim_{\beta\to 0} \lambda'(\beta) \ge C \sum_{x\in\mathbb{N}_0} (1+x)^{(\gamma-1)\wedge 0} = \infty. \tag{2.61}$$

8. Let us stress that the scaling $n^{\gamma/(\gamma+2)}$ that appears in Theorem 2.3 is different from the scaling of the PAM in a bounded i.i.d. potential. In this case [90, Example 5.10] states that the correct scaling is n up to a logarithmic correction (see also (2.13)). Hence we are in a case where the space correlations of the potential have a drastic effect on the asymptotic behaviour of the survival probability.

2.3 Perspectives

In this section we briefly discuss two natural extensions of our work in Section 2.2: path properties and higher-dimensional models.

2.3.1 Renewal interfaces: path localization

We found in Theorem 2.3 the asymptotic behaviour of the partition function for the model of renewal interfaces. The proof we sketched suggests that the polymer follows a *confinement strategy* inside a gap which solves the variational problem in (2.26). However, path properties do not directly follow from the derivation of the leading term in the asymptotics of the partition function and new arguments are necessary. This task has been taken up in a joint work with F. Simenhaus and we present here some early results extracted from our work in progress.

First of all, let us recall a general fact for RWRO which states that, under the polymer measure $\mathrm{P}_{n,\beta}$ (see (2.2)) the random walk $(S_k)_{1\le k\le n}$ follows an inhomogeneous Markov chain in a (random) drift field. Indeed, one can easily check that for all $1 \le k < n$ and $x_1, \dots, x_{k+1} \in \mathbb{Z}^d$,

$$\mathrm{P}_{n,\beta}(S_{k+1} = x_{k+1} | S_k = x_k, \dots, S_1 = x_1) = \frac{e^{-\beta \mathbf{1}_{\{x_{k+1}\in\tau\}}} Z_{n-k-1,x_{k+1}}}{\sum_{|y-x_k|=1} e^{-\beta \mathbf{1}_{\{y\in\tau\}}} Z_{n-k-1,y}}, \tag{2.62}$$

which may depend on n, k, x_k, x_{k+1} but not on $x_1, \dots, x_{k-1}$. For the rest of the paragraph we will refer to the index of the optimal gap as

$$\ell_0 = \ell_0(n, \tau, \beta) = \mathrm{argmin}_{\ell\in\mathbb{N}_0} \left\{ \lambda(\beta, \ell)\ell + \frac{\pi^2 n}{2T^2_{\ell+1}} \right\}. \tag{2.63}$$

As the confinement strategy is done in two steps, two questions arise:

(1) How long does it take for the random walk to reach τ_{ℓ_0} (left endpoint of the optimal gap)?

(2) What amount of the remaining time does the random walk spend in the gap $(\tau_{\ell_0}, \tau_{\ell_0+1})$?

We find that the answer to (1) depends on the value of γ. If $0 < \gamma < 2$ then, conditional on survival and in the limit of large ℓ, the time to reach τ_ℓ is of order τ_ℓ to the power $\frac{2}{\gamma\vee 1}$, that is diffusive for $0 < \gamma < 1$ and super-diffusive but sub-ballistic for $1 < \gamma < 2$. In any of these cases, the time to reach τ_ℓ is also of order $\ell^{2/\gamma}$. If $\gamma > 2$ then we enter a ballistic regime as this time becomes of order τ_ℓ (hence also of order ℓ by the Law of Large Numbers). These asymptotics are derived by comparing the random walk conditioned to hit τ_ℓ before it dies to a simpler random walk in random environment.

The answer to (2) seems to reveal a dichotomy: when $\gamma < 1$, the random walk, after it has reached τ_{ℓ_0}, spends essentially all of its remaining time (up to $O(1)$ steps) inside the optimal gap, which means that the interfaces induce a strong localization effect. This does not seem to be the case anymore if $\gamma > 1$. In this case, we conjecture that the walk does not spend all its time in the optimal gap but does not wander off too far from it (weaker localization). This dichotomy between $\gamma < 1$ and $\gamma > 1$ echoes that of Caravenna and Pétrélis [34] in the case of interfaces distributed along the lattice $t\mathbb{Z}$, when $t = t(n) = n^a$ (see Section 2.1.2). Indeed, since the size of the relevant gaps in our renewal model is $N^{\frac{1}{\gamma}} = n^{\frac{1}{\gamma+2}}$, one sees that the borderline case corresponds to $a(\gamma) := \frac{1}{\gamma+2} = \frac{1}{3}$, that is precisely $\gamma = 1$.

2.3.2 Random walk in a polymer melt

It would be interesting to consider the effect of spatially correlated potentials in higher dimensions. The random interlacement set introduced by Sznitman [131] provides such an example of a random

subset of $\mathbb{Z}^d$ ($d \geq 3$) which could serve as a set of obstacles τ (joint project with Q. Berger). The random interlacement at level $u > 0$, denoted by $\mathrm{RI}(u)$, is characterized by the relation

$$\mathbb{P}(A \cap \mathrm{RI}(u) = \emptyset) = \exp(-u \ \mathrm{cap}(A)), \tag{2.64}$$

where A is any finite subset of $\mathbb{Z}^d$ and $\mathrm{cap}(A)$ its Newtonian capacity. It is known to exhibit long-range spatial correlations:

$$|\mathbb{P}(x, y \in \mathrm{RI}(u)) - \mathbb{P}(x \in \mathrm{RI}(u))\mathbb{P}(y \in \mathrm{RI}(u))| \sim (\mathrm{cst})\|x - y\|^{2-d}, \qquad x, y \in \mathbb{Z}^d, \tag{2.65}$$

as $\|x - y\|$ goes to infinity. Moreover, Sznitman proved that a percolation transition for the vacant set $\mathbb{Z}^d \setminus \mathrm{RI}(u)$ occurs at a critical level $u_c(d)$. Hence, the model suggested here is that of a random walk killed by a random interlacement (or random walk in a polymer melt) in the subcritical regime $u < u_c(d)$. A Flory argument reveals the size of the relevant vacant pocket (i.e. set free from obstacles) where the random walk might take shelter and the challenge is to determine whether the same kind of localization results hold as in the case of i.i.d. obstacles. Interestingly, when $\beta = \infty$ (hard obstacles) the annealed version of this model turns out to be the random walk penalized by its capacity:

$$\mathbb{E}(Z_{n,\infty}) = \mathrm{E}[\exp(-u \ \mathrm{cap}(\{S_1, \ldots, S_n\}))], \qquad u > 0. \tag{2.66}$$

This model is of course linked to the deviations of $\mathrm{cap}(\{S_1, \ldots, S_n\})$, a problem which has been recently investigated in [10] (see also the discussion in Section 4.4).

Chapter 3

Folding and unfolding of polymers.

This chapter is based on joint works with Q. Berger, F. Caravenna, F. den Hollander and N. Pétrélis [20, 32].

DNA and proteins are polyelectrolytes whose monomers are in a charged state that depends on the pH of the solution in which they are immersed. The charges may fluctuate in space ('quenched') and in time ('annealed'). In this chapter we consider the charged polymer chain introduced in Kantor and Kardar [87]. The polymer chain is modelled by the path of a simple random walk on $\mathbb{Z}^d$, $d \geq 1$. Each monomer in the polymer chain carries a *random electric charge*, drawn in an i.i.d. fashion from $\mathbb{R}$. Each self-intersection of the polymer chain contributes an energy that is equal to the product of the charges of the two monomers that meet (i.e., a negative energy when the charges have opposite sign and a positive energy when the charges have the same sign). The polymer chain has a probability distribution on path space that is given by the *Gibbs measure* associated with the energy. Our goal is to study the scaling properties of the polymer as its length tends to infinity.

What makes the charged polymer model challenging is that the *interaction is both attractive and repulsive*. This places it outside the range of models that have been studied with the help of subadditivity techniques (see Ioffe [84] for an overview), and makes it a testbed for the development of new approaches. The *collapse transition* of a charged polymer can be seen as a simplified version of the *folding transition* of a protein. Interactions between different parts of the protein cause it to fold into different configurations depending on the temperature.

This chapter is organized as follows. We first define the charged polymer model in Section 3.1 in both its annealed and quenched versions. In Section 3.2 we briefly discuss the weakly self-avoiding walk, which can be seen as a limiting case of the charged polymer model. Section 3.3 contains a few results about the quenched charged polymer, while in Section 3.4 we present a series of recent results about the collapse phase transition that occurs in the annealed model. A series of comments, open questions and conjectures is listed in Section 3.5. Finally, we discuss some perspectives on the issue of random walk folding in Section 3.6.

3.1 Charged polymers

Let $S = (S_i)_{i \in \mathbb{N}_0}$ be a simple random walk on $\mathbb{Z}^d$, $d \geq 1$, i.e., $S_0 = 0$ and $S_i = \sum_{j=1}^{i} X_j$, $i \in \mathbb{N}$, with $X = (X_j)_{j \in \mathbb{N}}$ i.i.d. random variables that are uniformly distributed on the $2d$ unit vectors. The path S models the configuration of the undirected polymer chain, i.e., S_i is the position of monomer i. We use the letters P and E for probability and expectation with respect to S.

Let $\omega = (\omega_i)_{i \in \mathbb{N}}$ be i.i.d. random variables taking values in $\mathbb{R}$. The sequence ω models the electric charges along the polymer chain, i.e., ω_i is the charge of monomer i (see Fig. 3.1). We use the letters

$\mathbb{P}$ and $\mathbb{E}$ for probability and expectation with respect to ω. Throughout the paper we assume that

$$M(t) = \mathbb{E}(e^{t\omega_1}) < \infty \qquad \forall\, t \in \mathbb{R}. \tag{3.1}$$

Without loss of generality we may take

$$\mathbb{E}(\omega_1) = 0, \qquad \mathbb{E}(\omega_1^2) = 1. \tag{3.2}$$

We shall sometimes abbreviate

$$m_k = \mathbb{E}[\omega_1^k], \qquad k \in \mathbb{N}, \tag{3.3}$$

so that $m_1 = 0$, $m_2 = 1$ by (3.2).

To allow for biased charges, we use a tilting parameter $\delta \in \mathbb{R}$ and write $\mathbb{P}^\delta$ for the i.i.d. law of ω with marginal

$$\mathbb{P}^\delta(\mathrm{d}\omega_1) = \frac{e^{\delta\omega_1}\,\mathbb{P}(\mathrm{d}\omega_1)}{M(\delta)}. \tag{3.4}$$

Note that $\mathbb{E}^\delta(\omega_1) = M'(\delta)/M(\delta)$. In what follows we may, without loss of generality, take $\delta \in [0,\infty)$.

Example 3.1. *The special case where the charges are $+1$ with probability p and -1 with probability $1-p$ for some $p \in (0,1)$ corresponds to $\mathbb{P} = [\frac{1}{2}(\delta_{-1} + \delta_{+1})]^{\otimes\mathbb{N}}$ and $\delta = \frac{1}{2}\log(\frac{p}{1-p})$.*

We will sometimes make a distinction between lattice and non-lattice charge distributions. To this end, we define

$$T := \sup\big\{t > 0\colon\ \mathbb{P}(\omega_1 \in t\,\mathbb{Z}) = 1\big\} \tag{3.5}$$

(with the convention $\sup\emptyset = 0$). Either $T > 0$ ('lattice case') or $T = 0$ ('non-lattice case'). The following additional assumption will be needed at a few places:

Assumption 3.2. *One of these two propositions holds:*

$$\begin{aligned} &(a)\ \omega_1 \text{ is discrete with a distribution that is lattice.}\\ &(b)\ \omega_1 \text{ is continuous with a density that is in } L^p \text{ for some } p > 1. \end{aligned} \tag{3.6}$$

Let Π denote the set of nearest-neighbour paths starting at 0. Given $n \in \mathbb{N}$, we associate with each $(\omega, S) \in \mathbb{R}^{\mathbb{N}} \times \Pi$ an energy given by the Hamiltonian (see Fig. 3.1)

$$H_n^\omega(S) = \sum_{1 \le i,j \le n} \omega_i\omega_j\, 1_{\{S_i = S_j\}}. \tag{3.7}$$

Let β denote the inverse temperature. Throughout the sequel the relevant space for the pair of parameters (δ,β) is the quadrant

$$\mathcal{Q} = [0,\infty) \times (0,\infty). \tag{3.8}$$

Given $(\delta,\beta) \in \mathcal{Q}$, the *quenched polymer measure of length n* is the Gibbs measure $\mathrm{P}_n^{\omega,\beta}$ defined as

$$\frac{\mathrm{dP}_n^{\omega,\beta}}{\mathrm{dP}}(S) = \frac{1}{Z_n^{\omega,\beta}}\, e^{-\beta H_n^\omega(S)}, \qquad S \in \Pi, \tag{3.9}$$

with

$$Z_n^{\omega,\beta} = \mathrm{E}\left[e^{-\beta H_n^\omega(S)}\right], \tag{3.10}$$

the *quenched partition function of length n*, whereas the *annealed polymer measure of length n* is the Gibbs measure $\mathbb{P}_n^{\delta,\beta}$ defined as

$$\frac{\mathrm{d}\mathbb{P}_n^{\delta,\beta}}{\mathrm{d}(\mathbb{P}^\delta \times \mathrm{P})}(\omega,S) = \frac{1}{\mathbb{Z}_n^{\delta,\beta}}\, e^{-\beta H_n^\omega(S)}, \qquad (\omega,S) \in \mathbb{R}^{\mathbb{N}} \times \Pi, \tag{3.11}$$

with

$$\mathbb{Z}_n^{\delta,\beta} = (\mathbb{E}^\delta \times \mathrm{E})\left[e^{-\beta H_n^\omega(S)}\right] \tag{3.12}$$

the *annealed partition function of length* n. The measure $\mathbb{P}_n^{\delta,\beta}$ is the joint probability distribution for the polymer chain and the charges at charge bias δ and inverse temperature β when the polymer chain has length n.

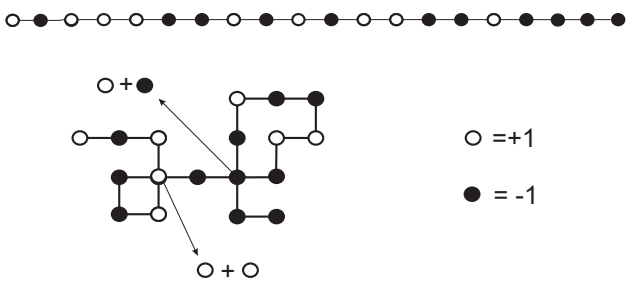

Figure 3.1: *Top:* A polymer chain carrying (± 1)-valued random charges. *Bottom:* The path may or may not be self-avoiding. The charges only interact at self-intersections.

Remark 3.3. *The Hamiltonian in* (3.7) *only picks up interaction between charges when the monomers carrying these charges meet at the same site. In other words, the long-range Coulomb interaction is* screened *to a short-range on-site interaction. This choice is mathematically convenient. In fact, so far no mathematically rigorous results have been obtained for long-range models. The short-range model considered here describes a charged polymer immersed in an* ionic fluid*, which surrounds the monomers and screens their charges.*

3.2 A special case: the weakly self-avoiding walk

The results of this section are based on a joint work with Q. Berger and F. den Hollander [20].

The quenched charged polymer model with $\mathbb{P} = [\frac{1}{2}(\delta_{-1} + \delta_{+1})]^{\otimes\mathbb{N}}$ interpolates between the simple random walk ($\beta = 0$), the self-avoiding walk ($\beta = \delta = \infty$) and the weakly self-avoiding walk ($\beta \in (0, \infty)$, $\delta = \infty$), for which an abundant literature is available (see den Hollander [53, Chapter 2] for references). The weakly self-avoiding walk model corresponds to the situation where all the charges are $+1$, in which case the Hamiltonian in (3.7) coincides with the self-intersection local time of the random walk, which we denote by

$$Q_n = \sum_{1\le i,j\le n} 1_{\{S_i = S_j\}} = \sum_{x\in\mathbb{Z}^d} \ell_n(x)^2, \tag{3.13}$$

where

$$\ell_n(x) = \sum_{1\le i\le n} 1_{\{S_i = x\}} \tag{3.14}$$

is the local time of the random walk at site x up to time n. A standard computation gives (see e.g. Spitzer [123, Section 7]), as $n \to \infty$,

$$\mathrm{E}[Q_n] = \sum_{1\le i,j\le n} \mathrm{P}(S_i = S_j) \sim \begin{cases} \lambda_1 n^{3/2}, & d = 1, \\ \lambda_2 n \log n, & d = 2, \\ \lambda_d n, & d \ge 3, \end{cases} \tag{3.15}$$

where

$$\lambda_2 = 2/\pi, \qquad \lambda_d = 2G_d - 1, \quad d \geq 3, \tag{3.16}$$

$G_d = \sum_{n\in\mathbb{N}_0} \mathrm{P}(S_n = 0)$ is the Green function at the origin of simple random walk on $\mathbb{Z}^d$, while λ_1 is a constant related to the one-dimensional Brownian self-intersection local time [46, Theorem 5.2.3].

For $u \geq 0$, let

$$Z_n^{\mathrm{wsaw}}(u) = \mathrm{E}\big[e^{-uQ_n}\big], \qquad u \in [0,\infty), \tag{3.17}$$

be the partition function of the weakly self-avoiding walk, which is a challenging and well-studied model. It would be too long to give a full account on the literature (see den Hollander [53]) but let us keep in mind that the penalization by the self-intersection local time tends to *unfold* the polymer. Here we shall only focus on results that came out as a by-product of our study of charged polymers [20]. More precisely, we consider:

- the free energy $\mathrm{F}^{\mathrm{wsaw}}$ of the weakly self-avoiding walk and its scaling in the limit of weak interaction (Proposition 3.4 below);
- the downward large deviations of the self-intersection local time Q_n as $n \to \infty$ (Proposition 3.5 below).

Weak-interaction limit of the free energy

First, we note that the partition function in (3.17) is sub-multiplicative because

$$Q_{n+m} \geq Q_n + Q_{n,n+m}, \qquad m, n \in \mathbb{N}, \tag{3.18}$$

where $Q_{n,n+m} := \#\{n < i, j \leq n+m \colon S_i = S_j\}$ is distributed as Q_m. Hence (minus) the free energy of the weakly self-avoiding walk

$$\mathrm{F}^{\mathrm{wsaw}}(u) = -\lim_{n\to\infty} \frac{1}{n} \log Z_n^{\mathrm{wsaw}}(u), \qquad u \in [0,\infty), \tag{3.19}$$

exists. The following proposition identifies the scaling behaviour of $\mathrm{F}^{\mathrm{wsaw}}(u)$ for $u \downarrow 0$.

Proposition 3.4. *As $u \downarrow 0$,*

$$\mathrm{F}^{\mathrm{wsaw}}(u) \sim \begin{cases} \lambda_1 u^{2/3}, & d = 1, \\ \lambda_2 u \log(1/u), & d = 2, \\ \lambda_d u, & d \geq 3, \end{cases} \tag{3.20}$$

where λ_d is given in (3.16). ∎

The anomalous scalings in $d = 1$ and $d = 2$ parallel that in (3.15). We comment further on this result in Section 3.5, see Conjecture 3.25.

Downward large deviations of the self-intersection local time

The case $d \geq 2$ in Proposition 3.4 may be deduced from Varadhan's lemma and the following proposition:

Proposition 3.5. *The limit*

$$I(t) = \lim_{n\to\infty} \left[-\frac{1}{n} \log \mathrm{P}(Q_n \leq tn)\right], \qquad t \in [1,\infty), \tag{3.21}$$

exists. Moreover, $t \mapsto I(t)$ is finite, non-negative, non-increasing and convex on $[1,\infty)$, and satisfies

$$d=2\colon \quad I(t)>0, \quad t\geq 1, \qquad d\geq 3\colon \quad I(t)\begin{cases} >0, & 1\leq t\leq \lambda_d,\\ =0, & t\geq \lambda_d. \end{cases} \tag{3.22}$$

Furthermore,

$$d=2\colon \quad \lim_{t\to\infty}\frac{-\log I(t)}{t}=\frac{1}{\lambda_2}. \tag{3.23}$$

■

Proposition 3.5 extends the downward moderate deviation result for Q_n derived by Chen [46, Theorem 8.3.2].

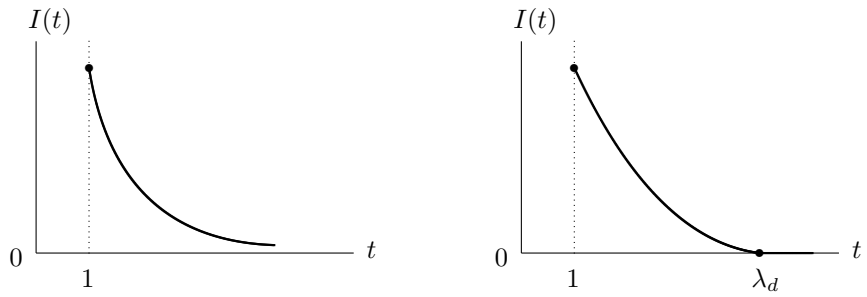

Figure 3.2: Qualitative plots of $t \mapsto I(t)$ for $d=2$ and $d\geq 3$.

3.3 The quenched charged polymer

Let us come back to the quenched polymer measure defined in (3.9). Very little is known mathematically about the quenched version of the model, where the charges are frozen. The two main questions of interest are:

(1) Is the free energy self-averaging in the disorder?

(2) Is there a phase transition from a 'collapsed phase' to an 'extended phase' at some critical value of the temperature?

We expect that the answer to (1) is yes and the answer to (2) is no.

In this section we prove two modest results about the quenched charged polymer. Recall (3.14). The range of the random walk up to time n is

$$R_n=\{S_1,\dots,S_n\}=\{x\in\mathbb{Z}\colon\ \ell_n(x)>0\}. \tag{3.24}$$

We first show that $|R_n|$ grows linearly in n when the average charge is non-zero.

Proposition 3.6. *Suppose that $\delta,\beta\in(0,\infty)$. Then there exist $c_1,c_2>0$ (depending on δ,β) such that, for $\mathbb{P}^\delta$-a.e. ω,*

$$\mathrm{P}_n^{\omega,\beta}(|R_n|\leq c_1 n)\leq e^{-c_2 n+o(n)}. \tag{3.25}$$

Proof. Let π be the one-sided path that takes steps only in one fixed direction (one of the $2d$ unit vectors). Estimate

$$Z_n^{\omega,\beta} \geq (\tfrac{1}{2d})^n \, \mathrm{E}\Big[e^{-\beta H_n^{\omega,\beta}(S)} \, 1_{\{S_i=\pi_i \, \forall \, 1\leq i\leq n\}}\Big] = (\tfrac{1}{2d})^n \, e^{-\beta \sum_{i=1}^n \omega_i^2} = (\tfrac{1}{2d})^n \, e^{-\beta n + o(n)}. \tag{3.26}$$

Moreover, by Jensen's inequality we have

$$H_n^{\omega}(S) = \sum_{x\in R_n} \left(\sum_{i=1}^n \omega_i 1_{\{S_i=x\}} \right)^2 \geq \frac{\Omega_n^2}{|R_n|}, \qquad \text{where} \quad \Omega_n = \sum_{1\leq i\leq n} \omega_i. \tag{3.27}$$

Combining (3.26) and (3.27), we obtain

$$\mathrm{P}_n^{\omega,\beta}(R_n \leq c_1 n) \leq \exp\left\{ -\beta n \left[\frac{\Omega_n^2}{c_1 n^2} - 1 - \tfrac{\log(2d)}{\beta} \right] \right\}. \tag{3.28}$$

By the strong law of large numbers for ω, we have $\lim_{n\to\infty} n^{-1}\Omega_n = (\partial/\partial\delta) \log M(\delta) = m(\delta) > 0$ for $\mathbb{P}^\delta$-a.a. ω, and so the term between square brackets equals $c_3[1+o(1)]$ with $c_3 = \frac{1}{c_1} m(\delta)^2 - 1 - \frac{\log(2d)}{\beta}$. Therefore, by choosing $c_1 > 0$ small enough so that $c_3 > 0$, we get (3.25) with $c_2 = \beta c_3$. ∎

We next show that the polymer chain is ballistic when the charges are sufficiently biased and $d = 1$.

Proposition 3.7. *Let $d = 1$. For every $\beta \in (0,\infty)$ there exists $\delta_0 = \delta_0(\beta) \in (0,\infty)$ such that*

$$\forall \, \delta > \delta_0 \; \exists \varepsilon = \varepsilon(\delta) > 0 \colon \qquad \lim_{n\to\infty} \mathrm{P}_n^{\omega,\beta}\big(n^{-1} S_n > \varepsilon \mid S_n > 0\big) = 1. \tag{3.29}$$

Proof. Fix $\beta \in (0,\infty)$. Pick δ_0 such that

$$m(\delta_0) = \sqrt{\tfrac{1}{2}\left(1 + \tfrac{\log 2}{\beta}\right)}. \tag{3.30}$$

If $\delta > \delta_0$, then we can choose $c_1 > \frac{1}{2}$ in Proposition 3.6 and use the inequality

$$\frac{\#\{x \in \mathbb{Z} \colon \, \ell_n(x) = 1\}}{n} \geq \frac{2|R_n|}{n} - 1 \tag{3.31}$$

to conclude that a positive fraction of the sites are visited precisely once. Consequently, if the polymer chain chooses to go to the right, then S_n/n has a strictly positive lim inf. ∎

3.4 The annealed charged polymer

In the present section we focus on the *annealed* version of the model, where the charges are averaged out. This version is somehow easier to deal with, yet turns out to exhibit an interesting phase transition between a 'collapsed phase' and an 'extended phase', which we explain in Section 3.4.1. In Section 3.4.2 we focus on the one-dimensional case [32], where a lot can be said. Section 3.4.3 is specific to dimensions $d \geq 2$.

3.4.1 The general picture

The goal of this section is to give a general picture of the phase diagram that is valid in all dimensions. Recall first the definition of the local times in (3.14). By integrating out the charges in (3.12), the annealed partition function can be rewritten as

$$\mathbb{Z}_n^{\delta,\beta} = \mathrm{E}\Big[\prod_{x\in\mathbb{Z}^d} g_{\delta,\beta}\big(\ell_n(x)\big) \Big], \tag{3.32}$$

where

$$g_{\delta,\beta}(\ell) = \mathbb{E}^{\delta}\big[\exp(-\beta\Omega_\ell^2)\big], \qquad \Omega_\ell = \sum_{i=1}^{\ell} \omega_i, \qquad \ell \in \mathbb{N}_0, \tag{3.33}$$

with the convention $g_{\delta,\beta}(0) = 1$. The *annealed free energy* per monomer is defined by

$$\mathrm{F}(\delta,\beta) = \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{Z}_n^{\delta,\beta}. \tag{3.34}$$

Remark 3.8. *We are able to prove that the limes superior in (3.34) is a limit in $d = 1$ (see Theorem 3.15 below) and we expect that the same holds in higher dimensions. Convergence appears to be hard to settle, due to the competition between attractive and repulsive interactions. Nonetheless, we are able to prove convergence in higher dimensions for large enough β and for charge distributions that are non-lattice with a bounded density (see Theorem 3.23 below).*

Our first theorem provides relevant upper and lower bounds on the free energy. Recall (3.1) and abbreviate

$$f(\delta) = -\log M(\delta) \in (-\infty, 0]. \tag{3.35}$$

Theorem 3.9. *The limes superior in (3.34) takes values in $(-\infty, 0]$ and satisfies the inequality*

$$\mathrm{F}(\delta,\beta) \geq f(\delta). \tag{3.36}$$

The *excess annealed free energy* per monomer is then defined by

$$\mathrm{F}^*(\delta,\beta) = \mathrm{F}(\delta,\beta) - f(\delta). \tag{3.37}$$

It follows from (3.32)–(3.37) that

$$\mathrm{F}^*(\delta,\beta) = \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{Z}_n^{*,\delta,\beta} \tag{3.38}$$

with

$$\mathbb{Z}_n^{*,\delta,\beta} = \mathrm{E}\Big[\prod_{x\in\mathbb{Z}^d} g_{\delta,\beta}^*\big(\ell_n(x)\big)\Big], \tag{3.39}$$

and

$$g_{\delta,\beta}^*(\ell) = \mathbb{E}\big[\exp\big(\delta\Omega_\ell - \beta\Omega_\ell^2\big)\big], \qquad \ell \in \mathbb{N}_0, \tag{3.40}$$

with the convention $g_{\delta,\beta}^*(0) = 1$. We may think of $g_{\delta,\beta}^*(\ell)$ as a single-site partition function for a site that is visited ℓ times.

Example 3.10. *If the distribution of the charges is standard normal, then*

$$g_{\delta,\beta}^*(\ell) = \sqrt{\frac{1}{1+2\beta\ell}}\, \exp\left[\frac{\delta^2\ell}{2(1+2\beta\ell)}\right], \qquad \ell \in \mathbb{N}_0. \tag{3.41}$$

Note that $-\log g_{\delta,\beta}^*$ can be decomposed as $-\log g_{\delta,\beta}^* = -\log g_{\delta,\beta}^{*,\mathrm{att}} - \log g_{\delta,\beta}^{*,\mathrm{rep}}$ with

$$-\log g_{\delta,\beta}^{*,\mathrm{att}}(\ell) = \frac{1}{2}\log(1+2\beta\ell), \qquad -\log g_{\delta,\beta}^{*,\mathrm{rep}}(\ell) = -\frac{\delta^2\ell}{2(1+2\beta\ell)}. \tag{3.42}$$

The former is an attractive interaction (positive concave function), the latter is a repulsive interaction (negative convex function).

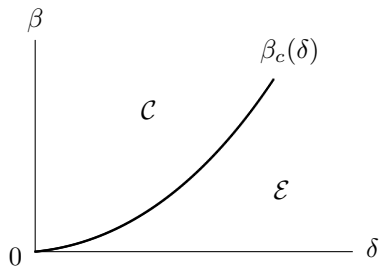

Figure 3.3: Qualitative plot of the critical curve $\delta \mapsto \beta_c(\delta)$ where the excess free energy $\mathrm{F}^*(\delta,\beta)$ changes from being zero ($\mathcal{C}$) to being strictly positive ($\mathcal{E}$). The critical curve is part of $\mathcal{C}$.

Recall (3.8). Because $\mathrm{F}^*(\delta,\beta) \geq 0$, it is natural to define two phases:

$$\begin{aligned} \mathcal{C} &= \{(\delta,\beta) \in \mathcal{Q} \colon \mathrm{F}^*(\delta,\beta) = 0\}, \\ \mathcal{E} &= \{(\delta,\beta) \in \mathcal{Q} \colon \mathrm{F}^*(\delta,\beta) > 0\}. \end{aligned} \tag{3.43}$$

For reasons that will become clear later, we refer to these as the *collapsed phase*, respectively, the *extended phase*. One may check that for every $\delta \in [0,\infty)$, $\beta \mapsto \mathrm{F}^*(\delta,\beta)$ is finite, non-negative, non-increasing and convex. Hence there is a critical threshold $\beta_c(\delta) \in [0,\infty]$ such that $\mathcal{C}$ is the region on and above the curve and $\mathcal{E}$ is the region below the curve (see Fig. 3.3).

Let us summarize what has been obtained until now for the critical curve (recall the definition of λ_d in (3.16) and that of m_3 in (3.3)):

Theorem 3.11. (i) $\delta \mapsto \beta_c(\delta)$ *is continuous, strictly increasing and convex on* $[0,\infty)$*, with* $\beta_c(0) = 0$.
(ii) *Let* $d = 1$ *and assume* (3.6)*. There exists a constant* $a \in (0,\infty)$ *such that, as* $\delta \downarrow 0$,

$$\beta_c(\delta) - \tfrac{1}{2}\delta^2 \sim -a(\tfrac{1}{2}\delta^2)^{\frac{4}{3}}. \tag{3.44}$$

(iii) *Let* $d \geq 2$*. As* $\delta \downarrow 0$,

$$\beta_c(\delta) = \tfrac{1}{2}\delta^2 - \tfrac{1}{3}m_3\delta^3 - \varepsilon_\delta \tag{3.45}$$

with

$$[\underline{\kappa} + o(1)]\,\delta^4 \leq \varepsilon_\delta \leq [1 + o(1)] \begin{cases} \kappa_2\delta^4 \log(1/\delta), & d = 2, \\ \kappa_d\delta^4, & d \geq 3, \end{cases} \tag{3.46}$$

where

$$\underline{\kappa} = \tfrac{1}{12}m_4 - \tfrac{1}{3}m_3^2, \qquad \kappa_d = \begin{cases} \frac{1}{4}\lambda_2, & d = 2 \\ \frac{1}{4}(\lambda_d - 1) + \underline{\kappa}, & d \geq 3. \end{cases} \tag{3.47}$$

Theorem 3.11 shows that there is indeed a phase transition at a non-trivial critical curve of which $\frac{1}{2}\delta^2$ is the first order. If the charge distribution is *symmetric*, then actually $\beta_c(\delta) \leq \frac{1}{2}\delta^2$ for *all* $\delta \in [0,\infty)$. The next order in the expansion depends on the dimension.

Theorem 3.11(ii) shows that the scaling of the critical curve in $d = 1$ is anomalous. The term in δ^3 that appears when $d \geq 2$ is not relevant in this case. A spectral representation of the multiplicative constant a will be given in Theorem 3.20 below.

Theorem 3.11(iii) identifies three terms in the upper bound of $\beta_c(\delta)$ for small δ, of which the last is *anomalous* for $d = 2$. The proof is based on an analysis of the *downward large deviations*

of the self-intersection local time Q_n defined in (3.13) under the law P of simple random walk, see Proposition 3.5. For the standard normal distribution $m_3 = 0$ and $m_4 = 3$, and so $\kappa_d = \frac{1}{4}\lambda_d$ for $d \geq 2$ in (3.47). Note that $\kappa_d \geq \underline{\kappa} > 0$ for $d \geq 3$ when $m_3 = 0$, but not necessarily when $m_3 \neq 0$. Indeed, if the distribution of the charges puts weight $\frac{1}{3N^2}$, $1 - \frac{1}{2N^2}$, $\frac{1}{6N^2}$ on the values $-N$, 0, $2N$, respectively, for some $N \in \mathbb{N}$, then $m_1 = 0$, $m_2 = 1$, $m_3 = N$, $m_4 = 3N^2$, in which case $-\frac{1}{3}m_3^2 + \frac{1}{12}m_4 = -\frac{1}{12}N^2$. This gives $\kappa_d < 0$ for N large enough and $\underline{\kappa} < 0 \leq \kappa_d$ for N small enough. A sharp version of Theorem 3.11(iii) is the object of Conjecture 3.26.

Let us now turn to the behaviour of the critical curve in the limit of a large charge bias. Interestingly, this depends on whether the charge distribution is lattice or not. Heuristically, the reason is that it is easier to build small absolute values of $\Omega_\ell = \sum_{k=1}^{\ell} \omega_k$ for small values of ℓ when the charge distribution is non-lattice rather than lattice. This is confirmed by the following theorem, which is valid in all dimensions.

Theorem 3.12. *As $\delta \to \infty$,*

$$\beta_c(\delta) \sim \frac{\delta}{T} \tag{3.48}$$

with T defined in (3.5). If $T = 0$ and ω_1 has a bounded density (with respect to the Lebesgue measure), then

$$\beta_c(\delta) \sim \frac{\delta^2}{4 \log \delta}. \tag{3.49}$$

Remark 3.13. *Biskup and König [26], Ioffe and Velenik [85], Kosygina and Mountford [91] deal with annealed versions of various models of simple random walk in a random potential. In all these models the interaction is either* attractive *or* repulsive*, meaning that the annealed partition function is the expectation of the exponential of a functional of the local times of simple random walk that is either* subadditive *or* superadditive*. In contrast, our annealed charged polymer model is neither attractive nor repulsive.*

3.4.2 One-dimensional case

Much more can be said in dimension one [32]. We obtain a spectral representation of the phase transition and show that the collapsed phase is actually a *subballistic phase* whereas the extended phase is a *ballistic phase*. Moreover, we show that the phase transition is first order and that the empirical speed and the empirical charge satisfy a law of large numbers. We identify the scaling of the critical curve in the limit of small average charge and that of the free energy in the limit of small inverse temperature, which exhibit anomalous behaviour. A key tool in our analysis is the Ray-Knight formula for the local times of the one-dimensional simple random walk. This tool, which has been used extensively in the literature, is exploited in full to obtain the fine details of the phase diagram of the charged polymer.

Spectral representation

Let $Q(i,j)$ be the probability matrix defined by

$$Q(i,j) = \begin{cases} 1_{\{j=0\}}, & \text{if } i = 0,\ j \in \mathbb{N}_0, \\ \binom{i+j-1}{i-1} \left(\frac{1}{2}\right)^{i+j}, & \text{if } i \in \mathbb{N},\ j \in \mathbb{N}_0, \end{cases} \tag{3.50}$$

which is the transition kernel of a critical Galton-Watson branching process with a geometric offspring distribution (of parameter $\frac{1}{2}$). For $(\mu, \delta, \beta) \in [0,\infty) \times \mathcal{Q}$, define the $\mathbb{N}_0 \times \mathbb{N}_0$ matrix $A_{\mu,\delta,\beta}$ by

$$A_{\mu,\delta,\beta}(i,j) = e^{-\mu(i+j+1)}\, g^*_{\delta,\beta}(i+j+1)\, Q(i+1,j), \qquad i,j \in \mathbb{N}_0. \tag{3.51}$$

Note that $A_{\mu,\delta,\beta}$ is symmetric. Let $\lambda_{\delta,\beta}(\mu)$ be the spectral radius of $A_{\mu,\delta,\beta}$ in $\ell^2(\mathbb{N}_0)$. One can check that, for every $(\delta,\beta)\in\mathcal{Q}$, $\mu\mapsto\lambda_{\delta,\beta}(\mu)$ is continuous, decreasing, log-convex on $[0,\infty)$ and tends to zero at infinity.

Definition 3.14. *Let $\mu(\delta,\beta)$ be the unique solution of the equation $\lambda_{\delta,\beta}(\mu)=1$ when it exists and $\mu(\delta,\beta)=0$ otherwise.*

One can actually prove that, for every $(\delta,\beta)\in\mathcal{Q}$, $\mu\mapsto\lambda_{\delta,\beta}(\mu)$ is analytic and strictly log-convex on $(0,\infty)$, and has a finite strictly negative right-slope at 0 (see Fig. 3.4).

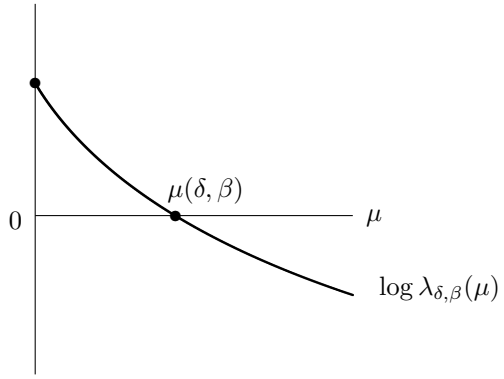

Figure 3.4: Qualitative plot of $\mu\mapsto\log\lambda_{\delta,\beta}(\mu)$. Only the case $\lambda_{\delta,\beta}(0)>1$ is shown. The interior of the ballistic phase $\mathrm{int}(\mathcal{B})$ corresponds to $\lambda_{\delta,\beta}(0)>1$, the subballistic phase $\mathcal{S}$ corresponds to $\lambda_{\delta,\beta}(0)<1$, the critical curve corresponds to $\lambda_{\delta,\beta}(0)=1$ (see (3.53)).

We begin with a *spectral representation* for the annealed free energy.

Theorem 3.15. *For all $(\delta,\beta)\in\mathcal{Q}$, the limes superior in (3.34) is a limit. Moreover, the excess free energy has the spectral representation*

$$\mathrm{F}^*(\delta,\beta)=\mu(\delta,\beta). \tag{3.52}$$

The latter generalizes the spectral representation derived in Greven and den Hollander [81] for the weakly self-avoiding walk (see also den Hollander [52, Chapter IX]). This in turn provides a spectral characterization of the critical curve.

Theorem 3.16. *For every $\delta\in[0,\infty)$, $\beta_c(\delta)$ is the unique solution of the equation $\lambda_{\delta,\beta}(0)=1$. Moreover, $\delta\mapsto\beta_c(\delta)$ is analytic on $(0,\infty)$ and $(\delta,\beta)\mapsto\mathrm{F}^*(\delta,\beta)$ is analytic on $\mathcal{E}$.*

Hence there is no other phase transition inside the extended phase.

First-order phase transition from a subballistic to a ballistic phase

Let

$$\mathcal{B}=\big\{(\delta,\beta)\in\mathcal{Q}\colon\ 0<\beta\le\beta_c(\delta)\big\},\qquad \mathcal{S}=\mathcal{Q}\backslash\mathcal{B}. \tag{3.53}$$

The set $\mathcal{B}$ will be referred to as the *ballistic phase*, the set $\mathcal{S}$ as the *subballistic phase*, for reasons we explain next. Namely, we proceed by stating a *law of large numbers* for the empirical speed (S_n/n) and the empirical charge (Ω_n/n), respectively. In the statement below the condition $S_n>0$ is put in to choose a direction for the endpoint of the polymer chain.

Theorem 3.17. *For every $(\delta,\beta)\in\mathcal{Q}$ there exists a $v(\delta,\beta)\in[0,1]$ such that*

$$\lim_{n\to\infty}\mathbb{P}_n^{\delta,\beta}\Big(\big|n^{-1}S_n-v(\delta,\beta)\big|>\varepsilon\,\Big|\,S_n>0\Big)=0\qquad\forall\,\varepsilon>0,\tag{3.54}$$

where

$$v(\delta,\beta)\begin{cases}>0, & (\delta,\beta)\in\mathcal{B},\\ =0, & (\delta,\beta)\in\mathcal{S}.\end{cases}\tag{3.55}$$

For every $(\delta,\beta)\in\mathcal{B}$,

$$\frac{1}{v(\delta,\beta)}=\left[-\frac{\partial}{\partial\mu}\log\lambda_{\delta,\beta}(\mu)\right]_{\mu=\mu(\delta,\beta)}=\left[-\frac{\partial}{\partial\mu}\lambda_{\delta,\beta}(\mu)\right]_{\mu=\mu(\delta,\beta)}.\tag{3.56}$$

(Take the right-derivative when $\mu(\delta,\beta)=0$.) Moreover, $(\delta,\beta)\mapsto v(\delta,\beta)$ is analytic on $\mathrm{int}(\mathcal{B})$.

Theorem 3.18. *For every $(\delta,\beta)\in\mathcal{Q}$, there exists a $\rho(\delta,\beta)\in[0,\infty)$ such that*

$$\lim_{n\to\infty}\mathbb{P}_n^{\delta,\beta}\left(\big|n^{-1}\Omega_n-\rho(\delta,\beta)\big|>\varepsilon\right)=0\qquad\forall\,\varepsilon>0,\tag{3.57}$$

where

$$\rho(\delta,\beta)\begin{cases}>0, & (\delta,\beta)\in\mathcal{B},\\ =0, & (\delta,\beta)\in\mathcal{S}.\end{cases}\tag{3.58}$$

For every $(\delta,\beta)\in\mathcal{B}$,

$$\rho(\delta,\beta)=\left[\frac{\frac{\partial}{\partial\delta}\log\lambda_{\delta,\beta}(\mu)}{-\frac{\partial}{\partial\mu}\log\lambda_{\delta,\beta}(\mu)}\right]_{\mu=\mu(\delta,\beta)}=\frac{\partial}{\partial\delta}\,\mu\big(\delta,\beta\big).\tag{3.59}$$

Moreover, $(\delta,\beta)\mapsto\rho(\delta,\beta)$ is analytic on $\mathrm{int}(\mathcal{B})$.

The speed $v(\delta,\beta)$ of the polymer chain is strictly positive in the ballistic phase and zero in the subballistic phase (which explains the names associated with these two phases). In the ballistic phase the speed is given by the spectral formula in (3.56). The latter generalizes the spectral formula derived in Greven and den Hollander [81] for the speed $v(\beta)=v(\infty,\beta)$ of the weakly self-avoiding walk. The charge $\rho(\delta,\beta)$ of the polymer chain is strictly positive in the ballistic phase and zero in the subballistic phase. In the ballistic phase the charge is given by the spectral formula in (3.59). Fig. 3.5 shows a numerical plot of $\beta\mapsto v(1,\beta)$ and $\beta\mapsto\rho(1,\beta)$ when ω_1 is standard normal. Interestingly, the speed is not monotone on $(0,\beta_c(1)]$. This is in contrast with the monotonicity that was found (but not proved) in [81] for the weakly self-avoiding walk (for which $\beta_c(\infty)=\infty$). Equally interesting, the charge is monotone on $(0,\beta_c(1)]$. Since the critical curve lies in the ballistic phase $\mathcal{B}$, Theorems 3.17–3.18 imply that $(\delta,\beta)\mapsto v(\delta,\beta)$ and $(\delta,\beta)\mapsto\rho(\delta,\beta)$ are *discontinuous at criticality*. This means that the *phase transition is first order*. The scaling behaviour of the excess free energy near the critical curve also shows that the phase transition is first order.

Theorem 3.19. *For every $\delta\in(0,\infty)$,*

$$\mathrm{F}^*(\delta,\beta)\sim K_\delta[\beta_c(\delta)-\beta],\qquad\text{as }\beta\uparrow\beta_c(\delta),\tag{3.60}$$

where $K_\delta\in(0,\infty)$ is given by

$$K_\delta=\left[\frac{\frac{\partial}{\partial\beta}\log\lambda_{\delta,\beta}(\mu)}{\frac{\partial}{\partial\mu}\log\lambda_{\delta,\beta}(\mu)}\right]_{\beta=\beta_c(\delta),\mu=0}.\tag{3.61}$$

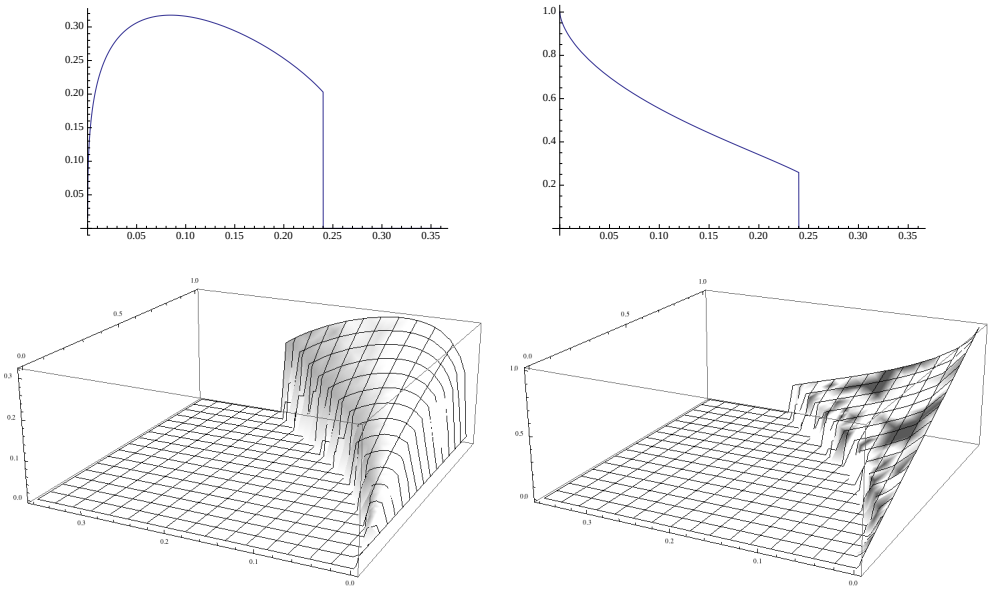

Figure 3.5: Numerical plots of the typical speed $v(\delta,\beta)$ and the typical charge $\rho(\delta,\beta)$ in Theorems 3.17 and 3.18, based on a 100×100 truncation of the matrix in (3.51), for the case where ω_1 is standard normal. *Above:* plot of $\beta\mapsto v(\delta,\beta)$ and $\beta\mapsto\rho(\delta,\beta)$ for $\delta=1$ and $\beta\in(0,0.36)$. *Below:* same for $\delta\in(0,1)$ and $\beta\in(0,0.36)$ (for graphical clarity the axes have been rotated: the δ-axis runs from front to back, the β-axis runs from right to left).

Weak coupling limits

For $a\in\mathbb{R}$ and $b\in(0,\infty)$, let $\mathcal{L}^{a,b}$ be the *Sturm-Liouville operator* defined by

$$(\mathcal{L}^{a,b}g)(x)=(2ax-4bx^2)g(x)+g'(x)+xg''(x),\qquad g\in C^2((0,\infty)). \tag{3.62}$$

This is a two-parameter version of a one-parameter family of operators considered in van der Hofstad and den Hollander [137]. Let

$$\mathfrak{C}=\left\{g\in L^2\big((0,\infty)\big)\cap C^\infty((0,\infty)\big):\ \|g\|_2=1,\ g>0,\ \int_0^\infty\Big[x\tfrac{9}{2}g(x)^2+xg'(x)^2\Big]\,\mathrm{d}x<\infty\right\}. \tag{3.63}$$

The largest eigenvalue problem

$$\mathcal{L}^{a,b}g=\chi g,\qquad \chi\in\mathbb{R},\ g\in\mathfrak{C}, \tag{3.64}$$

has a unique solution $(g^{a,b},\chi(a,b))$ with the following properties: for every $b\in(0,\infty)$,

$$\begin{aligned}&a\mapsto\chi(a,b)\text{ is analytic, strictly increasing and strictly convex on }\mathbb{R},\\ &\chi(0,b)<0,\ \lim_{a\to\infty}\chi(a,b)=\infty,\ \lim_{a\to-\infty}\chi(a,b)=-\infty,\\ &a\mapsto g^{a,b}\text{ is analytic as a map from }\mathbb{R}\text{ to }L^2((0,\infty)).\end{aligned} \tag{3.65}$$

(See Coddington and Levinson [48] for general background on Sturm-Liouville theory.)

Let $a^*=a^*(b)$ denote the unique solution of the equation $\chi(a,b)=0$ (see Fig. 3.6). We may now complement Theorem 3.11(ii).

Theorem 3.20. *The constant a in Theorem 3.11(ii) equals $a^*(1)$.*

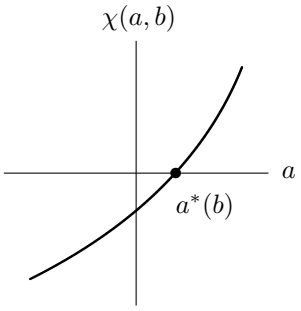

Figure 3.6: Qualitative plot of $a \mapsto \chi(a,b)$ for fixed $b \in (0,\infty)$.

We close by identifying the scaling behaviour of the free energy for small inverse temperature.

Theorem 3.21. *Assume (3.6). For every $\delta \in (0,\infty)$,*

$$\mathrm{F}(\delta,\beta) \sim -A_\delta \beta^{\frac{2}{3}}, \qquad v(\delta,\beta) \sim B_\delta \beta^{\frac{1}{3}}, \qquad \rho(\delta,\beta) - \rho_\delta \sim -C_\delta \beta^{\frac{2}{3}}, \qquad \text{as } \beta \downarrow 0, \tag{3.66}$$

where $\rho_\delta = \mathbb{E}^\delta(\omega_1) = -f'(\delta)$, and $A_\delta, B_\delta, C_\delta \in (0,\infty)$ are given by

$$A_\delta = a^*(\rho_\delta), \qquad \frac{1}{B_\delta} = \Big[\frac{\partial}{\partial a}\,\chi(a,b)\Big]_{a=a^*(\rho_\delta),\, b=\rho_\delta}, \qquad C_\delta = \frac{\mathrm{d}}{\mathrm{d}\delta}\,a^*(\rho_\delta). \tag{3.67}$$

The proofs of Theorems 3.20 and 3.21 follow van der Hofstad and den Hollander [137], but we have to address additional difficulties, due to our more complicated Hamiltonian. The third statement in (3.66) actually holds under an extra technical assumption related to the asymptotic variance of the charge distribution under the annealed polymer measure, which we do not detail here, for the sake of simplicity. We refer to [32, Theorem 1.10] for a precise statement.

3.4.3 Higher dimensions

The Ray-Knight formula is no longer available in $d \geq 2$ (at least not in such a tractable form as in $d = 1$). Our results [20] have been derived with the help of other tools, namely by expanding the partition function in the limit of small parameters and comparing it to the weakly self-avoiding walk. The phase diagram is qualitatively similar as in $d = 1$, but a detailed description of the scaling behaviour in the two phases is still missing.

Let us start with scaling bounds on the free energy for small inverse temperature and fixed charge bias.

Theorem 3.22. *For any $\delta \in (0,\infty)$, as $\beta \downarrow 0$,*

$$-\big[m(\delta)^2 + v(\delta) + o(1)\big]\,\beta \geq \mathrm{F}(\delta,\beta) \geq [1+o(1)] \begin{cases} -\lambda_2 m(\delta)^2\,\beta \log(1/\beta), & d=2, \\ -\big[\lambda_d m(\delta)^2 + v(\delta)\big]\,\beta, & d \geq 3, \end{cases} \tag{3.68}$$

where $m(\delta) = \mathbb{E}^\delta[\omega_1]$ and $v(\delta) = \mathbb{V}\mathrm{ar}^\delta[\omega_1]$. ∎

We finally settle the existence of the free energy in a subset of the collapsed phase (that is for large enough inverse temperature) and for a subclass of charge distributions.

Theorem 3.23. *Suppose that the charge distribution is non-lattice ($T = 0$) and has a bounded density. Then there exists a curve $\delta \mapsto \beta_0(\delta)$ such that, for all $\beta \geq \beta_0(\delta)$,*

(1) *the sequence* $\{\log g^*_{\delta,\beta}(\ell)\}_{\ell\in\mathbb{N}}$ *is super-additive,*

(2) *the limes superior in* (3.34) *is a limit, and equals* $-f(\delta)$*,*

(3) *the limes superior in* (3.38) *is a limit, and equals* 0*.*

Moreover, $\beta_0(\delta) \geq \beta_c(\delta)$ *and* $\beta_0(\delta) \sim \beta_c(\delta)$ *as* $\delta \to \infty$*.* ■

3.4.4 A selection of proof techniques and heuristics

Spectral representation of the free energy in $d = 1$.

As we have seen, the annealed partition function can be expressed as the expected value of a relatively simple function of the local times. The key idea behind the results in $d = 1$ is to use a Ray-Knight identity, which relates these local times to a Galton-Watson process. This identity can be conveniently stated in terms of the edge-crossing numbers of the random walk, which are defined by ($y \in \mathbb{N}_0$)

$$\begin{aligned} M_n^+(y) &= \left\lfloor \frac{1}{2} \sum_{1\leq k\leq n} 1_{\{\{S_{k-1},S_k\}=\{y,y+1\}\}} \right\rfloor \\ M_n^-(y) &= \left\lfloor \frac{1}{2} \sum_{1\leq k\leq n} 1_{\{\{S_{k-1},S_k\}=\{-y,-y-1\}\}} \right\rfloor, \end{aligned} \tag{3.69}$$

and which are linked to the local times by the following relation:

$$\text{On the event } \{S_n = x\} \text{ with } x \in \mathbb{N}_0\colon \quad \ell_n(y) = \begin{cases} M_n^+(y-1) + M_n^+(y), & \text{if } y > x, \\ M_n^+(y-1) + M_n^+(y) + 1, & \text{if } 1 \leq y \leq x, \\ M_n^+(0) + M_n^-(0), & \text{if } y = 0, \\ M_n^-(-y-1) + M_n^-(-y), & \text{if } y < 0. \end{cases} \tag{3.70}$$

Let us now fix $\ell, x \in \mathbb{N}_0$ and define a two-species *branching process*

$$(M^+, M^-) = (M^+(y), M^-(y))_{y\in\mathbb{N}_0} \tag{3.71}$$

with law $\mathcal{P}_{\ell,x}$ as follows:

- At generation 0 there are ℓ individuals, which are divided by fair coin tossing into two subpopulations, labelled $+$ and $-$.
- Each subpopulation evolves independently as a *critical Galton-Watson branching process with a geometric offspring distribution*, denoted by $\mathrm{Geo}_0(\frac{1}{2})$ and given by $\mathrm{Geo}_0(\frac{1}{2})(i) = 2^{-(i+1)}$, $i \in \mathbb{N}_0$.
- If $x \in \mathbb{N}$, then there is additional immigration of a $\mathrm{Geo}_0(\frac{1}{2})$-distributed number of individuals in the $+$ subpopulation, at each generation $1, \ldots, x$ (equivalently, the generations $0, \ldots, x-1$ have an additional "hidden" individual, which is not counted but produces offspring).
- Define $M^{\pm}(y)$ as the size of the $\pm$ subpopulation in the y-th generation.

Define the total population size

$$\Xi = \sum_{y\in\mathbb{N}_0} (M^+(y) + M^-(y)) \tag{3.72}$$

and note that $\Xi < \infty$ a.s. because a critical Galton-Watson process eventually dies out. The version of the Ray-Knight identity that follows is extracted from [32].

Theorem 3.24. *Fix $\ell, n, x \in \mathbb{N}_0$ such that $0 \le \ell \le \frac{1}{2}n$, $0 \le x \le n$ and $x - n$ is even. The edge-crossing numbers (M_n^+, M_n^-) of the simple random walk defined in (3.69) conditionally on $\{\ell_n(0) = \ell, S_n = x\}$ have the same joint distribution as the branching process with law $\mathcal{P}_{\ell,x}$ conditionally on $\{\Xi = \frac{1}{2}(n-x)\}$. In formulas,*

$$\begin{aligned} &\mathrm{P}\Big((M_n^+, M_n^-) = (m^+, m^-),\, \ell_n(0) = \ell,\, S_n = x\Big) \\ &\qquad = \mathcal{P}_{\ell,x}\Big((M^+, M^-) = (m^+, m^-),\, \Xi = \tfrac{1}{2}(n-x)\Big), \end{aligned} \tag{3.73}$$

for all sequences $(m^+, m^-) = (m_y^+, m_y^-)_{y\in\mathbb{N}_0} \in (\mathbb{N}_0 \times \mathbb{N}_0)^{\mathbb{N}_0}$.

Let us now introduce the grand-canonical partition function:

$$\mathcal{Z}(\mu, \delta, \beta) = \sum_{n\in\mathbb{N}_0} e^{-\mu n}\, \mathbb{Z}_n^{*,\delta,\beta}, \qquad \mu \in [0, \infty), \tag{3.74}$$

and observe that the (excess) free energy is then directly linked to the radius of convergence of this series. The key step of our analysis is to use Theorem 3.24 and express the grand-canonical partition function in terms of the matrix $A_{\mu,\delta,\beta}$ defined in (3.51):

$$\mathcal{Z}(\mu, \delta, \beta) = \left[\frac{1}{1 - \widetilde{A}_{\mu,\delta,\beta}^{\intercal}}\, \widehat{A}_{\mu,\delta,\beta}\, \frac{1 + A_{\mu,\delta,\beta}}{1 - A_{\mu,\delta,\beta}}\, \frac{1}{1 - \widetilde{A}_{\mu,\delta,\beta}}\right](0,0), \tag{3.75}$$

where:

- we write $(1 - A_{\mu,\delta,\beta})^{-1}$ as an abuse of notation for $\sum_{n\ge 0} A_{\mu,\delta,\beta}^n$, which is well-defined as a matrix with entries in $[0, \infty]$, since $A_{\mu,\delta,\beta}$ has non-negative entries;
- the matrices $\widetilde{A}_{\mu,\delta,\beta}$ and $\widehat{A}_{\mu,\delta,\beta}$ are modifications of $A_{\mu,\delta,\beta}$ whose precise value is irrelevant at this level of presentation;
- the relevant term for the understanding of what comes next is $(1 - A_{\mu,\delta,\beta})^{-1}$.

It turns out that when $\mu > \mu(\delta, \beta)$ the matrix $\mathrm{Id} - A_{\mu,\delta,\beta}$ is invertible, which leads to $\mathcal{Z}(\mu, \delta, \beta)$ being finite. On the contrary, we can prove that $\mathcal{Z}(\mu, \delta, \beta)$ is infinite as soon as $0 \le \mu < \mu(\delta, \beta)$, hence

$$\mu(\delta, \beta) = \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{Z}_n^{*,\delta,\beta}. \tag{3.76}$$

It remains to prove that the lim sup is actually a limit. To this end, we restrict the partition function to bridges, that are the random walk paths satisfying the condition $0 < S_k \le S_n$ for all $0 < k \le n$, and prove that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{Z}_n^{*,\delta,\beta} \ge \lim_{n\to\infty} \frac{1}{n} \log \mathbb{Z}_{n,\mathrm{bridge}}^{*,\delta,\beta} = \mu(\delta, \beta). \tag{3.77}$$

The equality above comes from the relation (with self-explanatory notation)

$$\mathcal{Z}_{\mathrm{bridge}}(\mu, \delta, \beta) = \left[\frac{A_{\mu,\delta,\beta}}{1 - A_{\mu,\delta,\beta}}\right](0,0), \tag{3.78}$$

and the fact that the limit exists for the bridge version comes from super-additivity of the sequence $(\log \mathbb{Z}_{n,\mathrm{bridge}}^{*,\delta,\beta})_{n\in\mathbb{N}}$, the concatenation of two bridges being again a bridge. The combination of (3.76) and (3.77) finally settles Theorem 3.15.

Large Deviations Principles and Law of Large Numbers in $d = 1$.

The Law of Large Numbers for the endpoint of the polymer and the empirical charge, which are stated in Theorems 3.17 and 3.18, actually come as a corollary of Large Deviations Principles. To obtain such a result, we begin by introducing the joint moment-generating function for the speed and the charge. Fix $(\delta, \beta) \in \mathcal{Q}$ and $(\gamma, \gamma') \in \mathbb{R}^2$. Let

$$\mathbb{Z}_n^{*,\delta,\beta}(\gamma, \gamma') := \mathrm{E}\Big[e^{\gamma S_n} \prod_{x\in\mathbb{Z}} g^*_{\delta+\gamma',\beta}(\ell_n(x))\Big]. \tag{3.79}$$

Then

$$\mathbb{E}_n^{\delta,\beta}\left[e^{\gamma S_n+\gamma'\Omega_n}\right] = \frac{\mathbb{Z}_n^{*,\delta,\beta}(\gamma, \gamma')}{\mathbb{Z}_n^{*,\delta,\beta}(0, 0)}, \tag{3.80}$$

where we recall that $\mathbb{E}_n^{\delta,\beta}$ is the expectation w.r.t. the annealed polymer measure of length n defined in (3.11–3.12). Next, let

$$\mathcal{Z}(\mu, \delta, \beta; \gamma, \gamma') = \sum_{n\in\mathbb{N}_0} e^{-\mu n}\mathbb{Z}_n^{*,\delta,\beta}(\gamma, \gamma'). \tag{3.81}$$

Then $\mathcal{Z}(\mu, \delta, \beta; \gamma, \gamma')$ has a spectral representation similar to the one in (3.75). Indeed, the only difference is that $A_{\mu,\delta,\beta}$ must be replaced by $e^{\gamma}A_{\mu,\delta+\gamma',\beta}$. From this spectral representation one can compute the limiting log-moment generating function

$$\Lambda_{\delta,\beta}(\gamma, \gamma') = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_n^{\delta,\beta}\left[e^{\gamma S_n+\gamma'\Omega_n}\right] \tag{3.82}$$

and deduce, after a suitable Legendre transform, the rate functions $v \mapsto I_{\delta,\beta}(v)$ and $\rho \mapsto J_{\delta,\beta}(\rho)$ for the Large Deviation Principles associated to S_n/n and Ω_n/n under the annealed polymer measure. Provided $\beta \neq \beta_c(\delta)$, these rate functions have each a unique zero, denoted by $v(\delta, \beta)$ and $\rho(\delta, \beta)$ respectively, which turn out to be positive inside the extended phase and zero in the collapsed phase. This gives the Law of Large Numbers outside of criticality, the case $\beta = \beta_c(\delta)$ requiring a separate argument for which we refer to the original paper [32]. Interestingly, the rate functions also contain flat pieces (i.e. convex but non strictly convex). These flat pieces indicate a Non-Cramer regime in which the polymer adopts an inhomogeneous strategy when it is squeezed too much, see Figure 3.7 below (we restrict to the rate function associated to S_n/n, for conciseness).

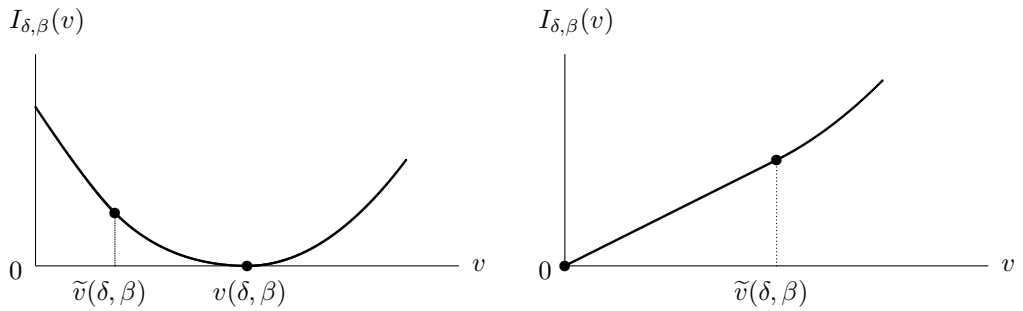

Figure 3.7: Qualitative plot of $v \mapsto I_{\delta,\beta}(v)$ for $(\delta, \beta) \in \mathrm{int}(\mathcal{B})$ (left) and $(\delta, \beta) \in \mathcal{S}$ (right). The flat pieces correspond to an inhomogeneous strategy for the polymer to realise a large deviation. For instance, in the flat piece on the left, if the speed is $v < \widetilde{v}(\delta, \beta)$, then the charge makes a large deviation on a stretch of the polymer of length $v/\widetilde{v}(\delta, \beta)$ times the total length, so as to allow it to move at speed $\widetilde{v}(\delta, \beta)$ along that stretch at zero cost, and then makes a large deviation on the remaining stretch, so as to allow it to be subballistic along that remaining stretch at zero cost.

Heuristics for the weak-interaction asymptotics of the critical curve (Theorem 3.11 (ii) and (iii)).

Let us consider Example 3.10 (standard normal case) and extend (3.41) to $\ell \in \mathbb{R}^+$. For convenience, we re-parametrize

$$t = t(\ell) = 2\beta\ell, \tag{3.83}$$

so that

$$G_{\delta,\beta}(\ell) := \log g^*_{\delta,\beta}(\ell) = \frac{1}{2}\Big\{\frac{\delta^2}{2\beta}\frac{t}{1+t} - \log(1+t)\Big\}. \tag{3.84}$$

By differentiating, we see that $G_{\delta,\beta}$ attains its maximum at

$$t_* = t_*(\delta,\beta) := \frac{\delta^2}{2\beta} - 1, \qquad \text{with} \quad t_* \geq 0 \quad \Leftrightarrow \quad \beta \leq \tfrac{1}{2}\delta^2. \tag{3.85}$$

A quick look at the shape of the function $G_{\delta,\beta}$, depending on the sign of t_* suggests that, at first order, $\beta_c(\delta) \approx \frac{1}{2}\delta^2$. In order to determine the next order term, we henceforth set

$$\beta = \tfrac{1}{2}\delta^2[1 - e(\delta)], \qquad \text{with} \quad \lim_{\delta\downarrow 0} e(\delta) = 0. \tag{3.86}$$

Assuming that t takes values close to $t_* \sim e(\delta)$ and expanding $G_{\delta,\beta}(\ell)$ for small values of δ (or β), we get

$$G_{\delta,\beta}(\ell) \approx \tfrac{1}{2}e(\delta)t - \tfrac{1}{4}t^2 = \beta e(\delta)\ell - \beta^2\ell^2. \tag{3.87}$$

Let us now apply the relation above to $\ell_n(x)$ and sum over $x \in \mathbb{Z}^d$. Recalling that

$$\sum_{x\in\mathbb{Z}^d} \ell_n(x) = n, \qquad \text{and} \qquad \sum_{x\in\mathbb{Z}^d} \ell_n(x)^2 = Q_n, \tag{3.88}$$

(see (3.13)) we get

$$\sum_{x\in\mathbb{Z}^d} G_{\delta,\beta}(\ell_n(x)) \approx \beta e(\delta)n - \beta^2 Q_n. \tag{3.89}$$

This leads to the following approximation:

$$\mathrm{F}^*(\delta,\beta) \approx \beta e(\delta) - \mathrm{F}^{\mathrm{wsaw}}(\beta^2). \tag{3.90}$$

By equating the right-hand side to zero and using Proposition 3.4, we eventually find

$$e(\delta) \sim \begin{cases} \lambda_1(\frac{1}{2}\delta^2)^{1/3} & (d=1),\\ \lambda_2(\frac{1}{2}\delta^2)|\log(\frac{1}{2}\delta^2)| & (d=2),\\ \lambda_d(\frac{1}{2}\delta^2) & (d\geq 3). \end{cases} \tag{3.91}$$

Let us stress, however, that (3.90) cannot be valid when the right-hand side becomes negative, since the excess free energy is always non-negative. Item (iii) of Theorem 3.11, that is for $d \geq 2$, actually follows from a rigorous adaptation of these heuristics [20] but the results obtained are not as sharp as those announced above, see Conjecture 3.26 below.

For $d = 1$, the results of Theorem 3.11 (ii) and its complement in Theorem 3.20 are derived in [32] with a different strategy, which we sketch now. Recall that $\beta_c(\delta)$ is the solution to $\lambda_{\delta,\beta}(0) = 1$. The starting point is the following Rayleigh-Ritz formula:

$$\lambda_{\delta,\beta}(\mu) = \sup_{\substack{v\in\ell_2(\mathbb{N}_0)\\ v\geq 0, \|v\|_2=1}} \langle v, A_{\mu,\delta,\beta}v\rangle. \tag{3.92}$$

By substracting $1 = \langle v, v\rangle$ and changing sequences to functions, we obtain a functional $\mathcal{I}_{\delta,\beta}$ such that

$$\lambda_{\delta,\beta}(0) - 1 = \sup_{\substack{f\in L^2(0,\infty)\\ f\geq 0, \|f\|_2=1}} \mathcal{I}_{\delta,\beta}(f). \tag{3.93}$$

It turns out that (provided f has enough regularity)

$$\lim_{\delta,\beta\downarrow 0} \mathcal{I}_{\delta,\beta}(f) = \mathcal{I}^{(a)}(f) := \langle f, \mathcal{L}^{a,1} f\rangle, \tag{3.94}$$

under the constraint

$$\beta = \tfrac{1}{2}\delta^2 - a\Big(\tfrac{1}{2}\delta^2\Big)^{4/3}, \tag{3.95}$$

with $\mathcal{L}^{a,1}$ defined in (3.62) and $a \in \mathbb{R}$. By using the tools of variational convergence (epi-convergence) we show that, under the same condition,

$$\lim_{\delta,\beta\downarrow 0} \beta^{-2/3}(\lambda_{\delta,\beta}(0) - 1) = \chi(a, 1) \tag{3.96}$$

with $\chi(a, 1)$ defined in (3.64). From this relation we finally infer that the critical value of a is indeed $a^*(1)$, as defined below (3.65). Interestingly, one can finally reconciliate this approach and the one leading to (3.91), by noticing that

$$a^*(1) = \lim_{\beta\downarrow 0} \frac{\mathrm{F}^{\mathrm{wsaw}}(\beta)}{\beta^{2/3}}, \tag{3.97}$$

see [137, Theorem 4].

3.5 Discussion and open questions

(1) Charges with long-range interaction. It would be interesting to deal with charges whose interaction extends beyond the 'on-site' interaction in (3.7), like a Coulomb potential (polynomial decay) or a Yukawa potential (exponential decay).

(2) A conjecture for the weakly self-avoiding walk. We complement Proposition 3.4 by stating a conjecture for the higher order terms in the asymptotic expansion of $\mathrm{F}^{\mathrm{wsaw}}(u)$ for $d \geq 3$.

Conjecture 3.25. *There are constants $a_d > 0$ such that*

$$\lambda_d u - \mathrm{F}^{\mathrm{wsaw}}(u) \sim \begin{cases} a_3\, u^{3/2}, & d = 3,\\ a_4\, u^2 \log(1/u), & d = 4,\\ a_d\, u^2, & d \geq 5, \end{cases} \qquad \text{as } u \downarrow 0. \tag{3.98}$$

This translates into a related conjecture for the rate function I in Proposition 3.5: we conjecture that there are constants $\widetilde{a}_d > 0$ such that

$$I(\lambda_d - s) \sim \begin{cases} \widetilde{a}_3\, s^3, & d = 3,\\ \widetilde{a}_4\, s^2/\log(1/s), & d = 4,\\ \widetilde{a}_d\, s^2, & d \geq 5, \end{cases} \qquad \text{as } s \downarrow 0. \tag{3.99}$$

We refer to [20] for some heuristic arguments that support Conjecture 3.25.

(3a) Scaling inside the collapsed phase [annealed, $d \geq 1$]. Theorem 3.11(i) corrects a mistake in den Hollander [53, Chapter 8], where it was argued that $\mathrm{F}^* \equiv 0$ (i.e., $\mathcal{C}$ covers the full quadrant,

or $\beta_c \equiv 0$). The mistake can be traced back to a failure of convexity of the function $\ell \mapsto G^*_{\delta,\beta}(\ell) = \log g^*_{\delta,\beta}(\ell)$. Is is stated in den Hollander [53, Chapter 8] that for every $d \geq 1$ and every $(\delta, \beta) \in \text{int}(\mathcal{C})$,

$$\lim_{n\to\infty} \frac{(\alpha_n)^2}{n} \log \mathbb{Z}_n^{*,\delta,\beta} = -\chi_d, \tag{3.100}$$

with $\alpha_n = (n/\log n)^{1/(d+2)}$ and with $\chi_d \in (0, \infty)$ a constant that is explicitly computable. The idea behind (3.100) is that the empirical charge makes a large deviation under the law $\mathbb{P}^\delta$ so that it becomes zero. The price for this large deviation is

$$\exp(-nH(\mathbb{P}^0 \,|\, \mathbb{P}^\delta)[1 + o(1)]), \qquad n \to \infty, \tag{3.101}$$

where $H(\mathbb{P}^0 \,|\, \mathbb{P}^\delta)$ denotes the specific relative entropy of $\mathbb{P}^0 = \mathbb{P}$ with respect to $\mathbb{P}^\delta$. Since the latter equals $\log M(\delta) = -f(\delta)$, this accounts for the term that is subtracted in the excess free energy. Conditional on the empirical charge being zero, the attraction between charged monomers with the same sign *wins* from the repulsion between charged monomers with opposite sign, making the polymer chain contract to a *subdiffusive* scale α_n. This accounts for the correction term in the free energy. It is also stated in [53] that, under the law $\mathbb{P}_n^{\delta,\beta}$,

$$\left(\frac{1}{\alpha_n} S_{\lfloor nt \rfloor}\right)_{0\leq t\leq 1} \Longrightarrow (U_t)_{0\leq t\leq 1}, \qquad n \to \infty, \tag{3.102}$$

where $\Longrightarrow$ denotes convergence in distribution and $(U_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^d$ conditioned not to leave a ball with a deterministic radius and a randomly shifted center (see Fig. 3.8). Compactification is a key step in the sketch of the proof provided in den Hollander [53, Chapter 8], which requires super-additivity of $\{\log g^*_{\delta,\beta}(\ell)\}_{\ell\in\mathbb{N}}$. From Theorem 3.23(1) we know that this property holds at least for β large enough. *

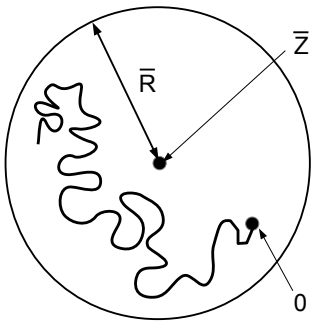

Figure 3.8: A Brownian motion starting at 0 conditioned to stay inside the ball with radius $\bar{R}$ and center $\bar{Z}$. A formula for $\bar{R}$ is given in [53, Chapter 8].

(3b) Scaling inside the extended phase [annealed, $d \geq 1$]. It is natural to expect that for every $(\delta, \beta) \in \mathcal{E}$ the polymer behaves like *weakly self-avoiding walk*. Once the empirical charge is strictly positive, the repulsion should win from the attraction, and the polymer should scale as if all the charges were strictly positive, with a change of time scale only. We have proved it for $d = 1$ but it remains open for $d \geq 2$.

* The scaling property in (3.100) was first noted by Biskup and König [26], as a by-product of their analysis of the parabolic Anderson model.

(4) Weak interaction limits [annealed, $d = 1$]. Theorem 3.21 deals with weak interaction limits and shows that near the horizontal axis in Fig. 3.3 the free energy, the speed and the charge exhibit an anomalous scaling. This is a generalization of the scaling found in van der Hofstad and den Hollander [137] for weakly self-avoiding walk. The constants $A_\delta, B_\delta, C_\delta$ are expected to represent the free energy, speed and charge of a Brownian version of the charged polymer with Hamiltonian

$$H_T^{\widetilde{\mathrm{W}}}(W[0,T]) = \int_{\mathbb{R}} L_T^{\widetilde{\mathrm{W}}}(x)^2 \,\mathrm{d}x, \quad L_T^{\widetilde{\mathrm{W}}}(x) = \int_0^T \mathrm{d}\widetilde{\mathrm{W}}_s \,\delta(W_s - x), \tag{3.103}$$

where $W[0,T]$ is the path of the polymer, $\mathrm{d}\widetilde{\mathrm{W}}_s$ is the charge of the interval $\mathrm{d}s$, $\widetilde{\mathrm{W}}[0,T]$ is an independent Brownian motion with drift δ, and the polymer measure has $\beta = 1$ with the Wiener measure as reference measure. The version without charges is known as the Edwards model (see van der Hofstad, den Hollander and König [138, 139]). The expressions in (3.103) are formal but it would be interesting to give them a meaning.

(5) Central limit theorems and fluctuations at criticality [annealed, $d = 1$]. In [32] we provide the central limit theorem for the speed and the charge in the interior of the ballistic regime. We do not deduce the central limit theorem from the large deviation principle, but rather exploit finer properties of the spectral representation for the excess free energy. The proof is actually inspired by König [89]. We have no result about the fluctuations of the polymer endpoint at criticality, but we expect these fluctuations to be of order $\sqrt{n}$ in the upward direction and of order $n^{2/3}$ in the downward direction.

(6a) The annealed critical curve in $d \geq 2$. The lower and upper bounds in Theorem 3.11(iii) differ by a multiplicative factor when $d \geq 3$ and by a logarithmic factor when $d = 2$. We expect that the upper bound gives the right asymptotic behaviour:

Conjecture 3.26. *As $\delta \downarrow 0$,*

$$\varepsilon_\delta \sim \begin{cases} \kappa_2 \delta^4 \log(1/\delta), & d = 2, \\ \kappa_d \delta^4, & d \geq 3. \end{cases} \tag{3.104}$$

In [20] we state a conjecture about trimmed local times that would imply Conjecture 3.26.

(6b) Annealed free energy in $d \geq 2$. We collect a list of open questions:

(1) Theorem 3.23 settles the existence of the free energy in a subset of the collapsed phase for a subclass of charge distributions. Is the limes superior in (3.34) always a limit, like for $d = 1$?

(2) Is $(\delta, \beta) \mapsto \mathrm{F}^*(\delta, \beta)$ analytic throughout the extended phase $\mathcal{E}$, as in $d = 1$? This would prove that there are no sub-phases inside the extended phase.

(3) How does $\mathrm{F}^*(\delta, \beta)$ behave as $\beta \uparrow \beta_c(\delta)$? Is the phase transition first order, as for $d = 1$, or higher order?

(4) In analogy with what we saw in Theorem 3.11(iii), the bounds in Theorem 3.22 do not match, but we expect the following:

Conjecture 3.27. *For any $\delta \in (0, \infty)$, as $\beta \downarrow 0$,*

$$\mathrm{F}(\delta, \beta) \sim \begin{cases} -\lambda_2 m(\delta)^2 \,\beta \log(1/\beta), & d = 2, \\ -\big[\lambda_d m(\delta)^2 + v(\delta)\big]\,\beta, & d \geq 3. \end{cases} \tag{3.105}$$

(7) More about the quenched charged polymer. Here are some open problems for the *quenched* version of the model:

(1) Does the quenched free energy exist for $\mathbb{P}^\delta$-a.e. ω, and is it constant? How does it depend on δ and β? Trivially, it is convex in β for all δ, but what more can be said?

(2) In $d = 1$, is the quenched charged polymer ballistic for all $\delta \in (0, \infty)$? How does the speed depend on β and δ?

(3) In the quenched model with $\delta = 0$, is the polymer chain subdiffusive (like in the annealed model; see discussion above)? The fluctuations of the charges are expected to push the polymer further apart than in the annealed model. Is there a scaling limit for $\mathbb{P}$-a.e. ω, or does the polymer chain fluctuate so much that there is a scaling limit only along ω-dependent subsequences ("sample dependence")?

(4) Is the scaling of the polymer always similar to that of the *self-avoiding walk* when the average charge is non-zero?

3.6 Perspectives on folding

In this final section we discuss two other random walk models that exhibit a folding mechanism.

3.6.1 Swiss Cheese : towards a microscopic limit

Let $d \geq 3$ and $B = (B_s)_{s\geq 0}$ be a d-dimensional standard Brownian motion. For $t > 0$ and $a > 0$, let $W^a(t) = \{x \in \mathbb{R}^d \colon \exists s \in [0,t],\ \|x - B_s\| \leq a\}$ be the Wiener sausage at time t and radius a. In [135], van den Berg, Bolthausen and den Hollander proved the following Moderate Deviations Principle for the volume of the Wiener sausage:

$$\lim_{t\to\infty} t^{\frac{2}{d}-1} \log \mathrm{P}(|W^a(t)| \leq bt) = -I^{\kappa_a}(b), \tag{3.106}$$

where

$$I^{\kappa_a}(b) := \inf_{\phi\in\Phi^{\kappa_a}(b)} \Big[\frac{1}{2}\int_{\mathbb{R}^d} |\nabla\phi|^2(x)\mathrm{d}x\Big], \qquad \kappa_a := \mathrm{cap}(\mathcal{B}(0,a)) = \lim_{t\to\infty}\frac{1}{t}\mathrm{E}(|W^a(t)|), \tag{3.107}$$

and

$$\Phi^{\kappa_a}(b) := \Big\{\phi \in H^1(\mathbb{R}^d) \colon \int \phi^2(x)\mathrm{d}x = 1, \int (1 - e^{-\kappa_a\phi^2(x)})\mathrm{d}x \leq b\Big\}. \tag{3.108}$$

Following the motto of [135], the picture to have in mind is that of *a Wiener sausage which folds into a Swiss cheese*: the Wiener sausage conditioned to have volume less that bt (where $b < \kappa_a$) spends a fraction of time $\phi^2(x)$ in the neighbourhood $xt^{1/d}$, covering only a fraction $1 - \exp(-\kappa_a\phi^2(x))$ of this neighbourhood and leaving random holes. However, the authors of [135] did not prove such a pathwise statement. Interestingly, in the case $3 \leq d \leq 4$ the variational problem in (3.107) has a minimiser for all $b \in (0, \kappa_a)$ whereas in the case $d \geq 5$, the existence of a minimiser is only guaranteed for $b < \kappa_a^* < \kappa_a$. It was also proved in [135] that any minimiser is (up to a shift) a positive radially symmetric and decreasing (RSD) function. The shift corresponds to a randomly located center at which the Swiss cheese strategy is expected to be performed.

The Moderate Deviations Principle has been adapted to the discrete case by Phetpradap in its PhD dissertation [110]. It reads, with the same rate function,

$$\lim_{n\to\infty} n^{\frac{2}{d}-1} \log \mathrm{P}(|R_n| \leq bn) = -\frac{1}{d} I^{\kappa_d}(b), \tag{3.109}$$

where $R_n = \{S_1, \ldots, S_n\}$ is the range of a simple random walk $S = (S_n)_{n\geq 0}$ on $\mathbb{Z}^d$ and, in this set-up,

$$\kappa_d := \mathrm{cap}(\{0\}) = \mathrm{P}(\forall n \in \mathbb{N},\ S_n \neq S_0) \in (0,1), \tag{3.110}$$

which is also the expected number of vertices visited by the walk per unit of time, in the large n limit. It was recently suggested in [97] that tilted interlacements (which is a space-modulated version of the random interlacement introduced by Sznitman and briefly presented in Sections 2.3.2 and 4.5.2) could give a microscopic description of the Swiss cheese in the discrete set-up. In an ongoing project with D. Erhard we attack the problem of determining the precise microscopic limit and conjecture the following:

Conjecture 3.28. *For all $b < \kappa_d$ if $d \in \{3, 4\}$, or for b small enough if $d \geq 5$, A any finite set in $\mathbb{Z}^d$ and $x \neq 0$,*

$$\lim_{n\to\infty} \mathrm{P}(R_n \cap (A + \lfloor xn^{1/d} \rfloor) = \emptyset \mid |R_n| \leq bn) = \frac{\int \exp(-\phi_b^2(x-y)\mathrm{cap}(A))\phi_b(y)\mathrm{d}y}{\int \phi_b(y)\mathrm{d}y}, \tag{3.111}$$

where ϕ_b is the RSD solution to the corresponding variational problem.

In words, this means that, conditional on having an atypically small volume, the trace of the random walk in the neighbourhood of $\lfloor xn^{1/d} \rfloor$ (up to a randomly located center) is that of a random interlacement with intensity $\phi_b^2(x)$. We do not use tilted interlacements to formulate our conjecture, but ϕ_b does modulate in space the intensity of the random interlacement. In order to simplify the problem and remove the question of the random center, we propose to consider a random walk running up to time $n = \lfloor uN^d \rfloor$ ($u > 0$) on a torus of size N. In this set-up, the conjecture becomes:

Conjecture 3.29. *Let $\mathrm{P}^{(N)}$ be the law of the simple random walk on $(\mathbb{Z}/N\mathbb{Z})^d$ and suppose that S_0 is uniformly distributed on $(\mathbb{Z}/N\mathbb{Z})^d$. Let*

$$\kappa_{d,u} := \lim_{N\to\infty} \frac{1}{n} \mathrm{E}^{(N)}(|R_n|). \tag{3.112}$$

For $b < \kappa_{d,u}$ we have

$$\lim_{N\to\infty} \mathrm{P}^{(N)}(R_n \cap A = \emptyset \mid |R_n| \leq bn) = \int_{y\in(\mathbb{R}/\mathbb{Z})^d} \exp(-u\phi_b^2(y)\mathrm{cap}(A))\mathrm{d}y. \tag{3.113}$$

where ϕ_b is the RSD solution to the corresponding variational problem on the torus.

We have implicitly stated that in each mentioned case there is a unique RSD minimiser to the variational problem (modulo space shifts). This statement, which has not yet been proved, is actually a substantial part of our ongoing work. We have a proof on the full space, but the case of the torus seems challenging. Finally, we also refer to [9,11] for recent developments in the pathwise description of folding in this context.

3.6.2 The Random Interacting Partially Directed Self-Avoiding Walk

The interacting partially directed self-avoiding walk (IPDSAW) is another instance of a polymer model that undergoes a folding (or collapse) transition. This model, which was introduced by Zwanzig and Lauritzen in 1968 [142] and later studied via transfer matrix methods and combinatorial tools, has been recently investigated with a probabilistic approach and a random walk representation which has led to new rigorous results [38,40,41,106]. The polymer is represented by a two-dimensional self-avoiding square lattice path whose steps are only in the right, up or bottom direction, or equivalently, by a series of oriented vertical stretches:

$$\Omega_L = \bigcup_{1\leq N\leq L} \mathcal{L}_{N,L}, \qquad \mathcal{L}_{N,L} = \Big\{\ell \in \mathbb{Z}^N \colon \sum_{1\leq n\leq N} |\ell_n| + N = L\Big\}. \tag{3.114}$$

Here, L is the total length of the polymer, which contains N vertical stretches with respective lengths $(\ell_n)_{1\leq n\leq N}$. The Hamiltonian of the IPDSAW counts the number of self-touchings, which

are defined as the pairs of non-consecutive monomers which are adjacent on the lattice:

$$H_L(\ell_1, \dots, \ell_N) = \sum_{1 \le n < N} (|\ell_n| \wedge |\ell_{n+1}|) \mathbf{1}_{\{\ell_n \ell_{n+1} < 0\}}. \tag{3.115}$$

Letting $\beta \in [0, \infty)$ be the strength of the interaction, the free energy of the model is defined by:

$$\mathrm{F}(\beta) = \lim_{L \to \infty} \frac{1}{L} \log Z_{L,\beta}, \qquad Z_{L,\beta} = \sum_{\ell \in \Omega_L} e^{\beta H_L(\ell)}. \tag{3.116}$$

As β increases the model exhibits a non-trivial phase transition, which can be written in terms of the excess free energy $\mathrm{F}_*(\beta) = \mathrm{F}(\beta) - \beta \ge 0$ and happens at a critical point $\beta_c \in (0, \infty)$. The system indeed goes from an extended regime, when $\beta < \beta_c$ and $\mathrm{F}_*(\beta) > 0$, to a collapsed regime, when $\beta > \beta_c$ and $\mathrm{F}_*(\beta) = 0$. In the extended regime, there are of order L vertical stretches which are of $O(1)$ length, while in the collapsed regime the number of self-touchings is saturated and the polymer forms a macroscopic bead with $O(\sqrt{L})$ vertical stretches which are of $O(\sqrt{L})$ length. Close to criticality, the free energy behaves like

$$\mathrm{F}_*(\beta) \sim (\mathrm{Cst})(\beta_c - \beta)^{3/2}, \qquad \beta \uparrow \beta_c, \tag{3.117}$$

with an explicit constant. We refer to the review paper [39] and references therein for a more complete account of all the available results, including scaling limits in the extended, collapsed and critical regimes.

Whether impurities would affect the phase transition of the IPDSAW is listed in [39] as one of the physically relevant and mathematically challenging open questions. This would amount to introduce inhomogeneities in the Hamiltonian (3.115). Does an arbitrary small disorder round up the phase transition in (3.117)? Does the Harris criterion apply in this context (see Section 1.2)? In an ongoing project with N. Pétrélis and N. Torri, we introduce a toy model of a disordered IPDSAW for which we hope to prove disorder relevance, via a chaos expansion of the partition function.

Chapter 4

Wiener sausage percolation

This chapter is based on joint works with D. Erhard and J. Martínez [66, 67].

We consider a continuum percolation model on $\mathbb{R}^d$, where $d \geq 1$. The occupied set is given by the union of independent Wiener sausages with radius $r \geq 0$ running up to time t and starting from random points that are distributed according to a homogeneous Poisson point process. This "Wiener sausage percolation" model could be seen as a simple model for defects randomly distributed in a medium and propagating at random. In Section 4.1 we give a more precise definition of the model and in Section 4.2 we present our two main results: (i) we establish a non-trivial percolation transition as t grows as soon as r is small enough and $d \geq 2$ (with $r > 0$ when $d \geq 4$) (ii) we investigate the asymptotic behaviour of the critical time when $d \geq 4$ and the radius r converges to 0. The proofs of these two results are respectively sketched in Sections 4.3 and 4.4. We close the chapter in Section 4.5 with a discussion on the results, related models and open questions.

Notation. For every $d \geq 1$, we denote by Leb_d the Lebesgue measure on $\mathbb{R}^d$. We use $\|\cdot\|$ for the Euclidean norm on $\mathbb{R}^d$. The open ball with center z and radius r with respect to the Euclidean norm is denoted by $\mathcal{B}(z, r)$. For all $I \subseteq \mathbb{R}^+$, we denote by B_I the set $\{B_t,\, t \in I\}$.

4.1 Definition of the model

The model we consider in this chapter falls into the category of Boolean models. This is a class of continuum percolation models which are obtained by putting independent copies of a (possibly random) set at the locations of a point process, usually chosen to be a Poisson point process. The most standard example is that of a Boolean model of disks, when the basic set is chosen as a Euclidean ball with a possibly random radius. The Boolean model of disks was first introduced by Gilbert [78] to study infinite communication networks, see also Meester and Roy [101] for an overview. Here we consider the case when the basic set is a Wiener sausage, that is the set of all points within a fixed distance to the trajectory of a standard Brownian motion.

We now give a more precise mathematical defintion. Let $\mathcal{E}$ be a Poisson point process with intensity $\lambda\, \mathrm{Leb}_d$, where $\lambda > 0$. Conditionally on $\mathcal{E}$, we give ourselves a collection of independent Brownian motions $\{(B_t^x)_{t\geq 0},\, x \in \mathcal{E}\}$ such that for each $x \in \mathcal{E}$, $B_0^x = x$ and $(B_t^x - x)_{t\geq 0}$ is independent of $\mathcal{E}$ Let P and E be the probability measure and expectation of Brownian motion, respectively. We denote by

$$W_{[0,t]}^{x,r} = \bigcup_{0\leq s\leq t} \mathcal{B}(B_s^x, r) = B_{[0,t]}^x \oplus \mathcal{B}(0, r) \tag{4.1}$$

the Wiener sausage with radius r, started at x and running up to time t.

The object of interest is the *occupied* set defined by

$$\mathcal{O}_{t,r} := \bigcup_{x\in\mathcal{E}} W^{x,r}_{[0,t]}, \qquad \mathcal{O}_t := \bigcup_{x\in\mathcal{E}} B^x_{[0,t]}, \qquad t \geq 0, \quad r > 0. \tag{4.2}$$

Two points x and y in $\mathbb{R}^d$ are said to be connected in $\mathcal{O}_{t,r}$ if and only if there exists a continuous function $\gamma : [0,1] \mapsto \mathcal{O}_{t,r}$ such that $\gamma(0) = x$ and $\gamma(1) = y$. A subset of $\mathcal{O}_{t,r}$ is connected if and only if all of its points are pairwise connected. In the following a connected subset of $\mathcal{O}_{t,r}$ is called a *component*. A component is bounded if it is contained in $\mathcal{B}(0,R)$ for some $R > 0$. Otherwise, the component is said to be unbounded. A *cluster* is a connected component which is maximal in the sense that it is not strictly contained in another connected component. We say that there is percolation at parameters (t,r) if $\mathcal{O}_{t,r}$ contains at least one unbounded cluster.

Remark 4.1. *A more rigorous definition of the model described above can be done along similar lines as in [101, Section 1.4] for the Boolean percolation model. One consequence of that construction is the ergodicity of $\mathcal{O}_{t,r}$ with respect to space shifts.*

Remark 4.2. *The case $t = 0$ and $r > 0$ coincides with the Boolean model of disks with deterministic radius r, for which we know there is a nontrivial percolation phase transition as soon as $d \geq 2$: there exists $\lambda_c(r) \in (0,\infty)$ (depending also on the dimension) such that for all $\lambda < \lambda_c(r)$ the set $\mathcal{O}_{0,r}$ has a.-s. no unbounded cluster, and such that for $\lambda > \lambda_c(r)$ it a.s. has one. We refer to Theorem 2.1 in Gouéré [79] for the fact that $\lambda_c(r) > 0$ and to Meester and Roy [101, Remark on p.52] for the fact that $\lambda_c(r) < \infty$.*

4.2 Results

We first investigate the percolation phase transition as t grows and λ, r are fixed.

Theorem 4.3. *We have the following three cases.*

- *Let $d = 1$. For all $t \geq 0$, the set $\mathcal{O}_t$ has almost surely no unbounded cluster.*
- *Let $d \in \{2,3\}$. There exists $t_c = t_c(\lambda, d) > 0$ such that for $t < t_c$, $\mathcal{O}_t$ has almost surely no unbounded cluster, whereas for $t > t_c$, $\mathcal{O}_t$ has almost surely a unique unbounded cluster.*
- *Let $d \geq 4$ and $r > 0$ small enough so that $\lambda < \lambda_c(r)$ (recall Remark 4.2). There exists $t_c = t_c(\lambda, d, r) > 0$ such that for $t < t_c$, $\mathcal{O}_{t,r}$ has almost surely no unbounded cluster, whereas for $t > t_c$, it has almost surely a unique unbounded cluster.*

Hence there is a non-trivial percolation phase transition as soon as $d \geq 2$, with the extra condition that the radius of the Wiener sausages is chosen positive but small enough when $d \geq 4$. The reason why the radius needs to be positive in this case is that the paths of two independent d-dimensional standard Brownian motions do not intersect when $d \geq 4$ (except at a possibly common starting point). They do when $d \leq 3$, see [104, Theorem 9.1].

We now assume that $d \geq 4$ and consider the behaviour of the critical time as the radius converges to zero and the intensity is kept fixed to $\lambda = 1$. For this reason, we shall now write $t_c(r)$ instead of $t_c(1,r)$. Let us mention that no simple scaling argument seems to immediately yield bounds on $t_c(r)$. Indeed, since for each d there are three parameters (λ, t and r), it is not possible to scale two parameters independently of the third one. One may expect that $t_c(r)$ goes to ∞ as $r \to 0$, since $t_c(0) = \infty$. Note that this is not an immediate consequence of continuity since the event $\{\mathcal{O}_t$ does not percolate$\}$ is not the increasing union of the events $\{\mathcal{O}_{t,r}$ does not percolate$\}$ for $r > 0$. The following theorem however confirms our intuition and determines at which speed the convergence takes place.

Theorem 4.4. *Let $d \geq 4$. There is a constant $c = c(d)$ and $r_0 \in (0,1)$ such that for all $r \in (0, r_0)$,*

$$\begin{aligned} c^{-1}\sqrt{\log(1/r)} \leq t_c(r) \leq c\sqrt{\log(1/r)}, \qquad & \text{if } d = 4, \\ c^{-1} r^{(4-d)/2} \leq t_c(r) \leq c\, r^{(4-d)/2}, \qquad & \text{if } d \geq 5. \end{aligned} \tag{4.3}$$

4.3 Sketch of the proof of Theorem 4.3

Non-percolative regime. The main idea to prove non-percolation at small times is to dominate $\mathcal{O}_{t,r}$ by a Boolean percolation model with radius distribution given by the maximal displacement of a Brownian motion before time t. Standard results on the Boolean model yield non-percolation at small times. Let us mention that in the case $d \geq 4$, additional work is required. Indeed, we need to discard the possibility that λ is supercritical for all $t > 0$ and subcritical at $t = 0$, which means proving continuity of the critical intensity of the Boolean model w.r.t. the radius distribution at δ_r (Dirac mass at r). This is obtained in [66, Proposition 2.2]. The proof requires a renormalization procedure and extends a finite box criterion for non-percolation in the Boolean model to radius distributions with an exponential tail. To our knowledge, such a criterion had only been proved for bounded radii. Moreover, we suspect that it could be extended to radius distributions with sufficiently thin polynomial tails.

Percolative regime. To establish the existence of a percolation phase, we distinguish between two cases:
(1) For $d \in \{2, 3\}$, we use a coarse-graining argument. More precisely, we divide $\mathbb{R}^d$ into boxes and we consider an edge percolation model on the coarse-grained graph whose vertices are identified as the centers of the boxes and the edges connect nearest neighbours. An edge connecting nearest neighbours, say x and x' in $\mathbb{Z}^d$, is said to be open if (i) both boxes associated to x and x' contain at least one point of the Poisson point process, say y and y', and (ii) the Brownian motions starting from y and y' intersect each other. A domination result by Liggett, Schonmann and Stacey [98] finally shows that percolation in that coarse-grained model occurs if one suitably chooses the size of the boxes and let time run for long enough. This implies percolation of our original model.
(2) For $d \geq 4$, our strategy is to construct a $(d-1)$-dimensional supercritical Boolean model included in $\mathcal{O}_{t,r}$. This supercritical model is obtained by considering the trace left by the Wiener sausages on some hyperplane.

Uniqueness of the unbounded cluster. The difficulty in the uniqueness proof lies in extending the Burton-Keane argument to the continuous setting. For this purpose, we exploit ideas from Meester and Roy [100, 101]. The case $d = 3$ turns out to be the most delicate one and requires new ideas such as a careful cutting-and-glueing procedure on the Brownian paths.

4.4 Sketch of the proof of Theorem 4.4

Let us first restrict to $d \geq 5$. We use a technique which has been used in the context of Boolean percolation. This technique consists in exploring the cluster containing the origin, which we denote by $\mathcal{C}(0)$, and comparing it to a (multitype) Galton-Watson branching process, see for instance [101, Section 3.3]. For simplicity, we assume that there is a Poisson point at the origin. The Wiener sausages intersecting the Wiener sausage starting at the origin are called first generation sausages, all other sausages intersecting the first generation sausages constitute the second generation sausages, and so on. Let us define

$$\mathcal{N}(x) = \Big\{ y \in \mathcal{E} \setminus \{x\} \colon W^{x,r}_{[0,t]} \cap W^{y,r}_{[0,t]} \neq \emptyset \Big\}, \qquad x \in \mathcal{E}. \tag{4.4}$$

From what we explained above, we get the following decomposition of $\mathcal{C}(0)$:

$$\mathcal{E}_0 = \{0\}, \qquad \mathcal{E}_{n+1} = \begin{cases} \bigcup_{y\in\mathcal{E}_n} \mathcal{N}(y) \setminus \bigcup_{k=0}^{n} \mathcal{E}_k & \text{if } \mathcal{E}_n \neq \emptyset \\ \emptyset & \text{if } \mathcal{E}_n = \emptyset \end{cases}, \quad n \in \mathbb{N}_0. \tag{4.5}$$

Here $\mathcal{E}_n$ is interpreted as the set of elements in $\mathcal{C}(0)$ at generation n. The idea is to dominate the process $\{|\mathcal{E}_n|\}_{n\in\mathbb{N}_0}$ by a branching process which eventually becomes extinct, thus proving that $\mathcal{C}(0)$ contains finitely many Poisson points, which in turn proves non-percolation. If this branching process would be close (in some reasonable sense) to a Galton Watson process, then it would be enough to control the mean number of offsprings of the Wiener sausage started at the origin.

In order to evaluate the mean number of sausages in the first generation, we use a relation between the probability that a Brownian motion intersects a Borel set $A \subseteq \mathbb{R}^d$ and the Newtonian capacity of A, which we denote by $\mathrm{cap}(A)$ (see e.g. [104, 117, 130] for reminders on potential theory and capacity). This relation writes:

$$\mathrm{P}(A \cap B^x_{[0,\infty)} \neq \emptyset) \sim (\mathrm{cst})\|x\|^{2-d}\mathrm{cap}(A), \qquad \text{as} \quad \|x\| \to \infty. \tag{4.6}$$

Let us then use the rough approximation

$$\mathrm{P}(W^{0,r}_{[0,t]} \cap W^{x,r}_{[0,t]} \neq \emptyset) \approx (\mathrm{cst})\|x\|^{2-d}\mathrm{E}[\mathrm{cap}(W^{0,2r}_{[0,t]})] \tag{4.7}$$

and make the following observations:

- The x's that contribute to the first generation should be at distance of order $\sqrt{t}$ from the origin, by Brownian scaling.
- The quantity $\mathrm{E}[\mathrm{cap}(W^{x,2r}_{[0,t]})]$ is of order tr^{d-4} (we come back below to this non-trivial fact).
- There are of order $\|x\|^{d-1}$ Wiener sausages starting at distance $\|x\|$ from the origin.

By combining all these elements we compute that there should be about t^2r^{d-4} Wiener sausages intersecting the origin. As a consequence, a transition should happen when $t^2r^{d-4} = 1$, that is the critical number of offsprings for a Galton-Watson process, which leads to a critical time of order $r^{\frac{4-d}{2}}$.

However, the Wiener sausages of the first generation are not distributed as Wiener sausages but as Wiener sausages conditioned to intersect $W^{0,r}_{[0,t]}$. These are subject to a *size biasing* effect, meaning that their capacities have a bias towards larger values, compared to the unconditioned Wiener sausage. To overcome this difficulty, we partition the set of Poisson points according to the capacities of their associated Wiener sausages:

$$\mathfrak{E}_j = \left\{ x \in \mathcal{E} \colon \mathrm{cap}(W^{x,2r}_{[0,t]}) \in [j, j+1)tr^{d-4} \right\}, \qquad j \in \mathbb{N}_0, \tag{4.8}$$

and we employ a multitype branching argument.

As we have seen, the proof of Theorem 4.4 strongly relies on the evaluation of the Newtonian capacity of a Wiener sausage, which is performed via moment and large deviations estimates, see [67, Sections 4.2 and 5]. When $d \geq 5$, the fact that $\mathrm{cap}(W^{0,1}_{[0,t]})$ grows *at most* linearly with t comes from sub-additivity arguments, while the fact that it grows *at least* linearly with t comes from evaluating the Coulomb energy of a Wiener sausage, see [67, Lemma 5.1]. The multiplicative factor r^{d-4} that appears when changing the radius from one to r is easily deduced from Brownian and capacity scaling.

The case $d = 4$ needs some amendments. The multitype Galton-Watson argument is essentially the same, except that we classify the Wiener sausages not only depending on the capacity but also on the outradius. Most importantly, the estimates on the capacity of one Wiener sausage are more subtle and therefore require a more careful analysis than in the high dimensional case $d \geq 5$. This is due to the logarithmic correction in the increase of the mutual intersection local time (intersection of two independent Wiener sausages) in four dimensions. We find in this case that $\mathrm{E}[\mathrm{cap}(W^{0,r}_{[0,t]})]$ is of order $t/\log(tr^{-2})$, which leads to a critical time of order $\sqrt{\log(1/r)}$.

Remark 4.5. *Limit theorems for the capacity of both random walks and Wiener sausages have been recently obtained in a series of papers by Asselah, Schapira and Sousi [10, 12–14]. Exponential moments for the inverse capacity of Wiener sausages are considered by van den Berg, Bolthausen and den Hollander [136]. For similar moment estimates in the case of simple random walk, we refer to Ráth and Sapozhnikov [119, 120] and Chang and Sapozhnikov [43, Equation (4)], which uses an estimate of Lawler [94]. See also Lawler's monograph [95] on the closely related question of random walk intersections.*

4.5 Discussion

4.5.1 Comments on the results

(1) In Theorem 4.3 we investigated a phase transition in t, but we could play with the intensity λ instead. Indeed, multiplying the intensity λ by a factor η changes the typical distance between two Poisson points by a factor $\eta^{-1/d}$. Thus, by scale invariance of Brownian motion, the percolative behaviour of the model is the same when we consider the Brownian paths up to time $\eta^{-2/d}t$ instead.

(2) Theorem 4.3 is stated only for $r = 0$ when $d \leq 3$, but the result would be the same for r positive and small enough. If r is too large then $\lambda > \lambda_c(r)$ so the model is supercritical at all times.

(3) If $d \geq 4$ and $\lambda > \lambda_c(r)$ then $\mathcal{O}_{0,r}$ already contains an unbounded component. Therefore there is percolation at all times. In that case, van den Berg, Meester and White [134] proved a stronger result: almost-surely, for all $t \geq 0$, the set $\cup_{x\in\mathcal{E}}\mathcal{B}(B^x_t, r)$ contains an unbounded component.

(4) For completeness, we state that $r \mapsto t_c(r)$ stays bounded as $r \to 0$ when $d \in \{2, 3\}$. Continuity at $r = 0$ is not immediate, but we expect that it follows from a finite-box criterion of percolation. Theorem 4.4 shows in particular that when $d \geq 4$ the critical time is continuous at $r = 0$, since $t_c(0) = \infty$.

4.5.2 Related models: dynamic Boolean percolation and random interlacement

In this section we present two related models.

Random plane networks and mobile sensors. The study of continuum percolation models can be traced back (at least) to the work of Gilbert [78] on random plane networks. Gilbert was interested in modeling infinite communication networks of stations with range $r > 0$. Another application that is mentioned in his work is the modeling of a contagious infection, when each individual gets infected when it is at distance less than r to an infected individual. Černý, Funken and Spodarev [42] describe the target detection area of a network of mobile sensors initially distributed at random and moving according to Brownian dynamics. However, in this work the focus is on numerical computations of coverage probabilities rather than on percolation. In a similar spirit, Kesidis, Kostantopoulos and Phoha [88] provide formulas for the detection time of a target particle positioned at the origin.

Peres, Sinclair, Sousi and Stauffer [108, 109] also study a network of mobile sensors. However, they consider $\cup_{x\in\mathcal{E}}\mathcal{B}(B_t^x, r)$ at *each fixed time* instead of $\mathcal{O}_{t,r}$ and focus on questions related to detection and coverage. The model of mobile sensors is also known under the name of *dynamic Boolean percolation*, see van den Berg, Meester and White [134].

Random interlacement. Random interlacement is a Poisson point process on infinite random walk paths obtained when looking at the trace of a simple random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$ started from the uniform distribution, running up to time uN^d and letting $N \nearrow \infty$, see Sznitman [131] and Section 2.3.2 of this *mémoire*. We expect that, as $t \uparrow \infty$, $\lambda \downarrow 0$ and λt stays constant, while r is fixed, our model shares features with a continuous version of random interlacements, see [132]. Indeed, in the regime described above, the number of Brownian trajectories entering a set A is a Poisson random variable with intensity proportional to $\lambda t\, \mathrm{cap}(A)$, which is a key feature of random interlacements. Moreover, the product of λt serves as an intensity parameter. This limiting regime exhibits long-range dependence, in the sense that if A_1 and A_2 are two bounded sets, then

$$\mathrm{Cov}(\mathbb{1}_{\{A_1\cap\mathcal{O}_t\neq\emptyset\}}, \mathbb{1}_{\{A_2\cap\mathcal{O}_t\neq\emptyset\}}) \sim c\ \mathrm{dist}(A_1, A_2)^{2-d}, \tag{4.9}$$

as $\mathrm{dist}(A_1, A_2) \nearrow \infty$, $t \nearrow \infty$ and λt stays constant. Indeed, the left-hand side becomes asymptotically equivalent to the difference between $\mathrm{cap}(A_1 \cup A_2)$ and $\mathrm{cap}(A_1) + \mathrm{cap}(A_2)$, which has the desired order.

We refer to Menshikov, Molchanov and Sidorenko [102] for other physical motivations of continuum percolation.

4.5.3 Open questions

We close the chapter with a few open questions.

(1) When $d \geq 2$, is there percolation at criticality?

(2) Can we prove a sharper version of Theorem 4.4? Namely, is there a constant $c_* \in (0, \infty)$ such that

$$\lim_{r\to 0} t_c(r)/f(r) = c_*, \quad \text{with} \quad f(r) = \begin{cases} r^{(4-d)/2}, & d \geq 5, \\ \sqrt{\log(1/r)}, & d = 4 \end{cases} \quad ? \tag{4.10}$$

(3) Is there a way to define a limiting random subset of $\mathbb{R}^d$ as we set time to $t(r) = cf(r)$ (see (4.10)) and intensity to $\lambda = 1$ in our model, and let $r \downarrow 0$? Would this limiting object have a percolation phase transition in c and if so, would the critical value of c coincide with the constant c_* in (4.10)? Note that one should beforehand perform a change of parameters. Indeed, for any pair of distinct points x and y in $\mathbb{R}^d$, the intersection of $W^{x,r}_{[0,\infty)}$ and $W^{y,r}_{[0,\infty)}$ becomes eventually empty as $d \geq 4$ and $r \downarrow 0$, meaning that percolation occurs out of arbitrarily large boxes and is thus not visible in the limit. To fix this issue, one may set time to a fixed value, say $t = 1$, so that intersections of Wiener sausages occur in a bounded space window and, in order to be consistent with the previous scaling, let the intensity parameter λ go to infinity while $r = (c^2/\lambda)^{\frac{1}{d-4}}$ if $d \geq 5$ and $r = \lambda^{-\frac{1}{4}} \exp(-\lambda/c)$ if $d = 4$, with the same constant c.

Contents

Bibliography

[1] Alexander, K. S. The effect of disorder on polymer depinning transitions. *Comm. Math. Phys. 279*, 1 (2008), 117–146.

[2] Alexander, K. S. Excursions and local limit theorems for Bessel-like random walks. *Electron. J. Probab. 16* (2011), no. 1, 1–44.

[3] Alexander, K. S., and Berger, Q. Pinning of a renewal on a quenched renewal. *Electron. J. Probab. 23* (2018), Paper No. 6, 48.

[4] Alexander, K. S., and Sidoravicius, V. Pinning of polymers and interfaces by random potentials. *Ann. Appl. Probab. 16*, 2 (2006), 636–669.

[5] Alexander, K. S., and Zygouras, N. Quenched and annealed critical points in polymer pinning models. *Comm. Math. Phys. 291*, 3 (2009), 659–689.

[6] Alexander, K. S., and Zygouras, N. Equality of critical points for polymer depinning transitions with loop exponent one. *Ann. Appl. Probab. 20*, 1 (2010), 356–366.

[7] Antal, P. Enlargement of obstacles for the simple random walk. *Ann. Probab. 23*, 3 (1995), 1061–1101.

[8] Asmussen, S. *Applied probability and queues*, second ed., vol. 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 2003. Stochastic Modelling and Applied Probability.

[9] Asselah, A., and Schapira, B. Moderate deviations for the range of a transient random walk: path concentration. *Ann. Sci. Éc. Norm. Supér. (4) 50*, 3 (2017), 755–786.

[10] Asselah, A., and Schapira, B. Deviations for the capacity of the range of a random walk. arXiv:1807.02325, 2018.

[11] Asselah, A., and Schapira, B. On the nature of the swiss cheese in dimension 3. *arXiv:1805.09194* (05 2018).

[12] Asselah, A., Schapira, B., and Sousi, P. Capacity of the range of random walk on $\mathbb{Z}^d$. *Trans. Amer. Math. Soc. 370*, 11 (2018), 7627–7645.

[13] Asselah, A., Schapira, B., and Sousi, P. Capacity of the range of random walk on $\mathbb{Z}^4$. *Ann. Probab. 47*, 3 (2019), 1447–1497.

[14] Asselah, A., Schapira, B., and Sousi, P. Strong law of large numbers for the capacity of the Wiener sausage in dimension four. *Probab. Theory Related Fields 173*, 3-4 (2019), 813–858.

[15] Aubry, S., and André, G. Analyticity breaking and Anderson localization in incommensurate lattices. In *Group theoretical methods in physics (Proc. Eighth Internat. Colloq., Kiryat Anavim, 1979)* (1980), vol. 3 of *Ann. Israel Phys. Soc.*, Hilger, Bristol, pp. 133–164.

[16] Berestycki, N., and Cerf, R. The random walk penalised by its range in dimensions $d \geq 3$. *arXiv:1811.04700* (2018).

[17] Berger, Q. Comments on the influence of disorder for pinning model in correlated Gaussian environment. *ALEA Lat. Am. J. Probab. Math. Stat. 10*, 2 (2013), 953–977.

[18] Berger, Q. Pinning model in random correlated environment: appearance of an *infinite disorder* regime. *J. Stat. Phys. 155*, 3 (2014), 544–570.

[19] Berger, Q., Caravenna, F., Poisat, J., Sun, R., and Zygouras, N. The critical curves of the random pinning and copolymer models at weak coupling. *Comm. Math. Phys. 326*, 2 (2014), 507–530.

[20] Berger, Q., den Hollander, F., and Poisat, J. Annealed scaling for a charged polymer in dimensions two and higher. *J. Phys. A 51*, 5 (2018), 054002, 37.

[21] Berger, Q., and Lacoin, H. Sharp critical behavior for pinning models in a random correlated environment. *Stochastic Process. Appl. 122*, 4 (2012), 1397–1436.

[22] Berger, Q., and Lacoin, H. Pinning on a defect line: characterization of marginal disorder relevance and sharp asymptotics for the critical point shift. *J. Inst. Math. Jussieu 17*, 2 (2018), 305–346.

[23] Berger, Q., and Poisat, J. On the critical curves of the pinning and copolymer models in correlated Gaussian environment. *Electron. J. Probab. 20* (2015), no. 71, 35.

[24] Berger, Q., and Toninelli, F. L. Hierarchical pinning model in correlated random environment. *Ann. Inst. Henri Poincaré Probab. Stat. 49*, 3 (2013), 781–816.

[25] Birkner, M., Greven, A., and den Hollander, F. Quenched large deviation principle for words in a letter sequence. *Probab. Theory Related Fields 148*, 3-4 (2010), 403–456.

[26] Biskup, M., and König, W. Long-time tails in the parabolic Anderson model with bounded potential. *Ann. Probab. 29*, 2 (2001), 636–682.

[27] Bolthausen, E. Localization of a two-dimensional random walk with an attractive path interaction. *Ann. Probab. 22*, 2 (1994), 875–918.

[28] Bolthausen, E., and den Hollander, F. Localization transition for a polymer near an interface. *Ann. Probab. 25*, 3 (1997), 1334–1366.

[29] Bolthausen, E., den Hollander, F., and Opoku, A. A. A copolymer near a selective interface: variational characterization of the free energy. *Ann. Probab. 43*, 2 (2015), 875–933.

[30] Brak, R., Owczarek, A. L., Rechnitzer, A., and Whittington, S. G. A directed walk model of a long chain polymer in a slit with attractive walls. *J. Phys. A 38*, 20 (2005), 4309–4325.

[31] Caravenna, F., and den Hollander, F. A general smoothing inequality for disordered polymers. *Electron. Commun. Probab. 18* (2013), no. 76, 15.

[32] Caravenna, F., den Hollander, F., Pétrélis, N., and Poisat, J. Annealed scaling for a charged polymer. *Math. Phys. Anal. Geom. 19*, 1 (2016), Art. 2, 87.

[33] Caravenna, F., and Giacomin, G. The weak coupling limit of disordered copolymer models. *Ann. Probab. 38*, 6 (2010), 2322–2378.

[34] Caravenna, F., and Pétrélis, N. Depinning of a polymer in a multi-interface medium. *Electron. J. Probab. 14* (2009), no. 70, 2038–2067.

[35] Caravenna, F., Sun, R., and Zygouras, N. The continuum disordered pinning model. *Probab. Theory Relat. Fields 164* (2016), 17–59.

[36] Caravenna, F., Sun, R., and Zygouras, N. Polynomial chaos and scaling limits of disordered systems. *J. Eur. Math. Soc. (JEMS) 19* (2017), 1–65.

[37] Caravenna, F., Toninelli, F. L., and Torri, N. Universality for the pinning model in the weak coupling regime. *Ann. Probab. 45*, 4 (2017), 2154–2209.

[38] Carmona, P., Nguyen, G. B., and Pétrélis, N. Interacting partially directed self avoiding walk. From phase transition to the geometry of the collapsed phase. *Ann. Probab. 44*, 5 (2016), 3234–3290.

[39] Carmona, P., Nguyen, G. B., Pétrélis, N., and Torri, N. Interacting partially directed self-avoiding walk: a probabilistic perspective. *J. Phys. A 51*, 15 (2018), 153001, 23.

[40] Carmona, P., and Pétrélis, N. Interacting partially directed self avoiding walk: scaling limits. *Electron. J. Probab. 21* (2016), Paper No. 49, 52.

[41] Carmona, P., and Pétrélis, N. A shape theorem for the scaling limit of the IPDSAW at criticality. *Ann. Appl. Probab. 29*, 2 (2019), 875–930.

[42] Černý, R., Funken, S., and Spodarev, E. On the Boolean model of Wiener sausages. *Methodol. Comput. Appl. Probab. 10*, 1 (2008), 23–37.

[43] Chang, Y., and Sapozhnikov, A. Phase transition in loop percolation. *Probab. Theory Related Fields 164*, 3-4 (2016), 979–1025.

[44] Cheliotis, D., Chino, Y., and Poisat, J. The random pinning model with correlated disorder given by a renewal set. *Ann. H. Lebesgue 2* (2019), 281–329.

[45] Cheliotis, D., and den Hollander, F. Variational characterization of the critical curve for pinning of random polymers. *Ann. Probab. 41*, 3B (2013), 1767–1805.

[46] Chen, X. *Random walk intersections*, vol. 157 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[47] Chen, X., Dagard, V., Derrida, B., Hu, Y., Lifshits, M., and Shi, Z. The derrida–retaux conjecture on recursive models. *arXiv:1907.01601* (07 2019).

[48] Coddington, E. A., and Levinson, N. *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[49] Cornfeld, I. P., Fomin, S. V., and Sinaĭ, Y. G. *Ergodic theory*, vol. 245 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskiĭ.

[50] de Gennes, P.-G. *Scaling Concepts in Polymer Physics*. Cornell University Press, 1979.

[51] Dembo, A., and Zeitouni, O. *Large deviations techniques and applications*, vol. 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[52] den Hollander, F. *Large deviations*, vol. 14 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2000.

[53] den Hollander, F. *Random polymers*, vol. 1974 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Lectures from the 37th Probability Summer School held in Saint-Flour, 2007.

[54] den Hollander, F., and Poisat, J. Large deviation principles for words drawn from correlated letter sequences. *Electron. Commun. Probab. 19* (2014), no. 12, 16.

[55] Derrida, B., Giacomin, G., Lacoin, H., and Toninelli, F. L. Fractional moment bounds and disorder relevance for pinning models. *Comm. Math. Phys. 287*, 3 (2009), 867–887.

[56] Derrida, B., Hakim, V., and Vannimenus, J. Effect of disorder on two-dimensional wetting. *J. Statist. Phys. 66*, 5-6 (1992), 1189–1213.

[57] Derrida, B., and Retaux, M. The depinning transition in presence of disorder : a toy model. *J. Stat. Phys.* (2014).

[58] Ding, J., Fukushima, R., Sun, R., and Xu, C. Biased random walk conditioned on survival among bernoulli obstacles: subcritical phase. *arXiv:1904.07433* (2019).

[59] Ding, J., Fukushima, R., Sun, R., and Xu, C. Distribution of the random walk conditioned on survival among quenched bernoulli obstacles. arXiv:1910.04449, 2019.

[60] Ding, J., Fukushima, R., Sun, R., and Xu, C. Geometry of the random walk range conditioned on survival among bernoulli obstacles. *arXiv:1806.08319* (2019).

[61] Ding, J., and Xu, C. Localization for random walks among random obstacles in a single Euclidean ball. *ArXiv e-prints* (July 2018).

[62] Ding, J., and Xu, C. Poly-logarithmic localization for random walks among random obstacles. *Ann. Probab. 47*, 4 (2019), 2011–2048.

[63] Donsker, M. D., and Varadhan, S. R. S. On the number of distinct sites visited by a random walk. *Comm. Pure Appl. Math. 32*, 6 (1979), 721–747.

[64] Durrett, R. *Probability: theory and examples*, fourth ed., vol. 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2010.

[65] Ellis, R. S. *Entropy, large deviations, and statistical mechanics*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1985 original.

[66] Erhard, D., Martínez, J., and Poisat, J. Brownian paths homogeneously distributed in space: percolation phase transition and uniqueness of the unbounded cluster. *J. Theoret. Probab. 30*, 3 (2017), 784–812.

[67] Erhard, D., and Poisat, J. Asymptotics of the critical time in Wiener sausage percolation with a small radius. *ALEA Lat. Am. J. Probab. Math. Stat. 13*, 1 (2016), 417–445.

[68] Fisher, M. E. Walks, walls, wetting, and melting. *J. Statist. Phys. 34*, 5-6 (1984), 667–729.

[69] Forgacs, G., Luck, J. M., Nieuwenhuizen, T. M., and Orland, H. Wetting of a disordered substrate: Exact critical behavior in two dimensions. *Phys. Rev. Lett. 57, 2184—2187* (1986).

[70] Frenk, J. B. G. The behavior of the renewal sequence in case the tail of the waiting-time distribution is regularly varying with index -1. *Adv. in Appl. Probab. 14*, 4 (1982), 870–884.

[71] Fukushima, R. From the Lifshitz tail to the quenched survival asymptotics in the trapping problem. *Electron. Commun. Probab. 14* (2009), 435–446.

[72] Giacomin, G. *Random polymer models*. Imperial College Press, London, 2007.

[73] Giacomin, G. *Disorder and critical phenomena through basic probability models*, vol. 2025 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lecture notes from the 40th Probability Summer School held in Saint-Flour, 2010, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[74] Giacomin, G., Lacoin, H., and Toninelli, F. Marginal relevance of disorder for pinning models. *Comm. Pure Appl. Math. 63*, 2 (2010), 233–265.

[75] Giacomin, G., Lacoin, H., and Toninelli, F. L. Disorder relevance at marginality and critical point shift. *Ann. Inst. Henri Poincaré Probab. Stat. 47*, 1 (2011), 148–175.

[76] Giacomin, G., and Toninelli, F. L. Smoothing effect of quenched disorder on polymer depinning transitions. *Comm. Math. Phys. 266*, 1 (2006), 1–16.

[77] Giacomin, G., and Toninelli, F. L. On the irrelevant disorder regime of pinning models. *Ann. Probab. 37*, 5 (2009), 1841–1875.

[78] Gilbert, E. N. Random plane networks. *J. Soc. Indust. Appl. Math. 9* (1961), 533–543.

[79] Gouéré, J.-B. Subcritical regimes in the Poisson Boolean model of continuum percolation. *Ann. Probab. 36*, 4 (2008), 1209–1220.

[80] Grenander, U., and Szegö, G. *Toeplitz forms and their applications*. University of California Press, Berkeley-Los Angeles, 1958.

[81] Greven, A., and den Hollander, F. A variational characterization of the speed of a one-dimensional self-repellent random walk. *Ann. Appl. Probab. 3*, 4 (1993), 1067–1099.

[82] Harris, A. B. Effect of random defects on the critical behaviour of ising models. *J. Phys.* (1974), 1671–1692.

[83] Hu, Y., Mallein, B., and Pain, M. An exactly solvable continuous-time derrida–retaux model, 2018.

[84] Ioffe, D. Multidimensional random polymers: a renewal approach. In *Random walks, random fields, and disordered systems*, vol. 2144 of *Lecture Notes in Math.* Springer, Cham, 2015, pp. 147–210.

[85] Ioffe, D., and Velenik, Y. Self-attractive random walks: the case of critical drifts. *Comm. Math. Phys. 313*, 1 (2012), 209–235.

[86] Jeon, J., Park, P., and Sung, W. The effect of sequence correlation on bubble statistics in double-stranded dna. *The Journal of chemical physics 125* (2006).

[87] Kantor, Y., and Kardar, M. Polymers with random self-interactions. *Europhys. Lett. 14* (1991), 421–426.

[88] Kesidis, G., Konstantopoulos, T., and Phoha, S. Surveillance coverage of sensor networks under a random mobility strategy. In *IEEE Sensors Conference*. 2003.

[89] König, W. A central limit theorem for a one-dimensional polymer measure. *Ann. Probab. 24*, 2 (1996), 1012–1035.

[90] König, W. *The parabolic Anderson model*. Pathways in Mathematics. Birkhäuser/Springer, 2016. Random walk in random potential.

[91] Kosygina, E., and Mountford, T. Crossing velocities for an annealed random walk in a random potential. *Stochastic Process. Appl. 122*, 1 (2012), 277–304.

[92] Lacoin, H. The martingale approach to disorder irrelevance for pinning models. *Electron. Commun. Probab. 15* (2010), 418–427.

[93] Lacoin, H., and Sohier, J. Disorder relevance without Harris criterion: the case of pinning model with γ-stable environment. *Electron. J. Probab. 22* (2017), Paper No. 50, 26.

[94] Lawler, G. F. The probability of intersection of independent random walks in four dimensions. *Comm. Math. Phys. 86*, 4 (1982), 539–554.

[95] Lawler, G. F. *Intersections of random walks*, reprint of the 1996 edition ed. Birkhäuser/Springer, New York, 2013.

[96] Lawler, G. F., and Limic, V. *Random walk: a modern introduction*, vol. 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[97] Li, X., and Sznitman, A.-S. A lower bound for disconnection by random interlacements. *Electron. J. Probab. 19* (2014), no. 17, 26.

[98] Liggett, T. M., Schonmann, R. H., and Stacey, A. M. Domination by product measures. *Ann. Probab. 25*, 1 (1997), 71–95.

[99] Martin, R., Orlandini, E., Owczarek, A. L., Rechnitzer, A., and Whittington, S. G. Exact enumeration and Monte Carlo results for self-avoiding walks in a slab. *J. Phys. A 40*, 27 (2007), 7509–7521.

[100] Meester, R., and Roy, R. Uniqueness of unbounded occupied and vacant components in Boolean models. *Ann. Appl. Probab. 4*, 3 (1994), 933–951.

[101] Meester, R., and Roy, R. *Continuum percolation*, vol. 119 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[102] Menshikov, M. V., Molchanov, S. A., and Sidorenko, A. F. Percolation theory and some applications. In *Probability theory. Mathematical statistics. Theoretical cybernetics, Vol. 24 (Russian)*, Itogi Nauki i Tekhniki. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1986, pp. 53–110, i. Translated in J. Soviet Math. **42** (1988), no. 4, 1766–1810.

[103] Monthus, C. On the localization of random heteropolymers at the interface between two selective solvents. *The European Physical Journal B - Condensed Matter and Complex Systems 13*, 1 (Jan 2000), 111–130.

[104] Mörters, P., and Peres, Y. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.

[105] Nelson, D., and Vinokur, V. Boson localization and correlated pinning of superconducting vortex arrays. *Phys. Rev. B 48* (1993), 13060–13097.

[106] Nguyen, G. B., and Pétrélis, N. A variational formula for the free energy of the partially directed polymer collapse. *J. Stat. Phys. 151*, 6 (2013), 1099–1120.

[107] Owczarek, A. L., Prellberg, T., and Rechnitzer, A. Finite-size scaling functions for directed polymers confined between attracting walls. *J. Phys. A 41*, 3 (2008), 035002, 16.

[108] Peres, Y., Sinclair, A., Sousi, P., and Stauffer, A. Mobile geometric graphs: detection, coverage and percolation. *Probab. Theory Related Fields 156*, 1-2 (2013), 273–305.

[109] Peres, Y., Sousi, P., and Stauffer, A. The isolation time of Poisson Brownian motions. *ALEA Lat. Am. J. Probab. Math. Stat. 10*, 2 (2013), 813–829.

[110] Phetpradap, P. *Intersections of random walks*. PhD thesis, University of Bath, 2011.

[111] Poisat, J. Random pinning model with finite range correlations: disorder relevant regime. *Stochastic Process. Appl. 122*, 10 (2012), 3560–3579.

[112] Poisat, J. On quenched and annealed critical curves of random pinning model with finite range correlations. *Ann. Inst. Henri Poincaré Probab. Stat. 49*, 2 (2013), 456–482.

[113] Poisat, J. Ruelle-Perron-Frobenius operator approach to the annealed pinning model with Gaussian long-range correlated disorder. *Markov Process. Related Fields 19*, 3 (2013), 577–606.

[114] Poisat, J., and Simenhaus, F. A limit theorem for the survival probability of a simple random walk among power-law renewal obstacles. *https://arxiv.org/abs/1809.08866 (To appear in Annals of Applied Probability)* (2019).

[115] Poland, D., and Scheraga, H. *Theory of helix-coil transitions in biopolymers: statistical mechanical theory of order-disorder transitions in biological macromolecules*. Academic Press, 1970.

[116] Pólya, G. Remarks on characteristic functions. In *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946* (1949), University of California Press, Berkeley and Los Angeles, pp. 115–123.

[117] Port, S. C., and Stone, C. J. *Brownian motion and classical potential theory*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. Probability and Mathematical Statistics.

[118] Povel, T. Confinement of Brownian motion among Poissonian obstacles in $\mathbf{R}^d$, $d \geq 3$. *Probab. Theory Related Fields 114*, 2 (1999), 177–205.

[119] Ráth, B., and Sapozhnikov, A. On the transience of random interlacements. *Electron. Commun. Probab. 16* (2011), 379–391.

[120] Ráth, B., and Sapozhnikov, A. Connectivity properties of random interlacement and intersection of random walks. *ALEA Lat. Am. J. Probab. Math. Stat. 9* (2012), 67–83.

[121] Resnick, S. I. *Extreme values, regular variation and point processes*. Springer Series in Operations Research and Financial Engineering. Springer, New York, 2008. Reprint of the 1987 original.

[122] Schmock, U. Convergence of the normalized one-dimensional Wiener sausage path measures to a mixture of Brownian taboo processes. *Stochastics Stochastics Rep. 29*, 2 (1990), 171–183.

[123] Spitzer, F. *Principles of random walk*, second ed. Springer-Verlag, New York-Heidelberg, 1976.

[124] Sznitman, A.-S. Lifschitz tail and Wiener sausage. I, II. *J. Funct. Anal. 94*, 2 (1990), 223–246, 247–272.

[125] Sznitman, A.-S. On long excursions of Brownian motion among Poissonian obstacles. In *Stochastic analysis (Durham, 1990)*, vol. 167 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 1991, pp. 353–375.

[126] Sznitman, A.-S. On the confinement property of two-dimensional Brownian motion among Poissonian obstacles. *Comm. Pure Appl. Math. 44*, 8-9 (1991), 1137–1170.

[127] Sznitman, A.-S. Brownian asymptotics in a Poissonian environment. *Probab. Theory Related Fields 95*, 2 (1993), 155–174.

[128] Sznitman, A.-S. Brownian confinement and pinning in a Poissonian potential. I, II. *Probab. Theory Related Fields 105*, 1 (1996), 1–29, 31–56.

[129] Sznitman, A.-S. Fluctuations of principal eigenvalues and random scales. *Comm. Math. Phys. 189*, 2 (1997), 337–363.

[130] Sznitman, A.-S. *Brownian motion, obstacles and random media.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[131] Sznitman, A.-S. Vacant set of random interlacements and percolation. *Ann. of Math. (2) 171*, 3 (2010), 2039–2087.

[132] Sznitman, A.-S. On scaling limits and Brownian interlacements. *Bull. Braz. Math. Soc. (N.S.) 44*, 4 (2013), 555–592.

[133] Toninelli, F. L. A replica-coupling approach to disordered pinning models. *Comm. Math. Phys. 280*, 2 (2008), 389–401.

[134] van den Berg, J., Meester, R., and White, D. G. Dynamic Boolean models. *Stochastic Process. Appl. 69*, 2 (1997), 247–257.

[135] van den Berg, M., Bolthausen, E., and den Hollander, F. Moderate deviations for the volume of the Wiener sausage. *Ann. of Math. (2) 153*, 2 (2001), 355–406.

[136] van den Berg, M., Bolthausen, E., and den Hollander, F. Torsional rigidity for regions with a Brownian boundary. *Potential Anal. 48*, 3 (2018), 375–403.

[137] van der Hofstad, R., and den Hollander, F. Scaling for a random polymer. *Comm. Math. Phys. 169*, 2 (1995), 397–440.

[138] van der Hofstad, R., den Hollander, F., and König, W. Central limit theorem for a weakly interacting random polymer. *Markov Process. Related Fields 3*, 1 (1997), 1–62.

[139] van der Hofstad, R., den Hollander, F., and König, W. Weak interaction limits for one-dimensional random polymers. *Probab. Theory Related Fields 125*, 4 (2003), 483–521.

[140] Weinrib, A. Long-range correlated percolation. *Phys. Rev. B (3) 29*, 1 (1984), 387–395.

[141] Weinrib, A., and Halperin, B. I. Critical phenomena in systems with long-range correlated disorder. *Phys. Rev. B 27* (1983), 413–427.

[142] Zwanzig, R., and Lauritzen, J. Exact calculation of the partition function for a model of two dimensional polymer crystallization by chain folding. *J. Chem. Phys. 48* (1968), 3351–3360.