

PCA FOR POINT PROCESSES: PROOFS

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This supplementary material contains all proofs of theorems and propositions of the main article PCA for Point Processes.

S.1. Proofs of Section 3.

S.1.1. *Proof of Proposition 3.1.* Let $j \geq 1$. We observe that, since $\lambda_j > 0$,

$$(S.1.1.1) \quad \eta_j(t) = \lambda_j^{-1} \int_0^1 K_\Delta(t, s) \eta_j(s) ds, \quad t \in [0, 1].$$

We prove that η_j is of finite variation on $[0, 1]$. We recall that a function f is a function of finite variation on an interval $[a, b]$ if, for any grid $(t_0, \dots, t_p)$ of $[0, 1]$, the quantity

$$V_f := \sup_{a \leq t_0 < t_1 < \dots < t_p \leq b} \left\{ \sum_{\ell=1}^p |f(t_\ell) - f(t_{\ell-1})| \right\}$$

is finite. For any grid $(t_0, \dots, t_p)$ of $[0, 1]$, we have:

$$\begin{aligned} \sum_{\ell=1}^p |\eta_j(t_\ell) - \eta_j(t_{\ell-1})| &= \lambda_j^{-1} \sum_{\ell=1}^p \left| \int_0^1 (K_\Delta(s, t_\ell) - K_\Delta(s, t_{\ell-1})) \eta_j(s) ds \right| \\ &= \lambda_j^{-1} \sum_{\ell=1}^p \left| \int_0^1 (\mathbb{E}[F_\Delta(s) F_\Delta(t_\ell)] - \mathbb{E}[F_\Delta(s) F_\Delta(t_{\ell-1})]) \eta_j(s) ds \right| \\ &= \lambda_j^{-1} \sum_{\ell=1}^p \left| \int_0^1 \mathbb{E}[F_\Delta(s) (F_\Delta(t_\ell) - F_\Delta(t_{\ell-1}))] \eta_j(s) ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq \lambda_j^{-1} \int_0^1 \mathbb{E} \left[|F_\Delta(s)| \sum_{\ell=1}^p |F_\Delta(t_\ell) - F_\Delta(t_{\ell-1})| \right] |\eta_j(s)| ds \\
&\leq \lambda_j^{-1} \int_0^1 \mathbb{E} \left[|F_\Delta(s)| \sum_{\ell=1}^p |\Delta(\cdot]t_{\ell-1}, t_\ell])| \right] |\eta_j(s)| ds \\
&\leq \lambda_j^{-1} \int_0^1 \mathbb{E} \left[|F_\Delta(s)| \sum_{\ell=1}^p (\Pi(\cdot]t_{\ell-1}, t_\ell]) + W(\cdot]t_{\ell-1}, t_\ell]) \right] |\eta_j(s)| ds \\
&\leq \lambda_j^{-1} \int_0^1 \mathbb{E} [|F_\Delta(s)| (\Pi([0, 1]) + W([0, 1]))] |\eta_j(s)| ds,
\end{aligned}$$

which is independent of the grid. We observe that

$$\begin{aligned}
\mathbb{E} [|F_\Delta(s)| (\Pi([0, 1]) + W([0, 1]))] &\leq \sqrt{\mathbb{E} [|F_\Delta(s)|^2] \times \mathbb{E} [(\Pi([0, 1]) + W([0, 1]))^2]} \\
&\leq 2\sqrt{\mathbb{E} [|F_\Delta(s)|^2] \mathbb{E} [\Pi^2([0, 1])]}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \mathbb{E} [|F_\Delta(s)| (\Pi([0, 1]) + W([0, 1]))] |\eta_j(s)| ds \\
&\leq 2\|\eta_j\|_2 \sqrt{\mathbb{E} [\Pi^2([0, 1])]} \left(\int_0^1 \mathbb{E} [|F_\Delta(s)|^2] ds \right)^{1/2} \\
&\leq 4\|\eta_j\|_2 \mathbb{E} [\Pi^2([0, 1])] < \infty.
\end{aligned}$$

Then, η_j is a function of finite variation. Since K_Δ is continuous, η_j is also continuous (and then right-continuous). We also have that $\eta_j(0) = 0$. Then, applying Proposition 4.4.3 of [2], we obtain that there exists a signed measure μ_j such that

$$\eta_j(t) = \mu_j([0, t]), \quad t \in [0, 1].$$

From Fubini's Theorem we have that for any function $\varphi \in \mathcal{H}_0^1$

$$\begin{aligned}
\int_0^1 \varphi'(t) \eta_j(t) dt &= \int_0^1 \varphi'(t) \left(\int_0^t d\mu_j(s) \right) dt = \int_0^1 \left(\int_s^1 \varphi'(t) dt \right) d\mu_j(s) \\
\text{(S.1.1.2)} \quad &= - \int_0^1 \varphi(s) d\mu_j(s),
\end{aligned}$$

since $\varphi(1) = 0$.

S.1.2. Proof of Theorem 3.2. By the Karhunen-Loève theorem (see e.g. [3], Theorem 7.3.5) applied to the stochastic process F_Π ,

$$\text{(S.1.2.3)} \quad \lim_{J \rightarrow +\infty} \sup_{t \in [0, 1]} \mathbb{E} \left[\left(F_\Pi(t) - F_W(t) - \sum_{j=1}^J \sqrt{\lambda_j} \xi_j \eta_j(t) \right)^2 \right] = 0.$$

Furthermore, similarly to (S.1.1.2), we have, for all $\varphi \in \mathcal{H}_0^1$,

$$\langle \varphi, \Pi - W \rangle = -\langle F_\Pi - F_W, \varphi' \rangle.$$

Then, using Equation (5) of the main file,

$$\begin{aligned}
\mathbb{E} \left[\left\| \Pi - W - \sum_{j=1}^J \sqrt{\lambda_j} \xi_j \mu_j \right\|_{\mathcal{H}^{-1}}^2 \right] &\leq \mathbb{E} \left[\sup_{\varphi \in \mathcal{H}_0^1, \|\varphi'\| \leq 1} \left(\langle \varphi, \Pi - W \rangle - \sum_{j=1}^J \sqrt{\lambda_j} \xi_j \langle \varphi, \mu_j \rangle \right)^2 \right] \\
&\leq \mathbb{E} \left[\sup_{\varphi \in \mathcal{H}_0^1, \|\varphi'\| \leq 1} \left\langle F_\Pi - F_W - \sum_{j=1}^J \sqrt{\lambda_j} \xi_j \eta_j, \varphi' \right\rangle^2 \right] \\
&\leq \mathbb{E} \left[\sup_{\varphi \in \mathcal{H}_0^1, \|\varphi'\| \leq 1} \|\varphi'\|^2 \left\| F_\Pi - F_W - \sum_{j=1}^J \sqrt{\lambda_j} \xi_j \eta_j \right\|^2 \right] \\
&\leq \mathbb{E} \left[\int_0^1 \left(F_\Pi(t) - F_W(t) - \sum_{j=1}^J \sqrt{\lambda_j} \xi_j \eta_j(t) \right)^2 dt \right]
\end{aligned}$$

and the result comes from Equation (S.1.2.3).

S.1.3. *Proof of Theorem 3.4.* A direct consequence of Mercer's theorem is that

$$\left\langle K_\Delta, \frac{\partial^2 \varphi}{\partial t \partial s} \right\rangle = \left\langle \sum_{j \geq 1} \lambda_j \eta_j \otimes \eta_j, \frac{\partial^2 \varphi}{\partial t \partial s} \right\rangle = \sum_{j \geq 1} \lambda_j \left\langle \eta_j \otimes \eta_j, \frac{\partial^2 \varphi}{\partial t \partial s} \right\rangle = \sum_{j \geq 1} \lambda_j \langle \varphi, \mu_j \otimes \mu_j \rangle.$$

Indeed, Fubini's theorem and the definition of μ_j implies

$$\begin{aligned}
\left\langle \eta_j \otimes \eta_j, \frac{\partial^2 \varphi}{\partial t \partial s} \right\rangle &= \int_{[0,1]^2} \eta_j(s) \eta_j(t) \frac{\partial^2 \varphi}{\partial t \partial s}(s, t) ds dt \\
&= \int_0^1 \eta_j(s) \left(\int_0^1 \eta_j(t) \frac{\partial^2 \varphi}{\partial t \partial s}(s, t) dt \right) ds \\
&= \int_0^1 \eta_j(s) \left\langle \eta_j, \frac{\partial^2 \varphi}{\partial t \partial s}(s, \cdot) \right\rangle ds \\
&= - \int_0^1 \eta_j(s) \left\langle \frac{\partial \varphi}{\partial s}(s, \cdot), \mu_j \right\rangle ds \\
&= - \int_0^1 \eta_j(s) \int_0^1 \frac{\partial \varphi}{\partial s}(s, t) d\mu_j(t) ds \\
&= \langle \varphi, \mu_j \otimes \mu_j \rangle,
\end{aligned}$$

using Fubini's theorem and the definition of μ_j a second time. Hence we can write

$$\left\langle \frac{\partial^2 K_\Delta}{\partial t \partial s}, \varphi \right\rangle = \sum_{j \geq 1} \lambda_j \langle \varphi, \mu_j \otimes \mu_j \rangle.$$

Now remark also that,

$$C_\Delta = \frac{\partial^2 K_\Delta}{\partial t \partial s}.$$

Indeed, for a function $\varphi \in \mathcal{H}_0^1$, we get

$$\begin{aligned}
\int_0^1 F_\Pi(t) \varphi'(t) dt &= \int_0^1 \sum_{T \in N} \mathbf{1}_{T \leq t} \varphi'(t) dt = \sum_{T \in N} \int_T^1 \varphi'(t) dt \\
&= \sum_{T \in N} (\varphi(1) - \varphi(T)) = -\langle \Pi, \varphi \rangle.
\end{aligned}$$

In particular

$$\mathbb{E} \left[\int_0^1 F_{\Pi}(t) \varphi'(t) dt \right] = -\mathbb{E}[\langle \Pi, \varphi \rangle] = -\langle W, \varphi \rangle.$$

This implies

$$\begin{aligned} \int_{[0,1]^2} K_{\Delta}(s,t) \frac{\partial^2 \varphi}{\partial t \partial s}(s,t) ds dt &= \int_{[0,1]^2} \mathbb{E}[F_{\Delta}(s) F_{\Delta}(t)] \frac{\partial^2 \varphi}{\partial t \partial s}(s,t) ds dt \\ &= \mathbb{E} \left[\int_{[0,1]^2} F_{\Pi}(s) F_{\Pi}(t) \frac{\partial^2 \varphi}{\partial t \partial s}(s,t) ds dt \right. \\ &\quad \left. - \int_{[0,1]^2} F_W(s) F_W(t) \frac{\partial^2 \varphi}{\partial t \partial s}(s,t) ds dt \right] \\ &= \mathbb{E} \left[\int_0^1 F_{\Pi}(s) \left(\int_0^1 F_{\Pi}(t) \frac{\partial^2 \varphi}{\partial t \partial s}(s,t) dt \right) ds \right. \\ &\quad \left. - \int_0^1 F_W(s) \mathbb{E} \left[\int_0^1 F_{\Pi}(t) \frac{\partial^2 \varphi}{\partial t \partial s}(s,t) dt \right] ds \right] \\ &= -\mathbb{E} \left[\int_0^1 F_{\Pi}(s) \sum_{T \in N} \frac{\partial \varphi}{\partial s}(s, T) ds \right] \\ &\quad + \int_0^1 F_W(s) \int_0^1 \frac{\partial \varphi}{\partial s}(s, t) W(t) dt ds \\ &= \mathbb{E} \left[\sum_{T, T' \in N} \varphi(T', T) \right] - \langle \varphi, W \otimes W \rangle \\ &= \langle C_{\Delta}, \varphi \rangle. \end{aligned}$$

Then, for all $\varphi \in \mathcal{H}_0^2$, from Cauchy-Schwarz's inequality we deduce

$$\left| \left\langle \sum_{j=1}^J \lambda_j \mu_j \otimes \mu_j - C_{\Delta}, \varphi \right\rangle \right| = \left| \left\langle \sum_{j=1}^J \lambda_j \eta_j \otimes \eta_j - K_{\Delta}, \frac{\partial^2 \varphi}{\partial s \partial t} \right\rangle \right| \leq \left\| K_{\Delta} - \sum_{j=1}^J \lambda_j \eta_j \otimes \eta_j \right\| \left\| \frac{\partial^2 \varphi}{\partial s \partial t} \right\|,$$

which implies

$$\left\| \sum_{j=1}^J \lambda_j \mu_j \otimes \mu_j - C_{\Delta} \right\|_{\mathcal{H}^{-2}} \leq \left\| \sum_{j=1}^J \lambda_j \eta_j \otimes \eta_j - K_{\Delta} \right\|,$$

and Mercer theorem implies the expected result.

S.2. Proofs of Section 4.

S.2.1. Proof of Theorem 4.1. First, assume that F_{μ_j} is an eigenfunction of the operator Γ_{Δ} associated to the eigenvalue λ_j . We have:

$$(S.2.1.4) \quad \lambda_j F_{\mu_j}(t) = \int_0^1 K_{\Delta}(t, s) F_{\mu_j}(s) ds, \quad t \in [0, 1].$$

We have for any $(t, t') \in [0, 1]^2$, since $\lambda_j > 0$,

$$|F_{\mu_j}(t) - F_{\mu_j}(t')| \leq \|F_{\mu_j}\|_2 \times \lambda_j^{-1} \left(\int_0^1 (K_\Delta(t, s) - K_\Delta(t', s))^2 ds \right)^{1/2}.$$

And since K_Δ is continuous, $|F_{\mu_j}(t) - F_{\mu_j}(t')|$ goes to 0 when $t' \rightarrow t$. So F_{μ_j} is continuous. Equation (S.2.1.4) gives

$$\begin{aligned} \lambda_j F_{\mu_j}(t) &= \int_0^1 \left(\int_0^{\min\{s; t\}} w(u) du \right) F_{\mu_j}(s) ds \\ (S.2.1.5) \quad &= \int_0^t F_{\mu_j}(s) \left(\int_0^s w(u) du \right) ds + \int_t^1 F_{\mu_j}(s) \left(\int_0^t w(u) du \right) ds. \end{aligned}$$

Now, consider

$$H_j(t) = \int_t^1 F_{\mu_j}(s) ds, \quad t \in [0, 1].$$

Since F_{μ_j} is continuous, H_j is a differentiable function and

$$H_j'(t) = -F_{\mu_j}(t).$$

Actually, previous computations show that F_{μ_j} is an infinitely-differentiable function and differentiating Equation (S.2.1.5) gives

$$(S.2.1.6) \quad \lambda_j F_{\mu_j}'(t) = w(t) \int_t^1 F_{\mu_j}(u) du$$

and then H_j is solution of a second-order differential equation,

$$(S.2.1.7) \quad -\lambda_j H_j''(t) = H_j(t) w(t),$$

with boundary conditions:

$$H_j(1) = 0, \quad H_j'(0) = -F_{\mu_j}(0) = 0.$$

Therefore, (λ_j, H_j) is solution of the system (15) of the main file.

Conversely, assume that (λ_j, H_j) is a solution of (15) of the main file. Then, setting

$$F_{\mu_j}(s) = -H_j'(s), \quad s \in [0, 1],$$

we obtain, by using integration by part, for any $t \in [0, 1]$,

$$\begin{aligned} \int_0^1 K_\Delta(t, s) F_{\mu_j}(s) ds &= \int_0^t F_{\mu_j}(s) \left(\int_0^s w(u) du \right) ds + \int_t^1 F_{\mu_j}(s) \left(\int_0^t w(u) du \right) ds \\ &= - \int_0^t H_j'(s) \left(\int_0^s w(u) du \right) ds - \left(\int_0^t w(u) du \right) \int_t^1 H_j'(s) ds \\ &= -H_j(t) \int_0^t w(u) du + \int_0^t H_j(s) w(s) ds - \left(\int_0^t w(u) du \right) (H_j(1) - H_j(t)) \\ &= \int_0^t H_j(s) w(s) ds, \end{aligned}$$

since $H_j(1) = 0$. Then, since $H_j'(0) = 0$, for any $t \in [0, 1]$,

$$\begin{aligned} \int_0^1 K_\Delta(t, s) F_{\mu_j}(s) ds &= -\lambda_j \int_0^t H_j''(s) ds \\ &= -\lambda_j H_j'(t) = \lambda_j F_{\mu_j}(t). \end{aligned}$$

Then, (λ_j, F_{μ_j}) are eigenelements of the operator of kernel K_Δ .

S.2.2. *Proofs of Theorem 4.8 and of the result of Remark 4.9.* In the sequel, we set

$$\theta = \beta - \alpha, \quad C_1 = \frac{\beta^3 w_0}{(\beta - \alpha)^3}, \quad C_2 = -\frac{w_0 \alpha \beta (2\beta - \alpha)}{2(\beta - \alpha)^4}.$$

We observe

$$\begin{aligned} (\alpha - \beta)^{-2} (2C_2(\alpha - \beta) - C_1) &= (\alpha - \beta)^{-2} \left(\frac{w_0 \alpha \beta (2\beta - \alpha)}{(\beta - \alpha)^3} - \frac{\beta^3 w_0}{(\beta - \alpha)^3} \right) \\ &= w_0 \beta (\beta - \alpha)^{-5} (\alpha (2\beta - \alpha) - \beta^2) \\ &= -\frac{w_0 \beta}{(\beta - \alpha)^3} = -C_1 \beta^{-2} \end{aligned}$$

and

$$(S.2.2.8) \quad 2C_2 \theta + C_1 = C_1 \beta^{-2} \theta^2.$$

Using Equation (20) of the main file, we have, by symmetry of the bivariate kernel

$$K_\Delta(s, t) = C_1 \min\{s, t\} - C_2 \left(-1 - e^{-\theta|t-s|} + e^{-\theta t} + e^{-\theta s} \right), \quad s, t \in [0, 1].$$

Furthermore, using definitions of C_1 and C_2 , (λ_j, H_j) is solution of (21) of the main file if and only if we have:

$$\begin{cases} -\lambda_j H_j''(s) = C_1 \beta^{-2} \theta^2 H_j(s) - C_2 \theta^2 \int_0^1 e^{-\theta|t-s|} H_j(t) dt, & s \in (0, 1), \\ H_j(1) = 0, & H_j'(0) = 0 \end{cases}$$

Then, let us express $\Gamma_\Delta F_{\mu_j}$. For any $s \in [0, 1]$,

$$\begin{aligned} \Gamma_\Delta F_{\mu_j}(s) &= \int_0^1 K_\Delta(s, t) F_{\mu_j}(t) dt \\ &= \int_0^1 \left(C_1 \min\{s, t\} - C_2 \left(-1 - e^{-\theta|t-s|} + e^{-\theta t} + e^{-\theta s} \right) \right) F_{\mu_j}(t) dt \\ &= C_1 \int_0^s t F_{\mu_j}(t) dt + C_1 s \int_s^1 F_{\mu_j}(t) dt + C_2 (1 - e^{-\theta s}) \int_0^1 F_{\mu_j}(t) dt \\ &\quad - C_2 \int_0^1 e^{-\theta t} F_{\mu_j}(t) dt + C_2 e^{\theta s} \int_s^1 e^{-\theta t} F_{\mu_j}(t) dt + C_2 e^{-\theta s} \int_0^s e^{\theta t} F_{\mu_j}(t) dt. \end{aligned}$$

We have

$$H_j(s) = \int_s^1 F_{\mu_j}(t) dt, \quad s \in [0, 1].$$

Since K_Δ is continuous, the eigenfunctions are continuous as well (eigenvalues are positive). Since F_{μ_j} is continuous, we have

$$F_{\mu_j}(s) = -H_j'(s), \quad s \in [0, 1]$$

with boundary conditions

$$H_j(1) = 0, \quad H_j'(0) = 0.$$

Then, we can write

$$\Gamma_\Delta F_{\mu_j}(s) = (I) + (II) - (III) + (IV) + (V)$$

with

$$\begin{aligned}
 (I) &= C_1 \int_0^s t F_{\mu_j}(t) dt + C_1 s \int_s^1 F_{\mu_j}(t) dt \\
 &= C_1 \int_0^s t (-H_j'(t)) dt + C_1 s H_j(s) \\
 &= C_1 \int_0^s H_j(t) dt,
 \end{aligned}$$

$$\begin{aligned}
 (II) &= C_2 (1 - e^{-\theta s}) \int_0^1 F_{\mu_j}(t) dt \\
 &= C_2 (1 - e^{-\theta s}) H_j(0),
 \end{aligned}$$

$$\begin{aligned}
 (III) &= C_2 \int_0^1 e^{-\theta t} F_{\mu_j}(t) dt \\
 &= C_2 \int_0^1 e^{-\theta t} (-H_j'(t)) dt \\
 &= C_2 H_j(0) - C_2 \theta \int_0^1 e^{-\theta t} H_j(t) dt
 \end{aligned}$$

and

$$-(III) = -C_2 H_j(0) + C_2 \theta \int_0^1 e^{-\theta t} H_j(t) dt.$$

Now,

$$\begin{aligned}
 (IV) &= C_2 e^{\theta s} \int_s^1 e^{-\theta t} F_{\mu_j}(t) dt \\
 &= C_2 e^{\theta s} \int_s^1 e^{-\theta t} (-H_j'(t)) dt \\
 &= C_2 e^{\theta s} \left(e^{-\theta s} H_j(s) - \theta \int_s^1 e^{-\theta t} H_j(t) dt \right) \\
 &= C_2 H_j(s) - C_2 \theta e^{\theta s} \int_s^1 e^{-\theta t} H_j(t) dt
 \end{aligned}$$

and, finally,

$$\begin{aligned}
 (V) &= C_2 e^{-\theta s} \int_0^s e^{\theta t} F_{\mu_j}(t) dt \\
 &= C_2 e^{-\theta s} \int_0^s e^{\theta t} (-H_j'(t)) dt \\
 &= C_2 e^{-\theta s} (H_j(0) - e^{\theta s} H_j(s)) + C_2 \theta e^{-\theta s} \int_0^s e^{\theta t} H_j(t) dt \\
 &= C_2 e^{-\theta s} H_j(0) - C_2 H_j(s) + C_2 \theta e^{-\theta s} \int_0^s e^{\theta t} H_j(t) dt.
 \end{aligned}$$

Therefore

$$\begin{aligned}
\Gamma_{\Delta} F_{\mu_j}(s) &= (I) + (II) - (III) + (IV) + (V) \\
&= C_1 \int_0^s H_j(t) dt + C_2(1 - e^{-\theta s})H_j(0) \\
&\quad - C_2 H_j(0) + C_2 \theta \int_0^1 e^{-\theta t} H_j(t) dt + C_2 H_j(s) - C_2 \theta e^{\theta s} \int_s^1 e^{-\theta t} H_j(t) dt \\
&\quad + C_2 e^{-\theta s} H_j(0) - C_2 H_j(s) + C_2 \theta e^{-\theta s} \int_0^s e^{\theta t} H_j(t) dt \\
&= C_1 \int_0^s H_j(t) dt + C_2 \theta \int_0^1 e^{-\theta t} H_j(t) dt - C_2 \theta e^{\theta s} \int_s^1 e^{-\theta t} H_j(t) dt \\
&\quad + C_2 \theta e^{-\theta s} \int_0^s e^{\theta t} H_j(t) dt.
\end{aligned}$$

Setting

$$U(s) = e^{-\theta s} \int_0^s e^{\theta t} H_j(t) dt - e^{\theta s} \int_s^1 e^{-\theta t} H_j(t) dt, \quad s \in [0, 1],$$

we finally obtain that for any $s \in [0, 1]$,

$$(S.2.2.9) \quad \Gamma_{\Delta} F_{\mu_j}(s) = C_1 \int_0^s H_j(t) dt + C_2 \theta \int_0^1 e^{-\theta t} H_j(t) dt + C_2 \theta U(s).$$

We then obtain that

$$(S.2.2.10) \quad \Gamma_{\Delta} F_{\mu_j}(s) = \lambda_j F_{\mu_j}(s), \quad s \in [0, 1]$$

if and only if

$$\lambda_j F_{\mu_j}(s) = C_1 \int_0^s H_j(t) dt + C_2 \theta \int_0^1 e^{-\theta t} H_j(t) dt + C_2 \theta U(s), \quad s \in [0, 1].$$

In the sequel, we use the following lemma.

LEMMA S.2.1. *Setting*

$$(S.2.2.11) \quad Z(s) = \int_0^1 e^{-\theta|t-s|} H_j(t) dt,$$

we have for any $s \in [0, 1]$,

$$U(s) = U(0) + 2 \int_0^s H_j(t) dt - \theta \int_0^s Z(t) dt.$$

PROOF OF THE LEMMA. We have for any $s \in [0, 1]$,

$$\begin{aligned}
U'(s) &= -\theta e^{-\theta s} \int_0^s e^{\theta t} H_j(t) dt + H_j(s) - \theta e^{\theta s} \int_s^1 e^{-\theta t} H_j(t) dt + H_j(s) \\
&= 2H_j(s) - \theta Z(s).
\end{aligned}$$

This provides the result of the lemma. □

Now, we assume that (S.2.2.10) is true. We have

$$\begin{aligned}\lambda_j F'_{\mu_j}(s) &= C_1 H_j(s) + C_2 \theta U'(s) \\ &= C_1 H_j(s) + C_2 \theta (2H_j(s) - \theta Z(s)) \\ &= \frac{C_1 \theta^2}{\beta^2} H_j(s) - C_2 \theta^2 Z(s),\end{aligned}$$

by using (S.2.2.8) and the lemma. This gives

$$(S.2.2.12) \quad -\lambda_j H_j''(s) = \frac{C_1 \theta^2}{\beta^2} H_j(s) - C_2 \theta^2 Z(s), \quad s \in [0, 1],$$

meaning that (λ_j, H_j) is solution of Equation (21) of the main file.

Conversely, assume that (λ_j, H_j) is solution of Equation (21) of the main file. We prove that (S.2.2.10) is satisfied. Indeed, using (S.2.2.9), the result of the lemma and

$$U(0) = - \int_0^1 e^{-\theta t} H_j(t) dt,$$

we have for any $s \in [0, 1]$,

$$\begin{aligned}\Gamma_{\Delta} F_{\mu_j}(s) &= C_1 \int_0^s H_j(t) dt + C_2 \theta \int_0^1 e^{-\theta t} H_j(t) dt + C_2 \theta U(s) \\ &= C_1 \int_0^s H_j(t) dt + C_2 \theta \int_0^1 e^{-\theta t} H_j(t) dt + C_2 \theta \left(U(0) + 2 \int_0^s H_j(t) dt - \theta \int_0^s Z(t) dt \right) \\ &= \frac{C_1 \theta^2}{\beta^2} \int_0^s H_j(t) dt - C_2 \theta^2 \int_0^s Z(t) dt.\end{aligned}$$

by using (S.2.2.8). Since (λ_j, H_j) is solution of Equation (21) of the main file, we finally obtain for any $s \in [0, 1]$,

$$\begin{aligned}\Gamma_{\Delta} F_{\mu_j}(s) &= -\lambda_j \int_0^s H_j''(t) dt \\ &= \lambda_j (H_j'(0) - H_j'(s)) \\ &= \lambda_j F_{\mu_j}(s)\end{aligned}$$

and the result is proved.

Now, we prove the result of Remark 4.9 of the main file. Using Equation (S.2.2.12), we obtain

$$-\lambda_j H_j^{(4)}(s) = \frac{C_1 \theta^2}{\beta^2} H_j''(s) - C_2 \theta^2 Z''(s), \quad s \in [0, 1].$$

By definition of Z given in (S.2.2.11), we have for any $s \in [0, 1]$,

$$Z'(s) = -\theta e^{-\theta s} \int_0^s e^{\theta u} H_j(u) du + \theta e^{\theta s} \int_s^1 e^{-\theta u} H_j(u) du$$

and

$$\begin{aligned} Z''(s) &= \theta^2 e^{-\theta s} \int_0^s e^{\theta u} H_j(u) du + \theta^2 e^{\theta s} \int_s^1 e^{-\theta u} H_j(u) du - 2\theta H_j(s) \\ &= \theta^2 Z(s) - 2\theta H_j(s). \end{aligned}$$

Now, we obtain, still using Equations (S.2.2.12) and (S.2.2.8),

$$\begin{aligned} -\lambda_j H_j^{(4)}(s) &= \frac{C_1 \theta^2}{\beta^2} H_j''(s) - C_2 \theta^4 Z(s) + 2C_2 \theta^3 H_j(s) \\ &= \frac{C_1 \theta^2}{\beta^2} H_j''(s) - \lambda_j \theta^2 H_j''(s) + \left(2C_2 \theta^3 - \frac{C_1 \theta^4}{\beta^2}\right) H_j(s) \\ &= \frac{C_1 \theta^2}{\beta^2} H_j''(s) - \lambda_j \theta^2 H_j''(s) - C_1 \theta^2 H_j(s). \end{aligned}$$

Finally, we obtain for any $s \in [0, 1]$,

$$(S.2.2.13) \quad -\lambda_j H_j^{(4)}(s) + \lambda_j \theta^2 H_j''(s) = \frac{C_1 \theta^2}{\beta^2} (H_j''(s) - \beta^2 H_j(s)).$$

In particular, if $\lambda_j = 0$, then

$$H_j(s) = c_1 e^{\beta s} + c_2 e^{-\beta s}, \quad s \in [0, 1],$$

for c_1 and c_2 two constants. The boundary condition $H_j'(0) = 0$ gives $c_1 = c_2$ and $H_j(1) = 0$ gives $c_1 = c_2 = 0$. Then $\lambda_j = 0$ entails $H_j = 0$ and then $F_{\mu_j} = 0$. We have assumed that F_{μ_j} is continuous, which allows to use previous computations.

S.2.3. Proof of Theorem 4.10. We set

$$w_1 = \frac{\beta w_0}{\beta - \alpha}, \quad c = \frac{w_0 \alpha \beta (2\beta - \alpha)}{2(\beta - \alpha)^2}, \quad \theta = \beta - \alpha$$

and

$$I(y)(t) = \frac{c}{w_1} \int_0^1 e^{-\theta|t-s|} y(s) ds, \quad t \in [0, 1].$$

In the sequel, we assume that Assumption (22) of the main file is satisfied, which is equivalent to

$$(S.2.3.14) \quad \frac{2c}{w_1} \left(2 + \frac{3 - 2e^{-\theta/2} - e^{-\theta}}{\theta}\right) < 1.$$

Now (21) of the main file writes

$$\begin{cases} -\lambda y''(t) = w_1 y(t) + w_1 I(y)(t), & t \in (0, 1), \\ y(1) = 0, & y'(0) = 0. \end{cases}$$

Let H_j be a solution of this ODE as expressed in Theorem 4.8. We have

$$\begin{cases} H_j''(t) + \rho_j^2 H_j(t) = -\rho_j^2 I(H_j)(t), & t \in (0, 1), \\ H_j(1) = 0, & H_j'(0) = 0 \end{cases}$$

with

$$(S.2.3.15) \quad \rho_j := \sqrt{\frac{w_1}{\lambda_j}} \xrightarrow{j \rightarrow +\infty} +\infty.$$

We write H_j and H'_j as

$$\begin{cases} H_j(t) = a(t) \cos(\rho_j t) + b(t) \sin(\rho_j t), \\ H'_j(t) = -\rho_j a(t) \sin(\rho_j t) + \rho_j b(t) \cos(\rho_j t), \end{cases}$$

with a and b C^2 -functions that must satisfy

$$\begin{cases} a'(t) \cos(\rho_j t) + b'(t) \sin(\rho_j t) = 0 \\ -a'(t) \rho_j \sin(\rho_j t) + b'(t) \rho_j \cos(\rho_j t) = -\rho_j^2 I(H_j)(t). \end{cases}$$

This means that

$$\begin{pmatrix} \cos(\rho_j t) & \sin(\rho_j t) \\ -\sin(\rho_j t) & \cos(\rho_j t) \end{pmatrix} \begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho_j I(H_j)(t) \end{pmatrix},$$

or equivalently

$$\begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix} = \begin{pmatrix} \cos(\rho_j t) & -\sin(\rho_j t) \\ \sin(\rho_j t) & \cos(\rho_j t) \end{pmatrix} \begin{pmatrix} 0 \\ -\rho_j I(H_j)(t) \end{pmatrix}.$$

Therefore

$$\begin{cases} a(t) = a + \rho_j \int_0^t \sin(\rho_j s) I(H_j)(s) ds \\ b(t) = b - \rho_j \int_0^t \cos(\rho_j s) I(H_j)(s) ds, \end{cases}$$

with a and b two constants. Finally, for any $t \in [0, 1]$,

$$H_j(t) = a \cos(\rho_j t) + b \sin(\rho_j t) + \rho_j \cos(\rho_j t) \int_0^t \sin(\rho_j s) I(H_j)(s) ds - \rho_j \sin(\rho_j t) \int_0^t \cos(\rho_j s) I(H_j)(s) ds,$$

and

$$H'_j(t) = -a \rho_j \sin(\rho_j t) + b \rho_j \cos(\rho_j t) - \rho_j^2 \sin(\rho_j t) \int_0^t \sin(\rho_j s) I(H_j)(s) ds - \rho_j^2 \cos(\rho_j t) \int_0^t \cos(\rho_j s) I(H_j)(s) ds,$$

with a and b two constants. The boundary conditions $H'_j(0) = 0$ implies that $b = 0$. We then obtain that H_j must write

(S.2.3.16)

$$H_j(t) = a \cos(\rho_j t) + \rho_j \cos(\rho_j t) \int_0^t \sin(\rho_j s) I(H_j)(s) ds - \rho_j \sin(\rho_j t) \int_0^t \cos(\rho_j s) I(H_j)(s) ds,$$

and

$$H'_j(t) = -a \rho_j \sin(\rho_j t) - \rho_j^2 \sin(\rho_j t) \int_0^t \sin(\rho_j s) I(H_j)(s) ds - \rho_j^2 \cos(\rho_j t) \int_0^t \cos(\rho_j s) I(H_j)(s) ds.$$

In particular, we have

$$\begin{aligned} H_j(t) &= a \cos(\rho_j t) - \rho_j \int_0^t \sin(\rho_j(t-s)) I(H_j)(s) ds \\ &= a \cos(\rho_j t) - \int_0^{\rho_j t} \sin(u) I(H_j)\left(t - \frac{u}{\rho_j}\right) du. \end{aligned}$$

We denote $J_j : \mathbb{L}_\infty[0, 1] \mapsto \mathbb{L}_\infty[0, 1]$ the linear operator defined for $f \in \mathbb{L}_\infty[0, 1]$, for $t \in [0, 1]$, by

$$\begin{aligned} J_j(f)(t) &:= \rho_j \cos(\rho_j t) \int_0^t \sin(\rho_j s) I(f)(s) ds - \rho_j \sin(\rho_j t) \int_0^t \cos(\rho_j s) I(f)(s) ds \\ &= -\rho_j \int_0^t \sin(\rho_j(t-s)) I(f)(s) ds \\ &= -\int_0^{\rho_j t} \sin(u) I(f)\left(t - \frac{u}{\rho_j}\right) du. \end{aligned}$$

Finally, if H_j satisfies Equation (21) of the main file, H_j satisfies

$$(S.2.3.17) \quad H_j(t) = a \cos(\rho_j t) + J_j(H_j)(t), \quad t \in [0, 1].$$

Conversely, if H_j satisfies Equation (S.2.3.17), for $t \in [0, 1]$,

$$\begin{aligned} H_j''(t) &= -a\rho_j^2 \cos(\rho_j t) + J_j''(H_j)(t) \\ &= -a\rho_j^2 \cos(\rho_j t) - \rho_j^3 \cos(\rho_j t) \int_0^t \sin(\rho_j s) I(H_j)(s) ds - \rho_j^2 \sin^2(\rho_j t) I(H_j)(t) \\ &\quad + \rho_j^3 \sin(\rho_j t) \int_0^t \cos(\rho_j s) I(H_j)(s) ds - \rho_j^2 \cos^2(\rho_j t) I(H_j)(t) \\ &= -a\rho_j^2 \cos(\rho_j t) - \rho_j^2 J_j(H_j)(t) - \rho_j^2 I(H_j)(t) \\ &= -\rho_j^2 H_j(t) - \rho_j^2 I(H_j)(t). \end{aligned}$$

It means that if H_j satisfies Equation (S.2.3.17), then it satisfies (21) of the main file. We have proved:

(S.2.3.18)

H_j satisfies Equation (21) of the main file $\iff H_j$ satisfies Equation (S.2.3.17).

Now, let us deal with $F_{\mu_j}(t) = -H_j'(t)$, for $t \in [0, 1]$. We have:

$$\begin{aligned} F_{\mu_j}(t) &= -H_j'(t) \\ &= a\rho_j \sin(\rho_j t) + \rho_j^2 \sin(\rho_j t) \int_0^t \sin(\rho_j s) I(H_j)(s) ds + \rho_j^2 \cos(\rho_j t) \int_0^t \cos(\rho_j s) I(H_j)(s) ds \\ &= a\rho_j \sin(\rho_j t) + \rho_j^2 \int_0^t \cos(\rho_j(t-s)) I(H_j)(s) ds \\ &= a\rho_j \sin(\rho_j t) + \rho_j \int_0^{\rho_j t} \cos(u) I(H_j)\left(t - \frac{u}{\rho_j}\right) du \\ &= a\rho_j \sin(\rho_j t) + \frac{c\rho_j}{w_1} \int_0^{\rho_j t} \cos(u) \left(\int_0^1 e^{-\theta|t-\rho_j^{-1}u-s|} H_j(s) ds \right) du. \end{aligned}$$

It gives

$$(S.2.3.19) \quad F_{\mu_j}(t) = a\rho_j \sin(\rho_j t) + \rho_j K_j(H_j)(t), \quad t \in [0, 1],$$

with for any bounded function f ,

$$K_j(f)(t) = \frac{c}{w_1} \int_0^{\rho_j t} \cos(u) \left(\int_0^1 e^{-\theta|t-\rho_j^{-1}u-s|} f(s) ds \right) du.$$

LEMMA S.2.2. We set

$$\|J_j\|_\infty := \sup_{f: \|f\|_\infty=1} \|J_j(f)\|_\infty.$$

If (S.2.3.14) is satisfied, we have

$$\|J_j\|_\infty < \frac{1}{2}.$$

Furthermore, there exist c_1 , c_2 and c_3 three positive constants only depending on c , θ and w_1 such that

$$\|J_j(\cos(\rho_j \cdot))\|_\infty \leq c_1 \rho_j^{-1}, \quad \|K_j(\cos(\rho_j \cdot))\|_\infty \leq c_2 \rho_j^{-1} \quad \text{and} \quad \|K_j\|_\infty \leq c_3.$$

PROOF OF LEMMA S.2.2. We have

$$\begin{aligned}
-J_j(f)(t) &= \int_0^{\rho_j t} \sin(u) I(f) \left(t - \frac{u}{\rho_j} \right) du \\
&= \frac{c}{w_1} \int_0^{\rho_j t} \sin(u) \left(\int_0^1 e^{-\theta |t - \rho_j^{-1} u - s|} f(s) ds \right) du \\
&=: \mathcal{I}m \left(J^{(1)}(f)(t) + J^{(2)}(f)(t) + J^{(3)}(f)(t) \right),
\end{aligned}$$

with

$$\begin{aligned}
J^{(1)}(f)(t) &= \frac{c}{w_1} \int_0^{\rho_j t} e^{iu} \left(\int_0^{t - \rho_j^{-1} u} e^{-\theta(t - \rho_j^{-1} u - s)} f(s) ds \right) du \\
&= \frac{c}{w_1} \int_0^t e^{-\theta(t-s)} f(s) \left(\int_0^{\rho_j(t-s)} e^{(i + \theta \rho_j^{-1})u} du \right) ds \\
&= \frac{c}{w_1(i + \theta \rho_j^{-1})} \int_0^t e^{-\theta(t-s)} f(s) \left(e^{i \rho_j(t-s) + \theta(t-s)} - 1 \right) ds \\
&= \frac{c(-i + \theta \rho_j^{-1})}{w_1 \sqrt{1 + \theta^2 \rho_j^{-2}}} \int_0^t f(s) \left(e^{i \rho_j(t-s)} - e^{-\theta(t-s)} \right) ds,
\end{aligned}$$

$$\begin{aligned}
J^{(2)}(f)(t) &= \frac{c}{w_1} \int_0^{\rho_j t} e^{iu} \left(\int_{t - \rho_j^{-1} u}^t e^{\theta(t - \rho_j^{-1} u - s)} f(s) ds \right) du \\
&= \frac{c}{w_1} \int_0^t e^{\theta(t-s)} f(s) \left(\int_{\rho_j(t-s)}^{\rho_j t} e^{(i - \theta \rho_j^{-1})u} du \right) ds \\
&= \frac{c}{w_1(i - \theta \rho_j^{-1})} \int_0^t e^{\theta(t-s)} f(s) \left(e^{i \rho_j t - \theta t} - e^{i \rho_j(t-s) - \theta(t-s)} \right) ds \\
&= \frac{c(-i - \theta \rho_j^{-1})}{w_1 \sqrt{1 + \theta^2 \rho_j^{-2}}} \int_0^t f(s) \left(e^{i \rho_j t - \theta s} - e^{i \rho_j(t-s)} \right) ds
\end{aligned}$$

and

$$\begin{aligned}
J^{(3)}(f)(t) &= \frac{c}{w_1} \int_0^{\rho_j t} e^{iu} \left(\int_t^1 e^{\theta(t - \rho_j^{-1} u - s)} f(s) ds \right) du \\
&= \frac{c}{w_1} \int_t^1 e^{\theta(t-s)} f(s) \left(\int_0^{\rho_j t} e^{(i - \theta \rho_j^{-1})u} du \right) ds \\
&= \frac{c}{w_1(i - \theta \rho_j^{-1})} \int_t^1 e^{\theta(t-s)} f(s) \left(e^{i \rho_j t - \theta t} - 1 \right) ds \\
&= \frac{c(-i - \theta \rho_j^{-1})}{w_1 \sqrt{1 + \theta^2 \rho_j^{-2}}} \int_t^1 f(s) \left(e^{i \rho_j t - \theta s} - e^{\theta(t-s)} \right) ds.
\end{aligned}$$

This yields

(S.2.3.20)

$$\begin{aligned} -J_j(f)(t) &= \frac{2c\theta\rho_j^{-1}}{w_1\sqrt{1+\theta^2\rho_j^{-2}}}\mathcal{I}m\left(\int_0^t f(s)e^{i\rho_j(t-s)}ds\right) + \frac{c}{w_1\sqrt{1+\theta^2\rho_j^{-2}}}\int_0^1 f(s)e^{-\theta|t-s|}ds \\ &\quad + \mathcal{I}m\left(\frac{c(-i-\theta\rho_j^{-1})}{w_1\sqrt{1+\theta^2\rho_j^{-2}}}\int_0^1 f(s)e^{i\rho_j t-\theta s}ds\right). \end{aligned}$$

We obtain for any $t \in [0, 1]$,

$$\begin{aligned} |J_j(f)(t)| &\leq \left(\frac{2c}{w_1} + \frac{c}{w_1}\int_0^1 e^{-\theta|t-s|}ds + \frac{c}{w_1}\int_0^1 e^{-\theta s}ds\right)\|f\|_\infty \\ &\leq \left(\frac{2c}{w_1} + \frac{ce^{-\theta t}}{w_1}\int_0^t e^{\theta s}ds + \frac{ce^{\theta t}}{w_1}\int_t^1 e^{-\theta s}ds + \frac{c}{w_1}\int_0^1 e^{-\theta s}ds\right)\|f\|_\infty \\ &\leq \frac{c}{w_1}\left(2 + \frac{3 - e^{-\theta t} - e^{\theta(t-1)} - e^{-\theta}}{\theta}\right)\|f\|_\infty \\ &\leq \frac{c}{w_1}\left(2 + \frac{3 - 2e^{-\theta/2} - e^{-\theta}}{\theta}\right)\|f\|_\infty \end{aligned}$$

and

$$\|J_j\|_\infty \leq \frac{c}{w_1}\left(2 + \frac{3 - 2e^{-\theta/2} - e^{-\theta}}{\theta}\right).$$

This yields the first result.

For the second result, observe that with $f = \cos(\rho_j \cdot)$, we obtain from (S.2.3.20),

$$|J_j(\cos(\rho_j \cdot))(t)| \leq \frac{2c\theta}{w_1\rho_j} + \frac{c}{w_1}\left|\int_0^1 e^{i\rho_j s}e^{-\theta|t-s|}ds\right| + \frac{c}{w_1}\left|\int_0^1 e^{i\rho_j s}e^{-\theta s}ds\right|.$$

But

$$\begin{aligned} \int_0^1 e^{i\rho_j s}e^{-\theta|t-s|}ds &= e^{-\theta t}\int_0^t e^{(i\rho_j+\theta)s}ds + e^{\theta t}\int_t^1 e^{(i\rho_j-\theta)s}ds \\ &= e^{-\theta t}\left(\frac{e^{(i\rho_j+\theta)t}-1}{i\rho_j+\theta}\right) + e^{\theta t}\left(\frac{e^{i\rho_j-\theta}-e^{(i\rho_j-\theta)t}}{i\rho_j-\theta}\right) \end{aligned}$$

and there exists c_1 only depending on c, θ and w_1 such that

$$\|J_j(\cos(\rho_j \cdot))\|_\infty \leq c_1\rho_j^{-1}.$$

For the third result, observe that

$$\begin{aligned} K_j(f)(t) &= \mathcal{R}e\left(\frac{c}{w_1}\int_0^{\rho_j t} e^{iu}\left(\int_0^1 e^{-\theta|t-\rho_j^{-1}u-s|}f(s)ds\right)du\right) \\ &= \mathcal{R}e\left(J^{(1)}(f)(t) + J^{(2)}(f)(t) + J^{(3)}(f)(t)\right) \\ &= \frac{2c\theta\rho_j^{-1}}{w_1\sqrt{1+\theta^2\rho_j^{-2}}}\mathcal{R}e\left(\int_0^t f(s)e^{i\rho_j(t-s)}ds\right) + \frac{c\theta\rho_j^{-1}}{w_1\sqrt{1+\theta^2\rho_j^{-2}}}\int_0^1 f(s)e^{-\theta|t-s|}ds \end{aligned}$$

$$+ \mathcal{R}e \left(\frac{c(-i - \theta \rho_j^{-1})}{w_1 \sqrt{1 + \theta^2 \rho_j^{-2}}} \int_0^1 f(s) e^{i \rho_j t - \theta s} ds \right).$$

Therefore, as previously, there exists c_2 only depending on c , θ and w_1 such that

$$\|K_j(\cos(\rho_j \cdot))\|_\infty \leq c_2 \rho_j^{-1}.$$

and there exists c_3 only depending on c , θ and w_1 such that

$$\| \| K_j \| \|_\infty \leq c_3.$$

□

Now, we end the proof of the theorem and use Equivalence (S.2.3.18). Since J_j is a linear operator, for any integer N , we have, for $t \in [0, 1]$,

$$\begin{aligned} H_j(t) &= a \cos(\rho_j t) + J_j(H_j)(t) \\ &= a \cos(\rho_j t) + a J_j(\cos(\rho_j \cdot))(t) + (J_j \circ J_j)(H_j)(t) \\ &= a \sum_{k=0}^{N-1} J_j^{k \circ}(\cos(\rho_j \cdot))(t) + (J_j^{N \circ})(H_j)(t), \end{aligned}$$

with for any bounded function f ,

$$J_j^{k \circ}(f) = \underbrace{(J_j \circ \dots \circ J_j)}_{k \text{ times}}(f).$$

Since H_j is bounded and $\| \| J_j \| \|_\infty < 1/2$ (see Lemma S.2.2), we have

$$\limsup_{N \rightarrow +\infty} \|(J_j^{N \circ})(H_j)\|_\infty \leq \limsup_{N \rightarrow +\infty} \| \| J_j \| \|_\infty^N \|H_j\|_\infty = 0.$$

Therefore

$$(S.2.3.21) \quad H_j(t) = a \sum_{k=0}^{+\infty} J_j^{k \circ}(\cos(\rho_j \cdot))(t), \quad t \in [0, 1].$$

This implies that, up to renormalization, we have at most one solution to the problem. Now, we obtain a solution of the system (21) of the main file if and only if the boundary condition $H_j(1) = 0$ is satisfied. Using (S.2.3.21), this is equivalent to

$$\cos(\rho_j) = - \sum_{k=1}^{+\infty} J_j^{k \circ}(\cos(\rho_j \cdot))(1).$$

We can write $\rho_j = (k_j - 1/2)\pi + u_j\pi$, with $k_j \in \mathbb{N}^*$ and $-1/2 < u_j \leq 1/2$ (k_j and u_j are uniquely defined). Then, $H_j(1) = 0$ writes

$$(S.2.3.22) \quad (-1)^{k_j} \sin(u_j\pi) + \sum_{k=1}^{+\infty} J_j^{k \circ}(\cos(((k_j - 1/2)\pi + u_j\pi) \cdot))(1) = 0.$$

Since $\| \| J_j \| \|_\infty < 1/2$ (see Lemma S.2.2), for k_j fixed, the left hand side is a continuous function of u_j taking positive and negative values when u_j describes the interval $(-1/2; 1/2]$. Therefore, for any positive integer k_j , there exists $u_j \in (-1/2; 1/2]$ such that Equation (S.2.3.22) is satisfied. Now, let us give a control of u_j . We have:

$$2|u_j| \leq |\sin(u_j\pi)| \leq \sum_{k=1}^{+\infty} \| \| J_j \| \|_\infty^{k-1} \times \| J_j(\cos(\rho_j \cdot)) \|_\infty \leq \sum_{k=1}^{+\infty} \left(\frac{1}{2}\right)^{k-1} \times \frac{c_1}{\rho_j} = \frac{2c_1}{\rho_j}.$$

leading to

$$|u_j| \leq \frac{c_1}{\rho_j}.$$

These computations hold for any integer k_j and in particular, we can consider $k_j = j$. In this case, we obtain

$$\rho_j = (j - 1/2)\pi + u_j\pi$$

and, using (S.2.3.15),

$$\lambda_j = \frac{w_1}{\rho_j^2} = \frac{w_1}{((j - 1/2)\pi + u_j\pi)^2} = \frac{w_1}{(j\pi - \pi/2)^2} \left(1 + \frac{u_j}{(j - 1/2)}\right)^{-2}.$$

We have finally proved that for any $j \geq 1$, System (21) of the main file has a solution (λ_j, H_j) and there exists a constant C_1 depending only on c, θ and w_1 such that for any $j \geq 1$,

$$\left| \lambda_j - \frac{w_1}{(j\pi - \pi/2)^2} \right| \leq C_1 j^{-4}.$$

Now, it remains to determine F_{μ_j} . Using (S.2.3.19), we have for any $t \in [0, 1]$,

$$\begin{aligned} F_{\mu_j}(t) &= a\rho_j \sin(\rho_j t) + \rho_j K_j(H_j)(t) \\ &= a\rho_j \sin(\rho_j t) + a\rho_j K_j(\cos(\rho_j \cdot))(t) + a\rho_j \sum_{k=1}^{+\infty} K_j(J_j^{k\circ}(\cos(\rho_j \cdot)))(t). \end{aligned}$$

and we have to determine the value of a by using the property $\|F_{\mu_j}\|_2 = 1$. Setting

$$v_j(t) := \rho_j K_j(\cos(\rho_j \cdot))(t) + \rho_j \sum_{k=1}^{+\infty} K_j(J_j^{k\circ}(\cos(\rho_j \cdot)))(t), \quad t \in [0, 1],$$

so that

$$(S.2.3.23) \quad F_{\mu_j}(t) = a\rho_j \sin(\rho_j t) + av_j(t), \quad t \in [0, 1],$$

we have, using Lemma S.2.2,

$$\|v_j\|_\infty \leq c_2 + c_3 \rho_j \sum_{k=1}^{+\infty} 2^{1-k} \|J_j(\cos(\rho_j \cdot))\|_\infty \leq c_2 + 2c_1 c_3$$

and, since $\rho_j^{-1} = O(j^{-1})$,

$$1 = \int F_{\mu_j}^2(t) dt = a^2 \rho_j^2 \int_0^1 \left(\sin(\rho_j t) + \frac{v_j(t)}{\rho_j} \right)^2 dt = a^2 \rho_j^2 \times \left(\frac{1}{2} + O(j^{-1}) \right).$$

This implies $a\rho_j = \sqrt{2} + O(j^{-1})$ and $a = \frac{\sqrt{2}}{\rho_j} + O(j^{-2})$. Using again Equation (S.2.3.23) yields the result stated in Equation (24) of the main file. Equation (S.2.3.21) shows that the eigenvalues $(\lambda_j)_{j \geq 1}$ are simple.

S.3. Proofs of Section 5.

S.3.1. Proof of Lemma 5.1. We first prove some intermediate results. First remark that given $i = 1, \dots, n$, if $T \in N_i$, there exists $k \in \{0, \dots, |\mathcal{T}|\}$ such that $T = T_k$, then for all $\ell \in \{1, \dots, |\mathcal{T}|\}$,

$$\int_{T_{\ell-1}}^{T_\ell} \mathbf{1}_{T \leq t} dt = \int_{T_{\ell-1}}^{T_\ell} \mathbf{1}_{T_k \leq t} dt = (T_\ell - T_{\ell-1}) \mathbf{1}_{T_k \leq T_{\ell-1}} = (T_\ell - T_{\ell-1}) \mathbf{1}_{T \leq T_{\ell-1}}.$$

This implies

$$\begin{aligned} \int_{T_{\ell-1}}^{T_\ell} F_{\Pi_i}(t) dt &= \int_{T_{\ell-1}}^{T_\ell} \sum_{T \in N_i} \mathbf{1}_{T \leq t} dt = \sum_{T \in N_i} \int_{T_{\ell-1}}^{T_\ell} \mathbf{1}_{T \leq t} dt \\ &= \sum_{T \in N_i} (T_\ell - T_{\ell-1}) \mathbf{1}_{T \leq T_{\ell-1}} = (T_\ell - T_{\ell-1}) F_{\Pi_i}(T_{\ell-1}), \end{aligned}$$

and consequently

$$\int_{T_{\ell-1}}^{T_\ell} F_{\widehat{W}}(t) dt = (T_\ell - T_{\ell-1}) \frac{1}{n} \sum_{i=1}^n F_{\Pi_i}(T_{\ell-1}) = (T_\ell - T_{\ell-1}) F_{\widehat{W}}(T_{\ell-1}).$$

and also

$$(S.3.1.24) \quad \int_{T_{\ell-1}}^{T_\ell} F_{\widehat{\Delta}_i}(t) dt = (T_\ell - T_{\ell-1}) F_{\widehat{\Delta}_i}(T_{\ell-1}).$$

We turn now to the proof of the first point. Let $\ell \in \{1, \dots, |\mathcal{T}|\}$,

$$\begin{aligned} \widehat{\Gamma}_{\widehat{\Delta}} e_\ell(s) &= \frac{1}{\sqrt{T_\ell - T_{\ell-1}}} \int_{T_{\ell-1}}^{T_\ell} \widehat{K}_{\widehat{\Delta}}(s, t) dt \\ &= \frac{1}{\sqrt{T_\ell - T_{\ell-1}}} \int_{T_{\ell-1}}^{T_\ell} \left(\frac{1}{n} \sum_{i=1}^n \sum_{T, T' \in N_i} \mathbf{1}_{T \leq s} \mathbf{1}_{T' \leq t} - F_{\widehat{W}}(s) F_{\widehat{W}}(t) \right) dt \\ &= \frac{1}{\sqrt{T_\ell - T_{\ell-1}}} \frac{1}{n} \sum_{i=1}^n \left(\sum_{T, T' \in N_i} \mathbf{1}_{T \leq s} \int_{T_{\ell-1}}^{T_\ell} \mathbf{1}_{T' \leq t} dt - F_{\widehat{W}}(s) \int_{T_{\ell-1}}^{T_\ell} F_{\widehat{W}}(t) dt \right) \\ &= \frac{1}{\sqrt{T_\ell - T_{\ell-1}}} \frac{1}{n} \sum_{i=1}^n \left(F_{\Pi_i}(s) \sum_{T' \in N_i} (T_\ell - T_{\ell-1}) \mathbf{1}_{T' \leq T_{\ell-1}} - F_{\widehat{W}}(s) F_{\widehat{W}}(T_{\ell-1}) (T_\ell - T_{\ell-1}) \right) \\ &= \frac{\sqrt{T_\ell - T_{\ell-1}}}{n} \sum_{i=1}^n (F_{\Pi_i}(s) F_{\Pi_i}(T_{\ell-1}) - F_{\widehat{W}}(s) F_{\widehat{W}}(T_{\ell-1})). \end{aligned}$$

Following the same lines, for all $\ell' \in \{1, \dots, |\mathcal{T}|\}$,

$$\begin{aligned} \langle \widehat{\Gamma}_{\widehat{\Delta}} e_\ell, e_{\ell'} \rangle &= \frac{1}{\sqrt{T_{\ell'} - T_{\ell'-1}}} \int_{T_{\ell'-1}}^{T_{\ell'}} \widehat{\Gamma}_{\widehat{\Delta}} e_\ell(s) ds \\ &= \frac{\sqrt{T_\ell - T_{\ell-1}}}{n \sqrt{T_{\ell'} - T_{\ell'-1}}} \sum_{i=1}^n \int_{T_{\ell'-1}}^{T_{\ell'}} (F_{\Pi_i}(s) F_{\Pi_i}(T_{\ell-1}) - F_{\widehat{W}}(s) F_{\widehat{W}}(T_{\ell-1})) ds \\ &= \frac{\sqrt{(T_\ell - T_{\ell-1})(T_{\ell'} - T_{\ell'-1})}}{n} \sum_{i=1}^n (F_{\Pi_i}(T_{\ell'-1}) F_{\Pi_i}(T_{\ell-1}) - F_{\widehat{W}}(T_{\ell'-1}) F_{\widehat{W}}(T_{\ell-1})). \end{aligned}$$

In particular, if for all ℓ , there exists a unique i such that $T_\ell \in N_i$, we have $F_{\widehat{W}}(T_{\ell-1}) = (\ell - 1)/n$ and

$$\int_{T_{\ell-1}}^{T_\ell} F_{\widehat{W}}(t)dt = \frac{\ell - 1}{n}(T_\ell - T_{\ell-1}),$$

which concludes the proof of the first point.

Now, we start the proof of the second point by proving that

$$(S.3.1.25) \quad \text{Im}(\widehat{\Gamma}_{\widehat{\Delta}}) \subset \text{span}\{e_1, \dots, e_{|\mathcal{T}|}\}.$$

Since $\widehat{\Gamma}_{\widehat{\Delta}}$ is a finite rank operator, both spaces $\text{Im}(\widehat{\Gamma}_{\widehat{\Delta}})$ and $\text{span}\{e_1, \dots, e_{|\mathcal{T}|}\}$ are finite-dimensional vector spaces and it is sufficient to prove

$$\text{span}\{e_1, \dots, e_{|\mathcal{T}|}\}^\perp \subset \text{Im}(\widehat{\Gamma}_{\widehat{\Delta}})^\perp.$$

Moreover, since $\widehat{\Gamma}_{\widehat{\Delta}}$ is a self-adjoint operator,

$$\text{Im}(\widehat{\Gamma}_{\widehat{\Delta}})^\perp = \text{Ker}(\widehat{\Gamma}_{\widehat{\Delta}}).$$

Then, to prove (S.3.1.25), it is sufficient to prove that

$$\text{span}\{e_1, \dots, e_{|\mathcal{T}|}\}^\perp \subset \text{Ker}(\widehat{\Gamma}_{\widehat{\Delta}}).$$

Let $f \in \text{span}\{e_1, \dots, e_{|\mathcal{T}|}\}^\perp$, we have for any $\ell \in \{1, \dots, |\mathcal{T}|\}$,

$$\langle f, e_\ell \rangle = 0,$$

then, by definition of e_ℓ ,

$$\int_{T_{\ell-1}}^{T_\ell} f(t)dt = 0.$$

This implies that, for all $i = 1, \dots, n$, for all $T \in N_i$, since, by construction of the grid $\mathcal{T}$, there exists ℓ such that $T = T_{\ell-1}$,

$$\int_T^1 f(t)dt = \sum_{\ell'= \ell}^{|\mathcal{T}|} \int_{T_{\ell'-1}}^{T_{\ell'}} f(t)dt = 0.$$

Hence

$$\langle F_{\widehat{W}}, f \rangle = \int_0^1 \widehat{W}([0, t])f(t)dt = \frac{1}{n} \sum_{i=1}^n \sum_{T \in N_i} \int_0^1 \mathbf{1}_{\{T \leq t\}} f(t)dt = \frac{1}{n} \sum_{i=1}^n \sum_{T \in N_i} \int_T^1 f(t)dt = 0$$

and

$$\begin{aligned} \widehat{\Gamma}_{\widehat{\Delta}} f(t) &= \frac{1}{n} \sum_{i=1}^n \sum_{T, T' \in N_i} \int_0^1 \mathbf{1}_{\{T \leq s, T' \leq t\}} f(s)ds - \widehat{W}([0, t]) \langle F_{\widehat{W}}, f \rangle, \\ &= \frac{1}{n} \sum_{i=1}^n \sum_{T, T' \in N_i} \mathbf{1}_{\{T' \leq t\}} \int_T^1 f(s)ds - \widehat{W}([0, t]) \langle F_{\widehat{W}}, f \rangle = 0. \end{aligned}$$

This concludes the proof of (S.3.1.25).

Now we identify the eigenfunctions of $\widehat{\Gamma}_{\widehat{\Delta}}$ with the eigenvectors of the matrix $\widehat{G}_{\widehat{\Delta}}$.

- Let $j \geq 1$. We consider $\hat{\lambda}_j$ a non-zero eigenvalue of $\hat{\Gamma}_{\hat{\Delta}}$. Let $\hat{\eta}_j$ an associated eigenfunction such that $\|\hat{\eta}_j\| = 1$. We set

$$\hat{v}_j = (\langle \hat{\eta}_j, e_\ell \rangle)_{\ell=1, \dots, |\mathcal{T}|}.$$

We have

$$\begin{aligned} \hat{G}_{\hat{\Delta}} \hat{v}_j &= \left(\sum_{\ell'=1}^{|\mathcal{T}|} \langle \hat{\Gamma}_{\hat{\Delta}} e_{\ell}, e_{\ell'} \rangle \langle \hat{\eta}_j, e_{\ell'} \rangle \right)_{\ell=1, \dots, |\mathcal{T}|} \\ (S.3.1.26) \quad &= \left(\left\langle \hat{\Gamma}_{\hat{\Delta}} e_{\ell}, \sum_{\ell'=1}^{|\mathcal{T}|} \langle \hat{\eta}_j, e_{\ell'} \rangle e_{\ell'} \right\rangle \right)_{\ell=1, \dots, |\mathcal{T}|}. \end{aligned}$$

Now remark that $\sum_{\ell'=1}^{|\mathcal{T}|} \langle \hat{\eta}_j, e_{\ell'} \rangle e_{\ell'}$ is the orthogonal projection of the function $\hat{\eta}_j$ on the space $\text{span}\{e_1, \dots, e_{|\mathcal{T}|}\}$. Then, since $\hat{\Gamma}_{\hat{\Delta}} \hat{\eta}_j = \hat{\lambda}_j \hat{\eta}_j$ and $\hat{\lambda}_j > 0$, from (S.3.1.25), we deduce

$$\hat{\eta}_j \in \text{Im}(\hat{\Gamma}_{\hat{\Delta}}) \subset \text{span}\{e_1, \dots, e_{|\mathcal{T}|}\},$$

meaning that

$$\sum_{\ell'=1}^{|\mathcal{T}|} \langle \hat{\eta}_j, e_{\ell'} \rangle e_{\ell'} = \hat{\eta}_j$$

and $\|\hat{v}_j\| = 1$. Then, since $\hat{\Gamma}_{\hat{\Delta}}$ is a self-adjoint operator, (S.3.1.26) can be written

$$\hat{G}_{\hat{\Delta}} \hat{v}_j = \left(\left\langle \hat{\Gamma}_{\hat{\Delta}} e_{\ell}, \hat{\eta}_j \right\rangle \right)_{\ell=1, \dots, |\mathcal{T}|} = \left(\left\langle e_{\ell}, \hat{\Gamma}_{\hat{\Delta}} \hat{\eta}_j \right\rangle \right)_{\ell=1, \dots, |\mathcal{T}|} = \hat{\lambda}_j (\langle e_{\ell}, \hat{\eta}_j \rangle)_{\ell=1, \dots, |\mathcal{T}|} = \hat{\lambda}_j \hat{v}_j.$$

This means that $\hat{\lambda}_j \in \text{Sp}(\hat{G}_{\hat{\Delta}})$ and, consequently

$$\text{Sp}(\hat{\Gamma}_{\hat{\Delta}}) \setminus \{0\} \subset \text{Sp}(\hat{G}_{\hat{\Delta}}).$$

- Conversely, let $\hat{v}_j = (\hat{v}_j^1, \dots, \hat{v}_j^{|\mathcal{T}|})^t$ a unit-norm eigenvector of $\hat{G}_{\hat{\Delta}}$ associated with the eigenvalue $\hat{\lambda}_j$ and let

$$\hat{\eta}_j = \sum_{\ell=1}^{|\mathcal{T}|} \hat{v}_j^{\ell} e_{\ell}.$$

We have:

$$\begin{aligned} \hat{\Gamma}_{\hat{\Delta}} \hat{\eta}_j &= \sum_{\ell'=1}^{|\mathcal{T}|} \langle \hat{\Gamma}_{\hat{\Delta}} \hat{\eta}_j, e_{\ell'} \rangle e_{\ell'} = \sum_{\ell, \ell'=1}^{|\mathcal{T}|} \hat{v}_j^{\ell} \langle \hat{\Gamma}_{\hat{\Delta}} e_{\ell}, e_{\ell'} \rangle e_{\ell'} = \sum_{\ell'=1}^{|\mathcal{T}|} \left[\hat{G}_{\hat{\Delta}} \hat{v}_j \right]_{\ell'} e_{\ell'} \\ &= \hat{\lambda}_j \sum_{\ell'=1}^{|\mathcal{T}|} \hat{v}_j^{\ell'} e_{\ell'} = \hat{\lambda}_j \hat{\eta}_j. \end{aligned}$$

Then, $\hat{\eta}_j$ is a unit-norm eigenfunction of the operator $\hat{\Gamma}_{\hat{\Delta}}$ associated with the eigenvalue $\hat{\lambda}_j$ and

$$\text{Sp}(\hat{G}_{\hat{\Delta}}) \subset \text{Sp}(\hat{\Gamma}_{\hat{\Delta}}).$$

To end the proof of the third point, it remains to remark that, since the functions e_ℓ 's are right continuous functions, then $\hat{\eta}_j$ also is a right-continuous function. Moreover, following the same lines than the proof of Proposition 3.1, $\hat{\eta}_j$ is a function with bounded variations and $\hat{\eta}_j(t) = 0$ for $t < 0$. Then, applying Proposition 4.4.3 of [2], we obtain the expected result: there exists a unique measure $\hat{\mu}_j$ such that $F_{\hat{\mu}_j} = \hat{\eta}_j$.

S.3.2. Proof of Theorem 5.6. We first prove the first point. From [1], Lemma 4.2, p. 103, we can deduce that

$$(S.3.2.27) \quad \sup_{j \geq 1} |\hat{\lambda}_j - \lambda_j| \leq \|\Gamma_\Delta - \hat{\Gamma}_\Delta\|,$$

where, for an operator S on $\mathbb{L}^2([0, 1])$,

$$\|S\| = \sup_{x \in \mathbb{L}^2([0, 1])} \frac{\|Sx\|}{\|x\|}$$

denotes the usual operator norm. Let $\hat{\Gamma}_\Delta = \frac{1}{n} \sum_{i=1}^n F_{\Delta_i} \otimes F_{\Delta_i}$ we get

$$(S.3.2.28) \quad \|\Gamma_\Delta - \hat{\Gamma}_\Delta\| \leq \|\hat{\Gamma}_\Delta - \Gamma_\Delta\| + \|\hat{\Gamma}_\Delta - \hat{\Gamma}_\Delta\|.$$

First, we deal with the first term on the right-hand side of (S.3.2.28). We get classically,

$$\hat{\Gamma}_\Delta - \Gamma_\Delta = \frac{1}{n} \sum_{i=1}^n F_{\Delta_i} \otimes F_{\Delta_i} - \mathbb{E}[F_\Delta \otimes F_\Delta].$$

We remark that $F_{\Delta_i} \otimes F_{\Delta_i} - \mathbb{E}[F_\Delta \otimes F_\Delta]$, $i = 1, \dots, n$ is an i.i.d. sequence of centered random variables taking values in the space of Hilbert-Schmidt operators equipped with its usual norm $\|\cdot\|_{HS}$ defined by $\|S\|_{HS}^2 = \sum_{j \geq 1} \|Se_j\|^2$ for $(e_j)_{j \geq 1}$ an orthonormal (hilbertian) basis of $\mathbb{L}^2([0, 1])$ (recall that the definition of the norm does not depend on the basis). Furthermore, we have

$$\|\hat{\Gamma}_\Delta - \Gamma_\Delta\| \leq \|\hat{\Gamma}_\Delta - \Gamma_\Delta\|_{HS}$$

and

$$\|\hat{\Gamma}_\Delta - \Gamma_\Delta\|_{HS}^2 = \frac{1}{n^2} \sum_{i,j=1}^n \langle F_{\Delta_i} \otimes F_{\Delta_i} - \mathbb{E}[F_\Delta \otimes F_\Delta], F_{\Delta_j} \otimes F_{\Delta_j} - \mathbb{E}[F_\Delta \otimes F_\Delta] \rangle_{HS},$$

where, for two Hilbert-Schmidt operators S and T , the Hilbert-Schmidt scalar product is defined by $\langle S_1, S_2 \rangle_{HS} = \sum_{j \geq 1} \langle S_1 e_j, S_2 e_j \rangle$ and does not depend on the considered basis. Then, taking $(e_j)_j = (\eta_j)_j$, we remark that

$$\begin{aligned} & \mathbb{E}[\langle F_{\Delta_i} \otimes F_{\Delta_i} - \mathbb{E}[F_\Delta \otimes F_\Delta], F_{\Delta_j} \otimes F_{\Delta_j} - \mathbb{E}[F_\Delta \otimes F_\Delta] \rangle_{HS}] \\ &= \sum_{\ell \geq 1} \mathbb{E}[\langle (F_{\Delta_i} \otimes F_{\Delta_i} - \mathbb{E}[F_\Delta \otimes F_\Delta])\eta_\ell, (F_{\Delta_j} \otimes F_{\Delta_j} - \mathbb{E}[F_\Delta \otimes F_\Delta])\eta_\ell \rangle] \\ &= \sum_{\ell \geq 1} \int_0^1 \text{Cov}(F_{\Delta_i} \otimes F_{\Delta_i} \eta_\ell(t), F_{\Delta_j} \otimes F_{\Delta_j} \eta_\ell(t)) dt \\ &= \mathbf{1}_{i=j} \sum_{\ell \geq 1} \int_0^1 \text{Var}(F_\Delta \otimes F_\Delta \eta_\ell(t)) dt \\ &= \mathbf{1}_{i=j} \sum_{\ell \geq 1} \int_0^1 \text{Var}(\langle F_\Delta, \eta_\ell \rangle F_\Delta(t)) dt \end{aligned}$$

$$\leq \mathbf{1}_{i=j} \sum_{\ell \geq 1} \mathbb{E} \left[\langle F_{\Delta}, \eta_{\ell} \rangle^2 \int_0^1 F_{\Delta}^2(t) dt \right] = \mathbf{1}_{i=j} \mathbb{E} [\|F_{\Delta}\|^4].$$

Then, finally, we get

$$(S.3.2.29) \quad \mathbb{E} [\|\widehat{\Gamma}_{\Delta} - \Gamma_{\Delta}\|^2] \leq \frac{\mathbb{E} [\|F_{\Delta}\|^4]}{n}.$$

Then for the second term on the right-hand side of (S.3.2.28),

$$\|\widehat{\Gamma}_{\Delta} - \widehat{\Gamma}_{\widehat{\Delta}}\| = \|(F_{\widehat{W}} - F_W) \otimes (F_{\widehat{W}} - F_W)\| = \|F_{\widehat{W}} - F_W\|^2.$$

Then, since $F_{\Delta}(t) = F_{\Pi}(t) - \mathbb{E}[F_{\Pi}(t)]$,

$$\begin{aligned} \mathbb{E}[\|\widehat{\Gamma}_{\Delta} - \widehat{\Gamma}_{\widehat{\Delta}}\|^2] &= \mathbb{E}[\|F_{\widehat{W}} - F_W\|^2] \\ &= \mathbb{E} \left[\int_0^1 (F_{\widehat{W}}(t) - F_W(t))^2 dt \right] \\ &= \int_0^1 \text{Var}(F_{\widehat{W}}(t)) dt \\ &= \int_0^1 \text{Var} \left(\frac{1}{n} \sum_{i=1}^n F_{\Pi_i}(t) \right) dt \\ (S.3.2.30) \quad &= \frac{1}{n} \int_0^1 \text{Var}(F_{\Pi}(t)) dt = \frac{1}{n} \int_0^1 \mathbb{E}[F_{\Delta}(t)^2] dt = \frac{1}{n} \mathbb{E}[\|F_{\Delta}\|^2]. \end{aligned}$$

Finally, combining eq. (S.3.2.28), (S.3.2.29) and (S.3.2.30), we get

$$(S.3.2.31) \quad \mathbb{E}[\|\Gamma_{\Delta} - \widehat{\Gamma}_{\widehat{\Delta}}\|^2] \leq 4 \frac{\mathbb{E} [\|F_{\Delta}\|^4]}{n},$$

and Inequality (26) of the main file follows from Bosq inequality (S.3.2.27).

From [1], Lemma 4.3, p.104, we have the upper-bound

$$\|\widehat{\eta}_j - \widetilde{\eta}_j\| \leq 2\sqrt{2}\delta_j^{-1} \|\Gamma_{\Delta} - \widehat{\Gamma}_{\widehat{\Delta}}\|,$$

and Inequality (27) of the main file follows from (S.3.2.31). Now, we also have, by Cauchy-Schwarz's inequality, for all $\varphi \in \mathcal{H}_0^1$,

$$|\langle \widehat{\mu}_j - \widetilde{\mu}_j, \varphi \rangle|^2 = |\langle \widehat{\eta}_j - \widetilde{\eta}_j, \varphi' \rangle|^2 \leq \|\widehat{\eta}_j - \widetilde{\eta}_j\|^2 \|\varphi'\|^2.$$

Then

$$\|\widehat{\mu}_j - \widetilde{\mu}_j\|_{\mathcal{H}^{-1}}^2 \leq \|\widehat{\eta}_j - \widetilde{\eta}_j\|^2$$

and Inequality (28) of the main file follows.

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