What do wavelets offer ?

1. Analysis of signals : wavelets provide signals decompositions of the form

$$f = A_{j_0} + \sum_{j \leq j_0} D_j = \sum_k \alpha_{j_0 k} \phi_{j_0 k} + \sum_{j \leq j_0} \sum_k \beta_{jk} \psi_{jk}$$

2. Denoising = functional estimation : Wavelet coefficients are estimated by thresholding. Thresholding is a “keep or kill” algorithm.

3. Data compression : Wavelet decompositions are sparse. Few informations have to be encoded (Ex : JPEG 2000, fingerprints storage by FBI)

The continuous Fourier transform

The Fourier transform provides analysis and reconstruction formula :

1. Analysis :

$$\hat{f}(w) = \mathcal{F}(f)(w) = \int_{\mathbb{R}} f(t) \exp(-2i\pi wt) dt$$

2. Reconstruction :

$$f(t) = \overline{\mathcal{F}}(\hat{f})(t) = \int_{\mathbb{R}} \hat{f}(w) \exp(2i\pi wt) dw$$

Theorem 1 *The Fourier Transform* $\mathcal{F} : \mathbb{L}_2(\mathbb{R}) \longrightarrow \mathbb{L}_2(\mathbb{R})$ *is linear, continuous, and bijective. We have for any* $f, g \in \mathbb{L}_2(\mathbb{R})$*,*

$$f = (\mathcal{F} \circ \overline{\mathcal{F}})(f) = (\overline{\mathcal{F}} \circ \mathcal{F})(f)$$

$$\int_{\mathbb{R}} f(t) g(t) dt = \int_{\mathbb{R}} \mathcal{F}(f)(w) \mathcal{F}(g)(w) dw = \int_{\mathbb{R}} \overline{\mathcal{F}}(f)(w) \overline{\mathcal{F}}(g)(w) dw$$

$$\|\mathcal{F} f\|_{\mathbb{L}_2} = \|\overline{\mathcal{F}} f\|_{\mathbb{L}_2} = \|f\|_{\mathbb{L}_2}$$

First drawback of the Fourier transform

Irregularities are not detected. Change points detection is not possible
Let us give illustrations by using step functions :

$$f_a(t) = 1_{[-a,a]}(t), \quad \hat{f}(w) = \mathcal{F}(f)(w) = \frac{\sin(2\pi a w)}{\pi w}$$

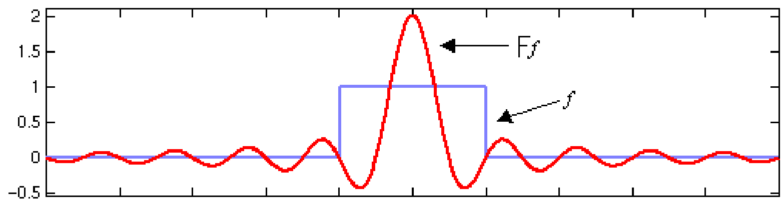


FIG. 1 – $f(t) = f_1(t)$

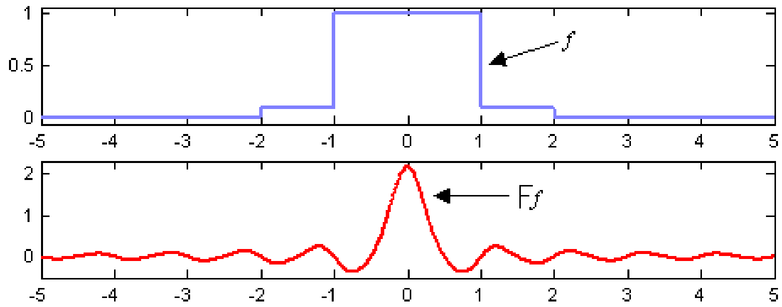
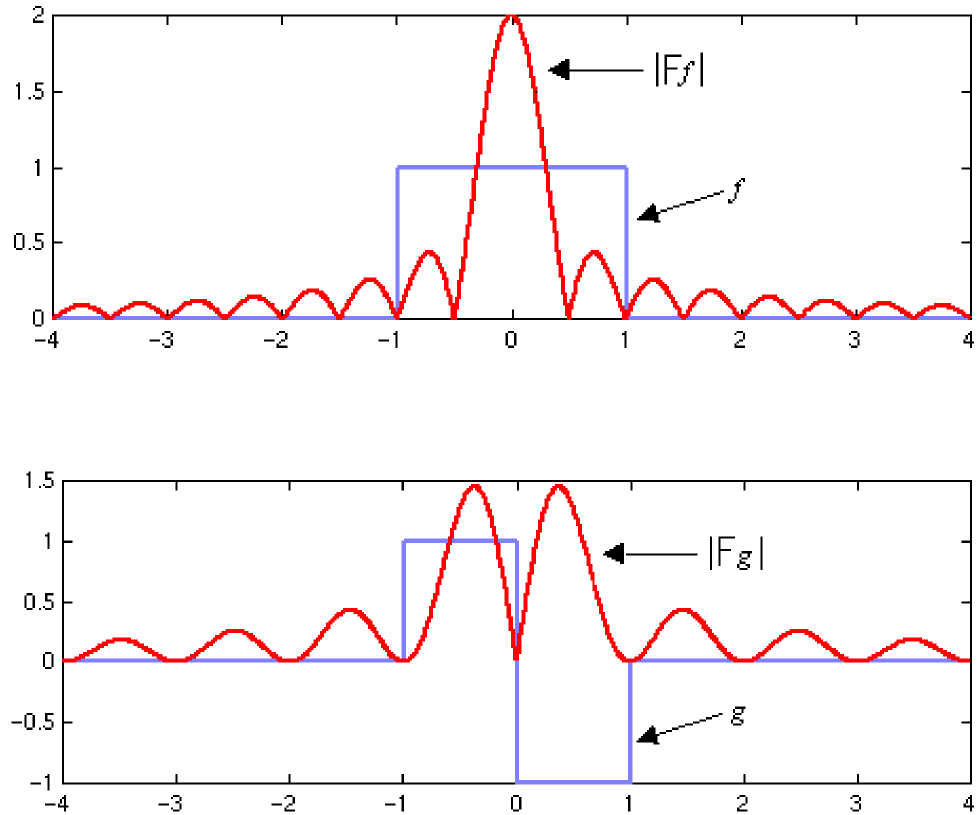


FIG. 2 – $f(t) = 0.9f_1(t) + 0.1f_2(t)$

Second drawback of the Fourier transform

The signal has to be known on the whole real line.



Third drawback of the Fourier transform

Heisenberg incertitude principle : Deviations of $|f|^2$ and $|\hat{f}|^2$ are related :

$$\sigma_f^2 \sigma_{\hat{f}}^2 \geq \frac{1}{4}$$

with

$$\sigma_f^2 = \frac{1}{\|f\|_{L_2}^2} \int_{\mathbb{R}} (t-u)^2 |f(t)|^2 dt, \quad \sigma_{\hat{f}}^2 = \frac{1}{\|f\|_{L_2}^2} \int_{\mathbb{R}} (w-\xi)^2 |\hat{f}(w)|^2 dw$$

and

$$u = \frac{1}{\|f\|_{L_2}^2} \int_{\mathbb{R}} t |f(t)|^2 dt, \quad \xi = \frac{1}{\|f\|_{L_2}^2} \int_{\mathbb{R}} w |\hat{f}(w)|^2 dw$$

So, f and $\hat{f}$ can't be both localized

Summary : Global transforms such as the Fourier transform have to be avoided

The Gabor transform

We replace the Fourier atoms

$$F_w(t) = \exp(-2i\pi wt)$$

with the localized Gabor atoms

$$G_{w,b}(t) = g(t-b)\exp(-2i\pi wt)$$

The Gabor transform provides analysis and reconstruction formula :

1. Analysis :

$$\mathcal{G}(f)(w,b) = \int_{\mathbb{R}} f(t) G_{w,b}(t) dt$$

2. Reconstruction :

$$f(t) = \int_{\mathbb{R}^2} \mathcal{G}(f)(w,b) \overline{G_{w,b}(t)} dw db$$

Theorem 2 *The Gabor Transform* $\mathcal{F} : \mathbb{L}_2(\mathbb{R}) \longrightarrow \mathbb{L}_2(\mathbb{R})$ *is linear, continuous, and bijective. It leaves norms and scalar products unchanged.*

Examples and drawbacks of the Gabor transform

Examples :

1. $g(t) = \frac{1}{\sqrt{2a}} 1_{[-a,a]}(t)$
2. $g(t) = \exp(-\pi t^2)$, $\mathcal{F}(g) = g$

With $g(t) = \exp(-\pi t^2)$, $G_{w,b}(t) = \exp(-\pi(t-b)^2 - 2i\pi w t)$,

$$\mathcal{G}(f)(w,b) = \int_{\mathbb{R}} f(t) e^{-\pi(t-b)^2 - 2i\pi w t} dt = \int_{\mathbb{R}} \hat{f}(\xi) e^{-\pi(\xi-w)^2 + 2i\pi b(\xi - w)} d\xi$$

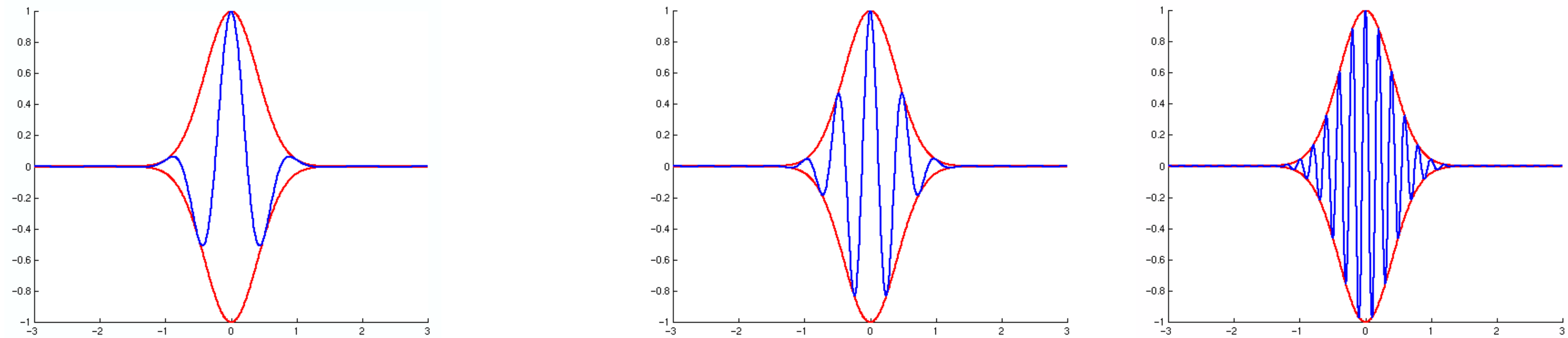


FIG. 3 – Envelop and real part of $G_{w,b}$ for $b = 0$ and $w = 1, 2, 5$

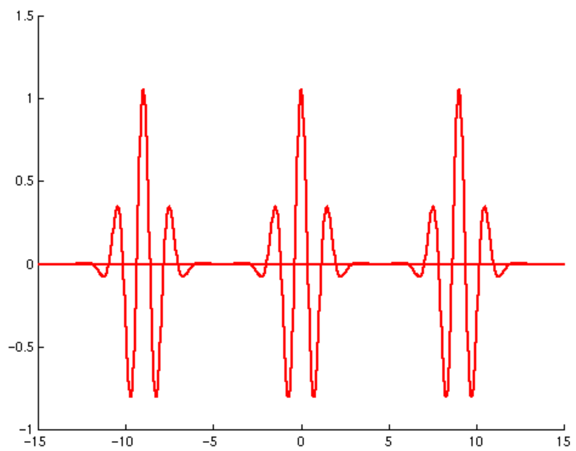
Wavelets

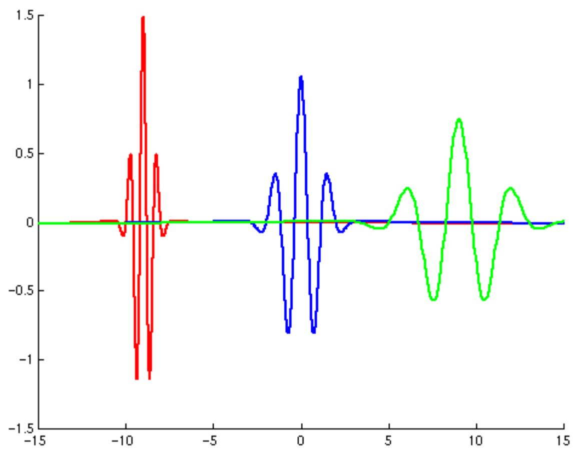
We are looking for a transform that acts in a whole range of temporal resolutions simultaneously. That's the purpose of the wavelet transform. A function $\psi \in \mathbb{L}_1(\mathbb{R}) \cap \mathbb{L}_2(\mathbb{R})$ such that for some $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} t^p \psi(t) dt = 0, \quad p = 0, \ldots, n$$

is called wavelet. We set for any $a > 0$ and any $b \in \mathbb{R}$,

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$$





The (continuous) wavelet transform

Let ψ be a wavelet.

1. Analysis :

$$\mathcal{C}(f)(a,b) = \int_{\mathbb{R}} f(t)\psi_{a,b}(t)dt$$

2. Reconstruction : if

$$\int_{\mathbb{R}} |t\psi(t)|dt < \infty,$$

   we have

$$f(t) = \frac{1}{K_\psi} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{C}(f)(a,b)\psi_{a,b}(t)\frac{dadb}{a^2},$$

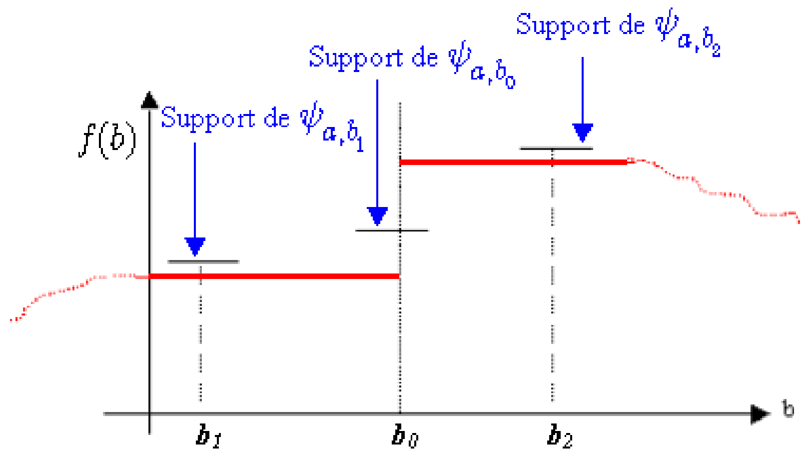
   with

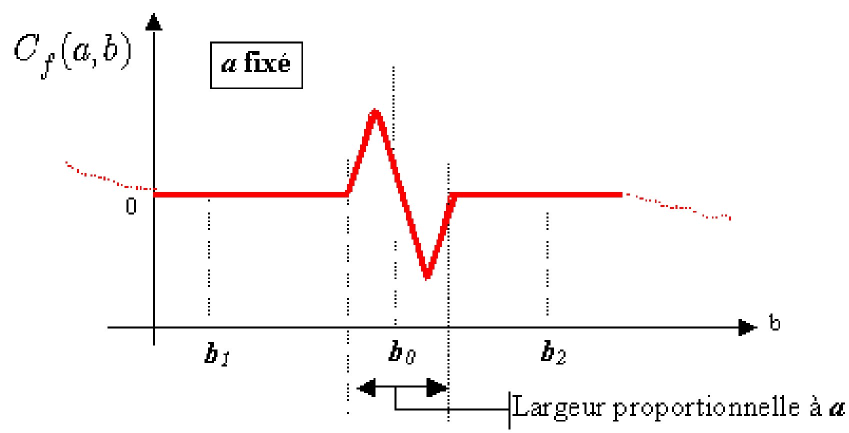
$$K_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(w)|^2}{|w|} dw.$$

Wavelets enable to detect irregularities and change points.

Change points detection by using wavelets

Observe that if $\psi(t) = 0$ for $t \in [-M, M]$, then $\psi_{a,b}(t) = 0$ if $t \in [b - aM, b + aM]$.





From continuous to discrete wavelet transform

We want to avoid redundancy.

We don't use all functions $\{\psi_{a,b}\}_{a>0, b\in\mathbb{R}}$. We only consider the values $a = 2^j$, $b = 2^j k$, for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$.

Fom now on, we change our notations and we set

$$\psi_{jk}(t) = 2^{-\frac{j}{2}}\psi(2^{-j}t - k).$$

We are looking for a function ψ such that $\{\psi_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ is an orthonormal basis of $\mathbb{L}_2(\mathbb{R})$.

This issue is related to the concept of multiresolution analysis.

Multiresolution analysis

A multiresolution analysis of $\mathbb{L}_2(\mathbb{R})$ is a family $\{V_j\}_{j\in\mathbb{Z}}$ of embedded linear subspaces with properties (A), (B) and (C) :

- Property (A) : For any $j \in \mathbb{Z}$

1. V_j is a closed subspace of $\mathbb{L}_2(\mathbb{R})$
2. $V_j \subset V_{j-1}$
3. Finally,

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = \mathbb{L}_2(\mathbb{R}), \qquad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}$$

So, the spaces V_j are approximation spaces

Multiresolution analysis (continued)

A multiresolution analysis of $\mathbb{L}_2(\mathbb{R})$ is a family $\{V_j\}_{j\in\mathbb{Z}}$ of embedded subspaces with properties (A), (B) and (C) :

- Property (B) : The spaces V_j are obtained by dyadic dilation or contraction of the functions of V_0. This implies :

$$\forall\, j \in \mathbb{Z}, \quad v(t) \in V_j \iff v(2t) \in V_{j-1}.$$

- Property (C) : There exists $g \in V_0$ such that $\{g(t-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of V_0.

It means that for any $h \in V_0$, there exists a unique $\alpha = (\alpha_k)_{k\in\mathbb{Z}} \in l_2(\mathbb{Z})$ such that

1.

$$\forall\, t \in \mathbb{R}, \quad h(t) = \sum_{k\in\mathbb{Z}} \alpha_k g(t-k)$$

2.

$$A\|\alpha\|_{l_2} \leq \|h\|_{L_2} \leq B\|\alpha\|_{l_2},$$

where A and B depend only on g.

Detail spaces

Previously, $\{V_j\}_{j\in\mathbb{Z}}$ could be viewed as approximation spaces. Detail spaces are denoted $\{W_j\}_{j\in\mathbb{Z}}$.

W_j is the orthogonal supplement of V_j in V_{j-1} :

$$V_{j-1} = V_j \oplus W_j, \quad V_j \perp W_j$$

Properties :

- $W_j \perp W_k,\ j \neq k$

- $V_J = V_K \oplus W_K \oplus \cdots \oplus W_{J+1},\ J < K$

- $V_J = \oplus_{j=J+1}^{+\infty} W_j$

- $\mathbb{L}_2(\mathbb{R}) = \oplus_{j=-\infty}^{+\infty} W_j = V_J \oplus \left\{\oplus_{j=-\infty}^{J} W_j\right\}$

Main theorem of this talk

Recall that we denote for any function f and for any $j, k \in \mathbb{Z}$

$$f_{jk}(t) = 2^{-\frac{j}{2}} f(2^{-j}t - k)$$

Theorem 3 *There exists a function ϕ, then a function ψ such that for any J,*

$$\{\{\phi_{Jk}\}_{k\in\mathbb{Z}}, \{\psi_{jk}\}_{j\leq J, k\in\mathbb{Z}}\}$$

is an orthonormal basis of $\mathbb{L}_2(\mathbb{R})$ and

$$\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$$

is an orthonormal basis of $\mathbb{L}_2(\mathbb{R})$. If for any j,

$$V_j = \overline{\text{span}\,(\phi_{jk} : \quad k \in \mathbb{Z})}$$

$$W_j = \overline{\text{span}\,(\psi_{jk} : \quad k \in \mathbb{Z})}$$

then $\{V_j\}_{j\in\mathbb{Z}}$ is a multiresolution analysis of $\mathbb{L}_2(\mathbb{R})$ and $\{W_j\}_{j\in\mathbb{Z}}$ are the detail spaces.

ϕ is called the father wavelet (or the scaling function)
ψ is called the mother wavelet (or the wavelet)

Consequences of the main theorem

Any $f \in \mathbb{L}_2(\mathbb{R})$ has following expansions. If we set, for any $j \in \mathbb{Z}$ and any $k \in \mathbb{Z}$,

$$\alpha_{jk} = \int_{\mathbb{R}} f(x)\phi_{jk}(x)dx, \quad \beta_{jk} = \int_{\mathbb{R}} f(x)\psi_{jk}(x)dx$$

we obtain the inhomogeneous wavelet expansion :

$$\forall\ j_0 \in \mathbb{Z}, \quad f(t) = \sum_{k\in\mathbb{Z}} \alpha_{j_0k}\phi_{j_0k}(t) + \sum_{j\le j_0}\sum_{k} \beta_{jk}\psi_{jk}(t)$$

and the homogeneous wavelet expansion :

$$f(t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \beta_{jk}\psi_{jk}(t)$$

So, f is of the form

$$f = A_{j_0} + \sum_{j\le j_0} D_j$$

$A_{j_0} = \text{proj}_{V_{j_0}}(f) = \sum_{k\in\mathbb{Z}} \alpha_{j_0k}\phi_{j_0k}$ is the approximation term ($\lim_{j_0\to-\infty} A_{j_0} = f$)
$D_j = \text{proj}_{W_j}(f) = \sum_{k\in\mathbb{Z}} \beta_{jk}\psi_{jk}$ for $j \le j_0$ are the detail terms ($W_j \perp V_{j_0}, j \le j_0$)

Sketch of the proof of the main theorem

We start from a trigonometric polynomial of the form :

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{k=N_0}^{N_1} h_k e^{-ik\xi},$$

for $N_0, N_1 \in \mathbb{Z}$ such that

1. $m_0(0) = 1$
2. $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$
3. $m_0(\xi) \neq 0$ for any $|\xi| \leq \frac{\pi}{2}$

then we set

$$\hat{\phi}(\xi) = \prod_{j=1}^{+\infty} m_0(2^{-j}\xi)$$

$$\hat{\psi}(\xi) = \overline{m_0\left(\frac{\xi}{2} + \pi\right)} e^{-i\frac{\xi}{2}} \hat{\phi}\left(\frac{\xi}{2}\right)$$

$$V_j = \overline{\text{span}\,(\phi_{jk} : \quad k \in \mathbb{Z})}$$

Simple important observations

We note

$$\phi(t) = \sqrt{2} \sum_l h_l \phi(2t - l)$$

$$\psi(t) = \sqrt{2} \sum_l \lambda_l \phi(2t - l), \quad \lambda_l = (-1)^{l+1} h_{1-l}$$

This implies for any j and any k,

$$\alpha_{jk} = \sum_l h_{l-2k} \alpha_{j-1\,l}$$

$$\beta_{jk} = \sum_l \lambda_{l-2k} \alpha_{j-1\,l}$$

This leads to the cascade algorithm. Furthermore ϕ is supported by $[N_0, N_1]$ (hard to establish). So, ψ is supported by $\left[\frac{1-N_1+N_0}{2}, \frac{1-N_0+N_1}{2}\right]$.

Construction of wavelets

Now, the question is : How to choose the filter $(h_k)_{N_0 \leq 1 \leq N_1}$?

If $N_1 - N_0 + 1 > 2$, then there exist many possible solutions. We choose $(h_k)_{N_0 \leq 1 \leq N_1}$ such that ϕ and ψ have

1. a prescribed number of continuous derivatives
2. a prescribed number of vanishing moments. More precisely, for a prescribed number N, we wish

$$\int_{\mathbb{R}} x^l \phi(x) = 0, \quad l = 1, \ldots, N$$

$$\int_{\mathbb{R}} x^l \psi(x) = 0, \quad l = 0, \ldots, N$$

Note that $\int_{\mathbb{R}} \phi(x) = 1$

First example : the Haar basis

The Haar system is associated with

$$m_0(\xi) = \frac{1 + e^{-i\xi}}{2}.$$

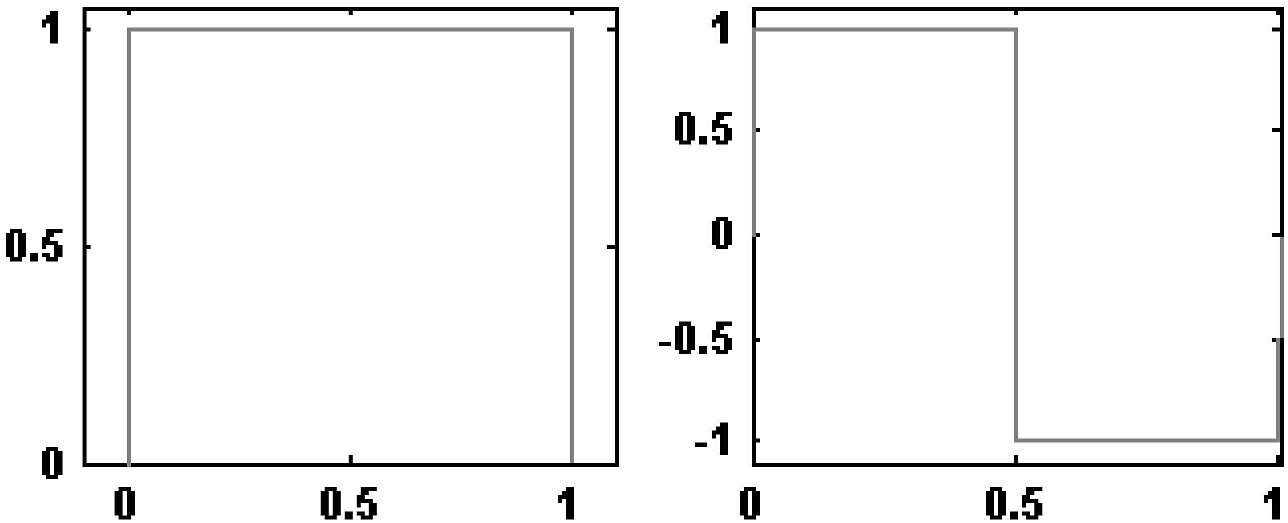


FIG. 4 – The father wavelet ϕ and the mother wavelet ψ for the Haar system

Second example : the Daubechies wavelet

For $N \in \mathbb{N}$, the Daubechies wavelet of order N, denoted DauN, is associated with

$$m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N P(\xi),$$

for P a convenient trigonometric polynomial.

1. If $N = 1$, we have to take $P = 1$, so obtain the Haar system
2. If $N = 2$, we have to take $P(\xi) = \frac{1+\sqrt{3}+(1-\sqrt{3})e^{-i\xi}}{2}$, so

$$m_0(\xi) = \frac{1}{8}\left((1+\sqrt{3}) + (3+\sqrt{3})e^{-i\xi} + (3-\sqrt{3})e^{-2i\xi} + (1-\sqrt{3})e^{-3i\xi}\right)$$

Properties :

$$\text{Supp}(\phi) \subset [0, 2N-1], \quad \text{Supp}(\psi) \subset [-N+1, N],$$

$$\int_{\mathbb{R}} x^l \psi(x) = 0, \quad l = 0, \ldots, N-1$$

$$\text{For } N \geq 2, \quad \phi, \psi \in C^{0.1936N}$$

Third example : the coiflets

For $N \in \mathbb{N}$, the coiflet of order N, denoted CoifN is associated with

$$m_0(\xi) = 1 + \left(1 - e^{-i\xi}\right)^{2N} Q(\xi),$$

for Q a convenient trigonometric polynomial.

Properties :

$$\text{Supp}(\phi) \subset [-2N, 4N-1], \quad \text{Supp}(\psi) \subset [-4N+1, 2N],$$

$$\int_{\mathbb{R}} x^l \phi(x) = 0, \quad l = 1, \ldots, 2N-1$$

$$\int_{\mathbb{R}} x^l \psi(x) = 0, \quad l = 0, \ldots, 2N-1$$

Forth example : the symlets

It was shown by Daubechies that except for the Haar system, no system (ϕ, ψ) can be at the same time compactly supported and symmetric. The symlets are almost symmetric.

Properties :

$$\text{Supp}(\phi) \subset [0, 2N-1], \quad \text{Supp}(\psi) \subset [-N+1, N],$$

$$\int_{\mathbb{R}} x^l \psi(x) = 0, \quad l = 0, \ldots, N-1$$

Analysis of an electrical consumption signal

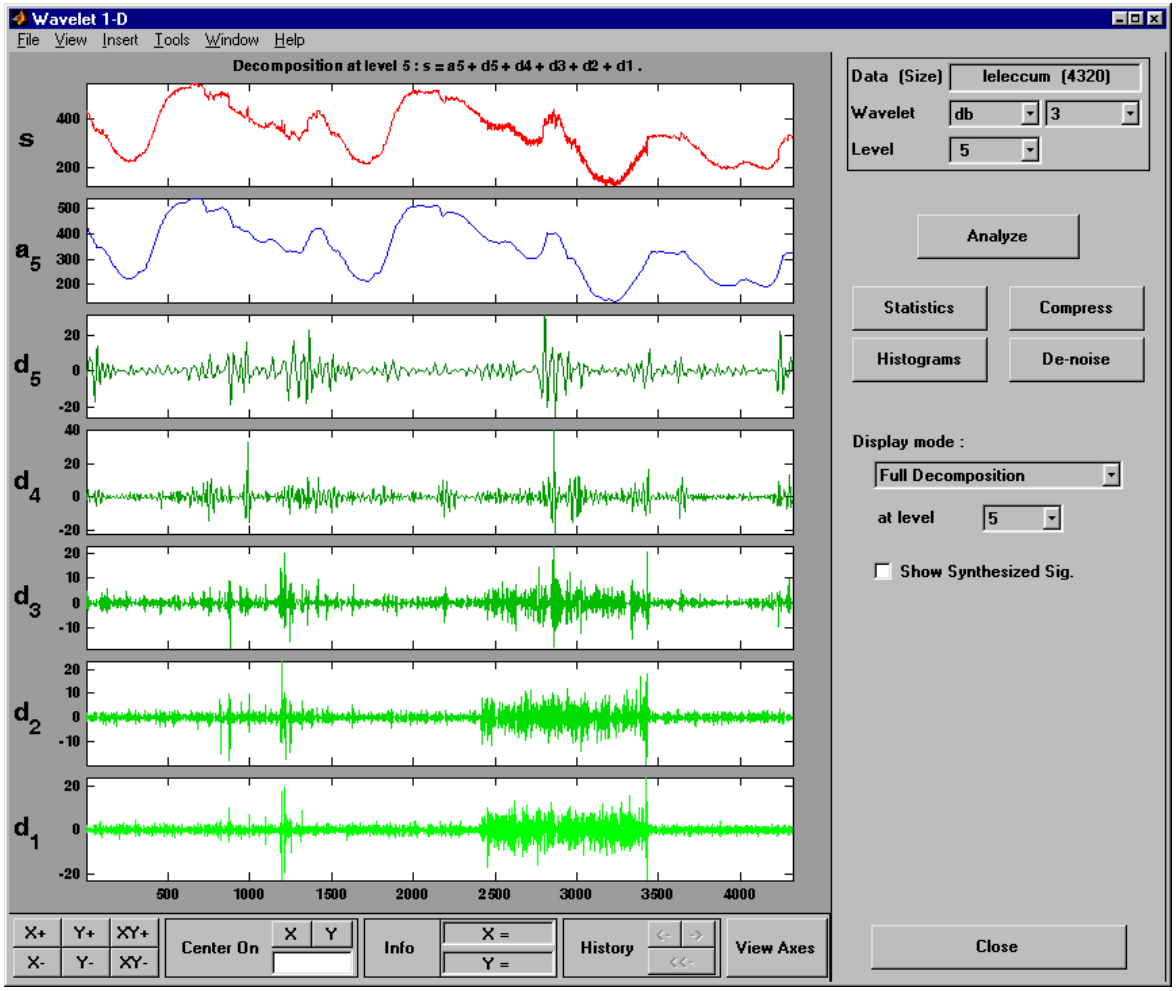
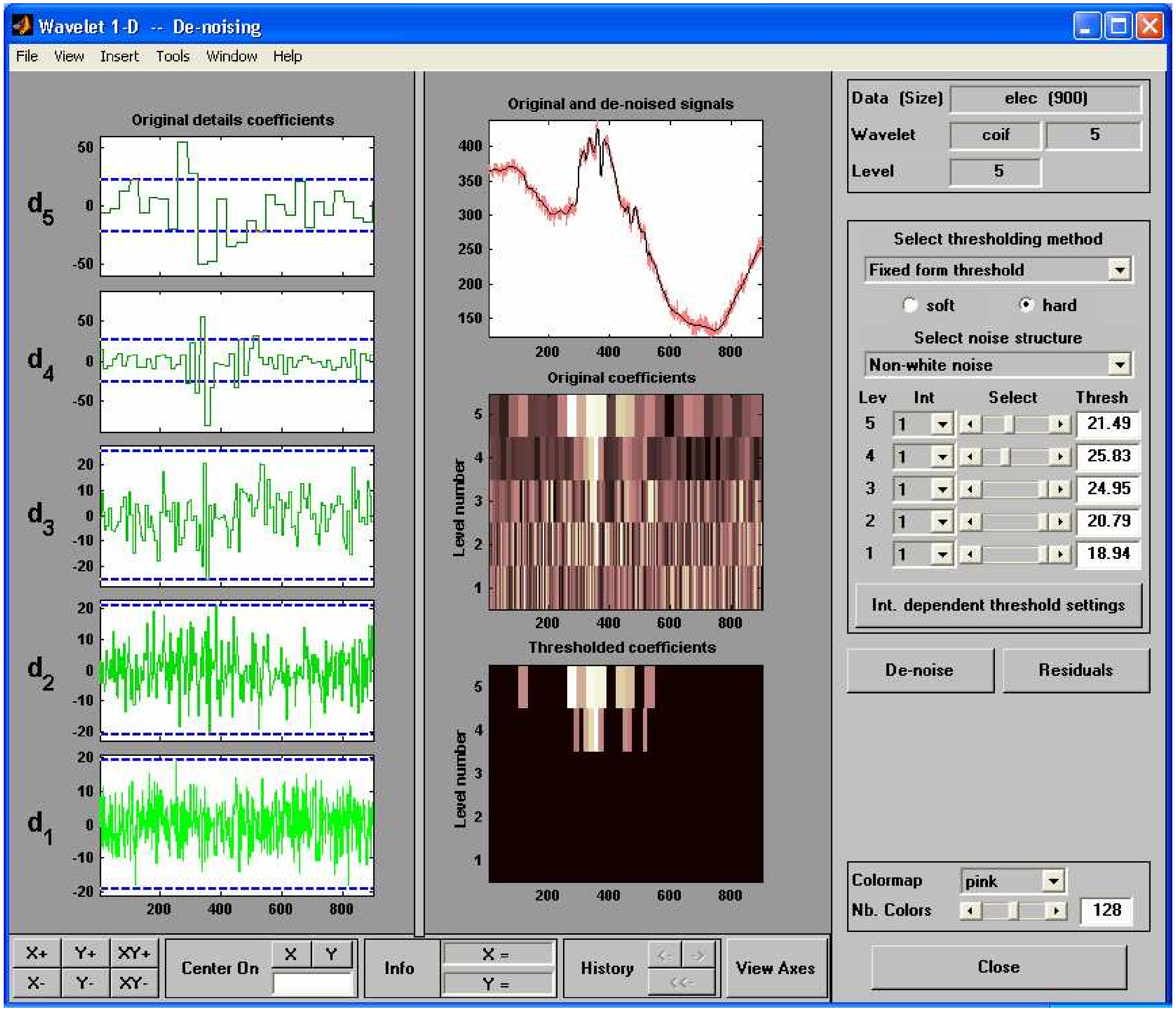
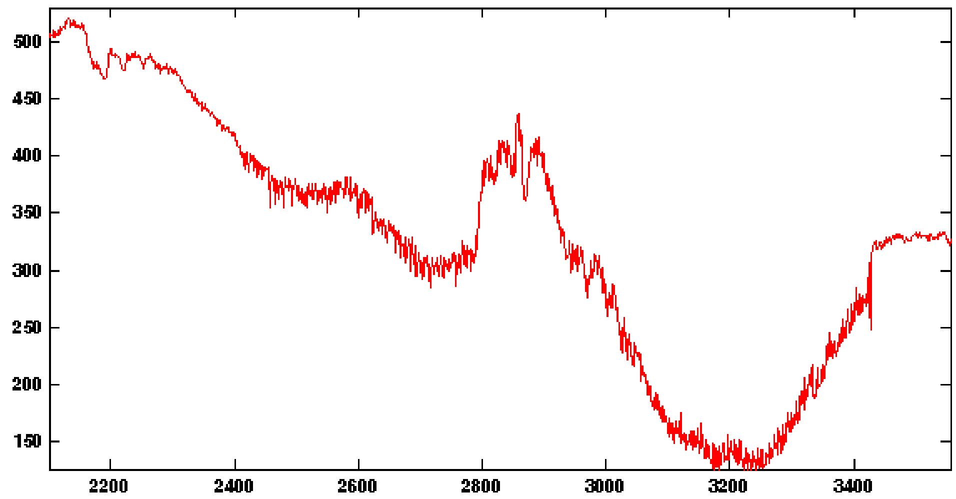


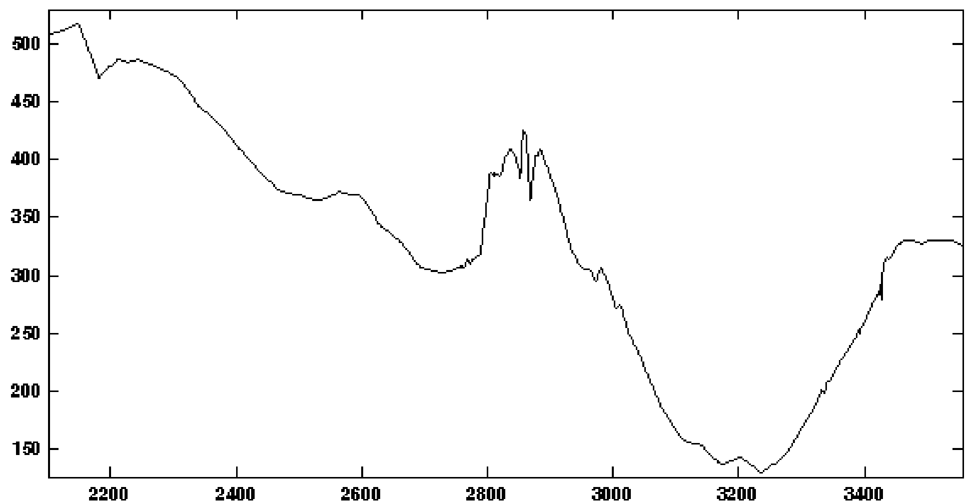
FIG. 5 – Electrical consumption signal during three days

Denoising the signal



Output


500
450
400
350
300
250
200
150
2200
2400
2600
2800
3000
3200
3400


500
450
400
350
300
250
200
150
2200
2400
2600
2800
3000
3200
3400

Analysis of the Doppler signal

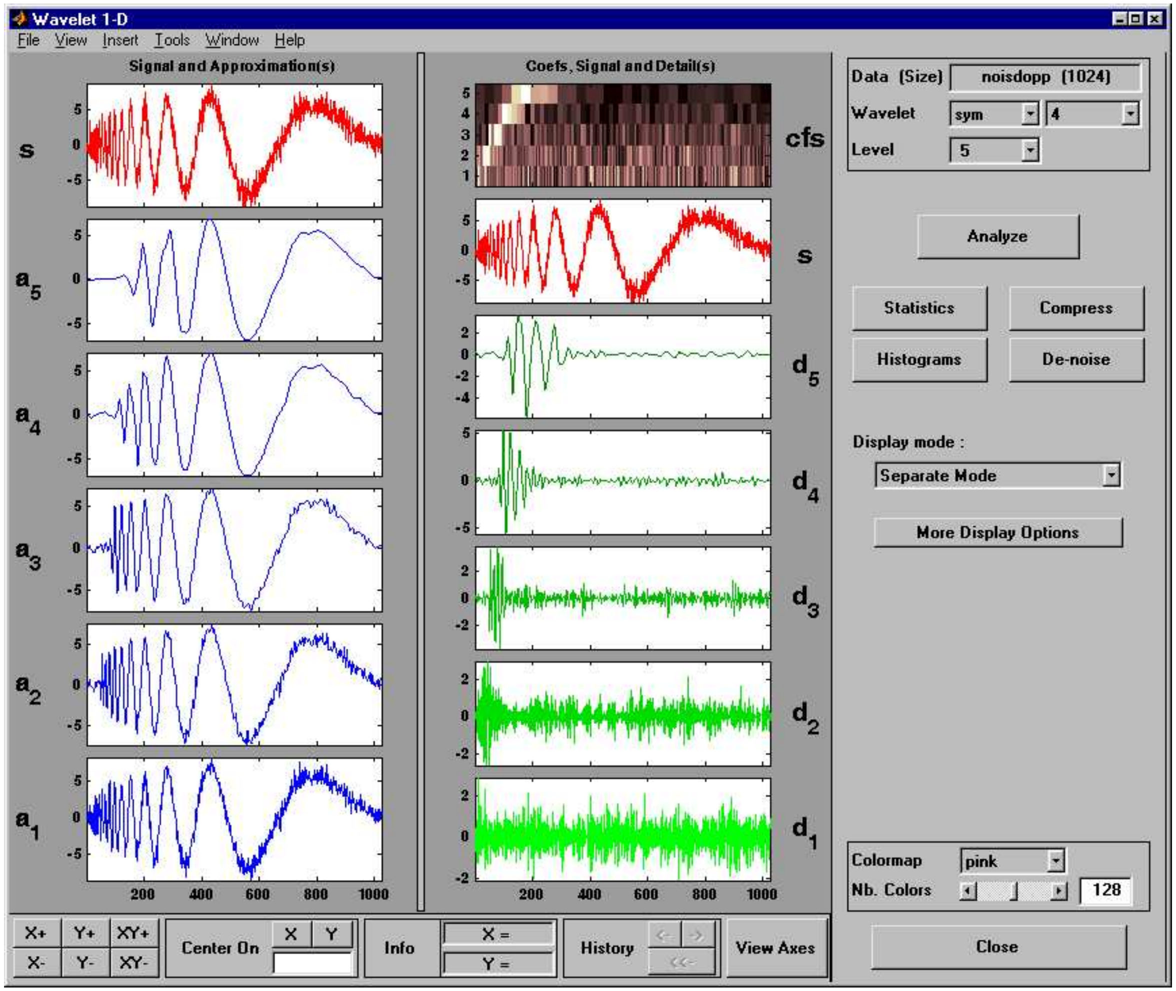


FIG. 6 – Doppler signal

Denoising the Doppler signal

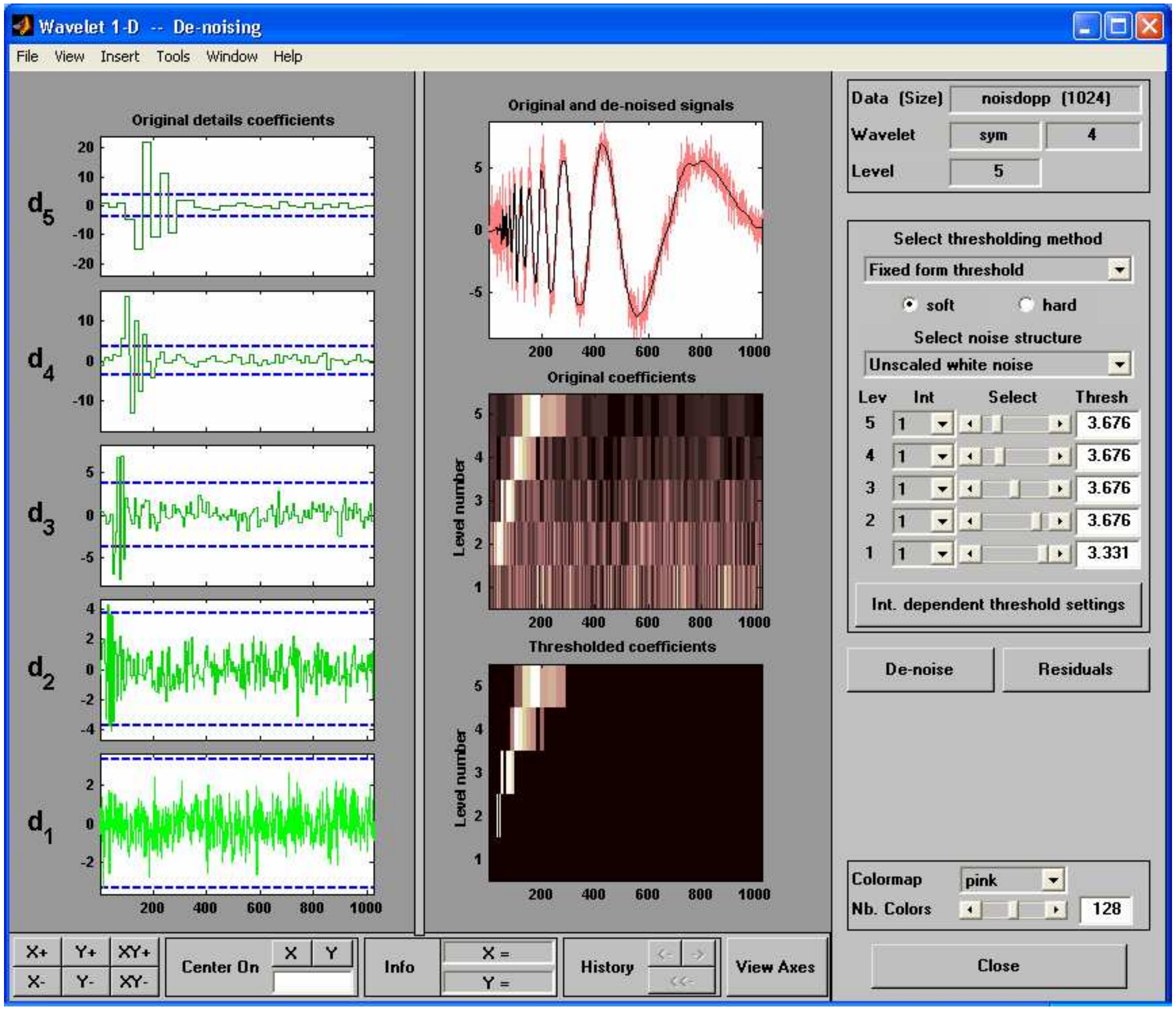


Image processing with wavelets

Major application of wavelets for bivariate signals : data compression

Famous examples :

- Fingerprints storage by FBI
- JPEG 2000 that is the standard algorithm for image compression

Other applications :

- Edge detection
- Denoising

Bidimensional multiresolution analyzes

Let us assume that (ϕ, ψ) generates a wavelet basis of $\mathbb{L}_2(\mathbb{R})$. Then, we can build (by using tensorisation) in $\mathbb{L}_2(\mathbb{R}^2)$

- one scaling function : $\phi^{2D}(x, y) = \phi(x)\phi(y)$
- three wavelets :

$$\psi_1^{2D}(x, y) = \phi(x)\psi(y), \quad \psi_2^{2D}(x, y) = \psi(x)\phi(y), \quad \psi_3^{2D}(x, y) = \psi(x)\psi(y).$$

If $\{V_j\}_{j\in\mathbb{Z}}$ and $\{W_j\}_{j\in\mathbb{Z}}$ are respectively the approximation and detail spaces associated with (ϕ, ψ), then approximation and detail spaces associated with $\phi^{2D}, \psi_1^{2D}, \psi_2^{2D}, \psi_3^{2D}$ are :

$$\{\overline{V_j \otimes V_j}\}_{j\in\mathbb{Z}}, \quad \{\overline{V_j \otimes W_j}\}_{j\in\mathbb{Z}}, \quad \{\overline{W_j \otimes V_j}\}_{j\in\mathbb{Z}}, \quad \{\overline{W_j \otimes W_j}\}_{j\in\mathbb{Z}}.$$

We have for any $j \in \mathbb{Z}$,

$$\begin{aligned} V_{j-1}^{2D} &= \overline{(V_j \otimes V_j)} \oplus \overline{(V_j \otimes W_j)} \oplus \overline{(W_j \otimes V_j)} \oplus \overline{(W_j \otimes W_j)} \\ &= V_j^{2D} \oplus \left[W_j^{2D}\right]_h \oplus \left[W_j^{2D}\right]_v \oplus \left[W_j^{2D}\right]_d \\ &= V_j^{2D} \oplus W_j^{2D} \end{aligned}$$

Decomposition of an image - 1

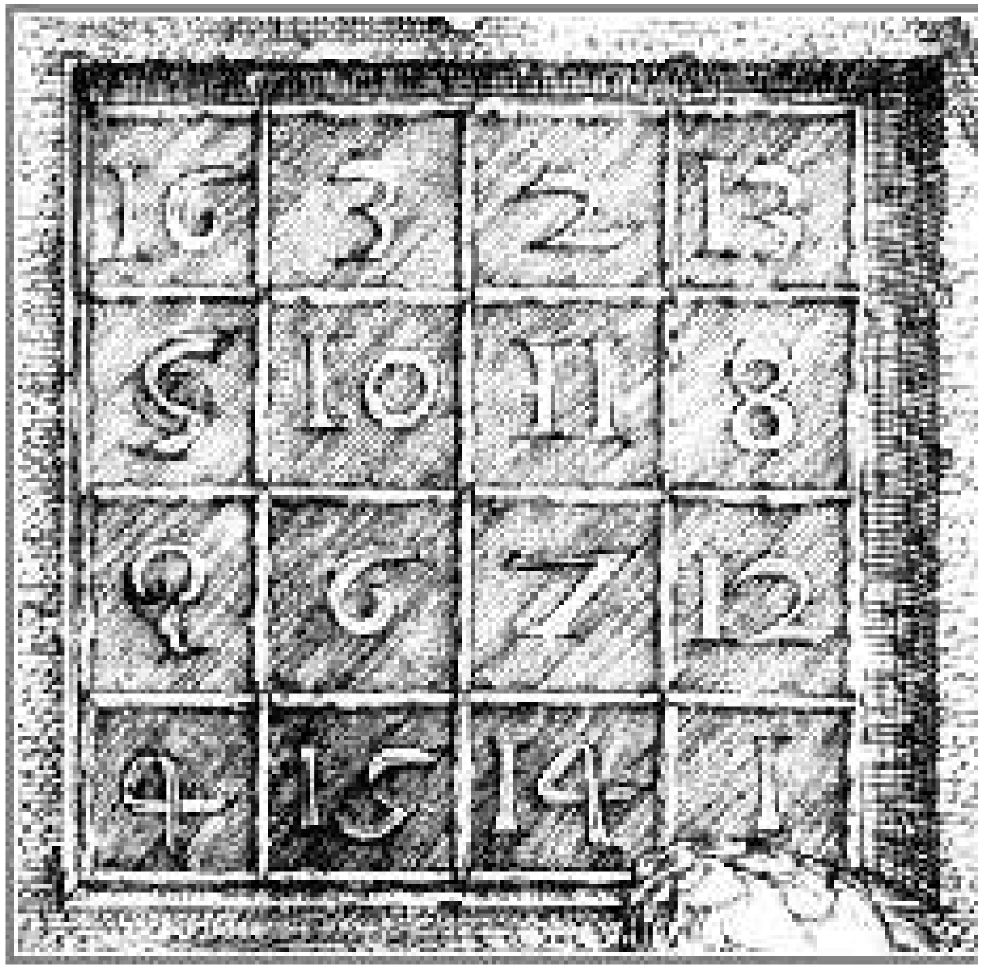


FIG. 1 – A magic square

Decomposition of an image - 2

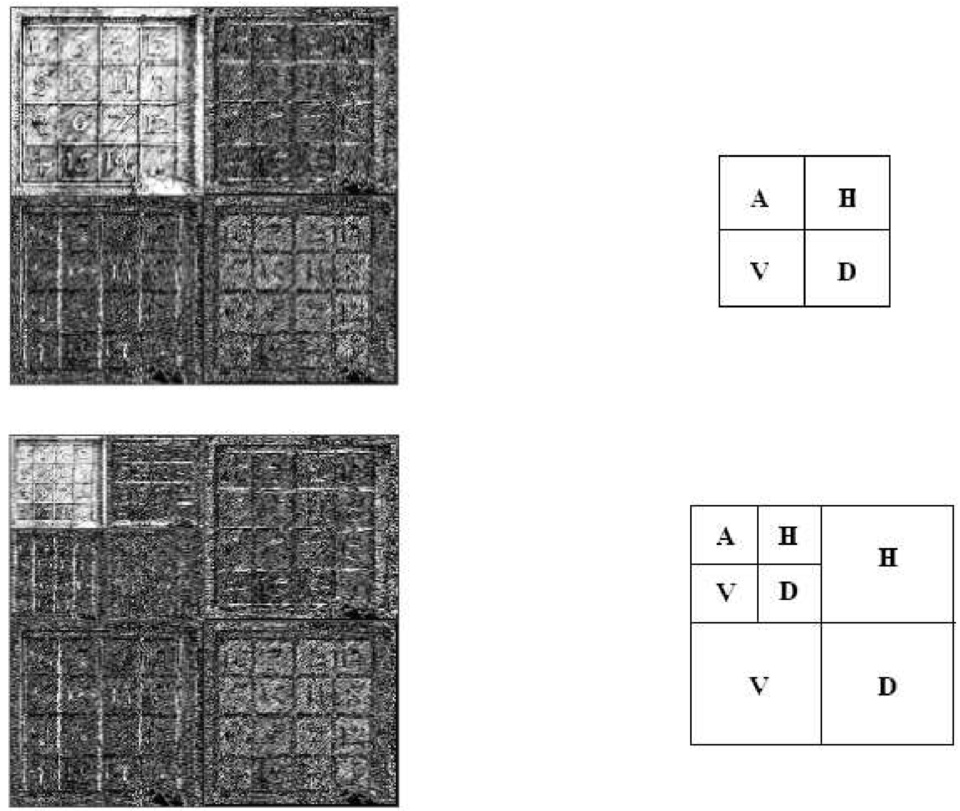


FIG. 2 – A magic square

Decomposition of an image - 3

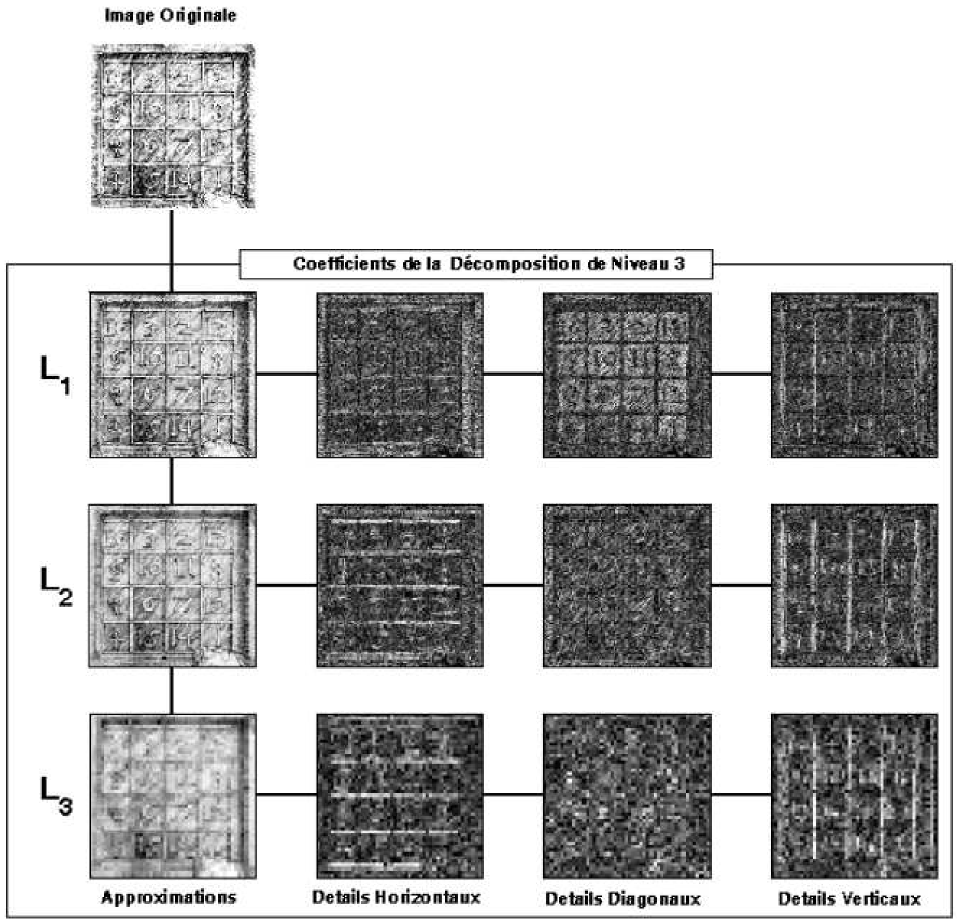


FIG. 3 – A magic square

Image compression

The need for storing large quantities of information and for fast transmission are key issues.

To address this problem, digital image compression minimizes the length of the series of bits necessary to represent images without damaging quality.

Wavelets offer sparse representations of images to be compressed : most of the wavelet coefficients are negligible.

Compression consists of keeping only the few significant coefficients.

Sparsity property of wavelets - 1

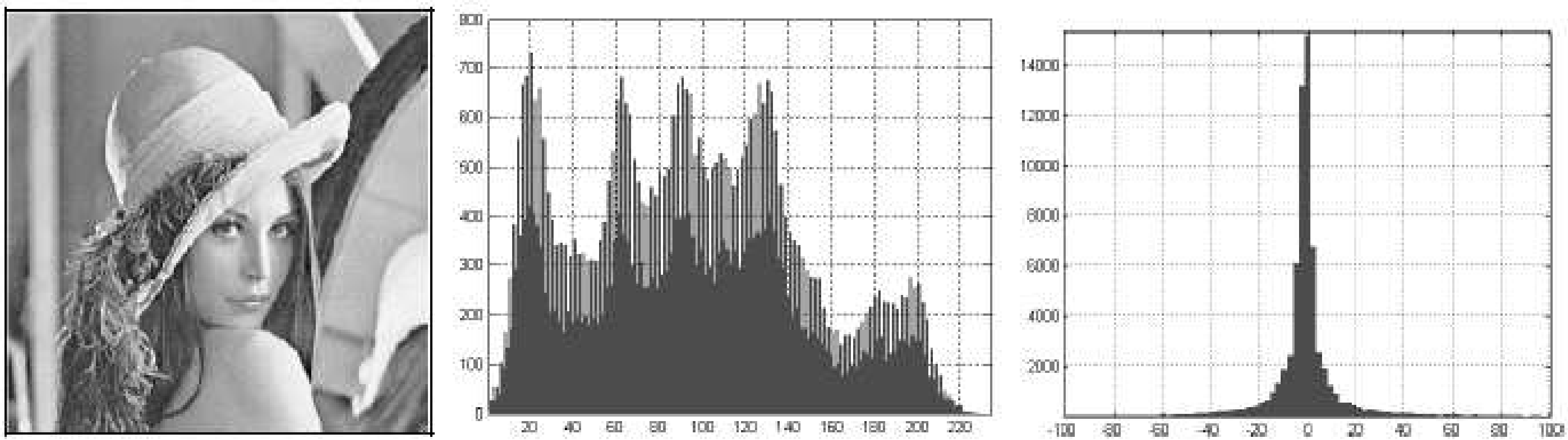


FIG. 4 – Histogram of values and histogram of wavelet coefficients

Sparsity property of wavelets - 2

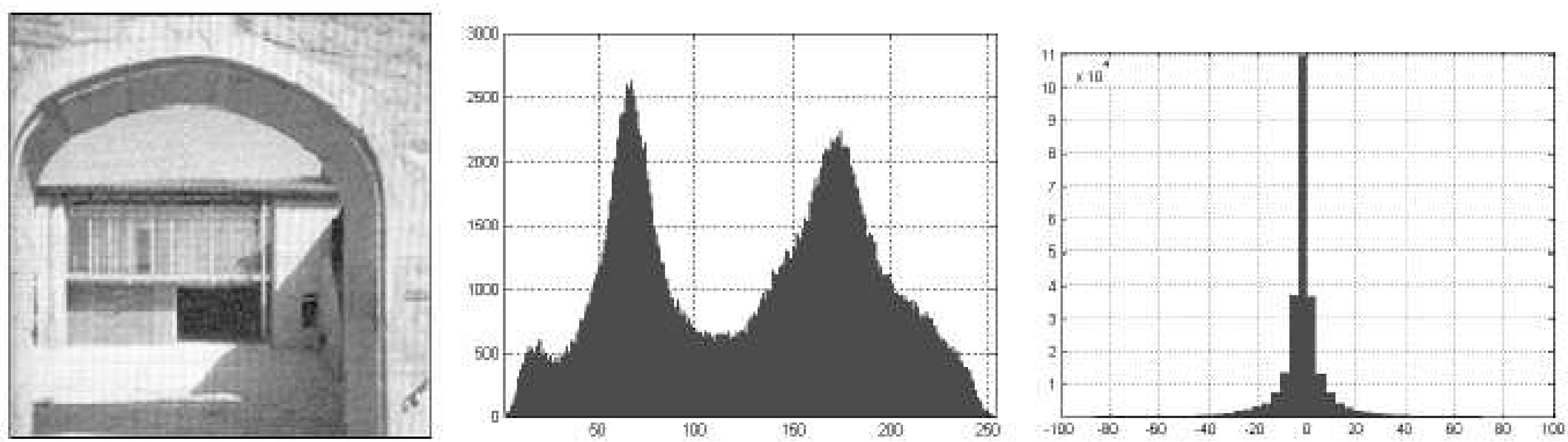


FIG. 5 – Histogram of values and histogram of wavelet coefficients

Sparsity property of wavelets - 3

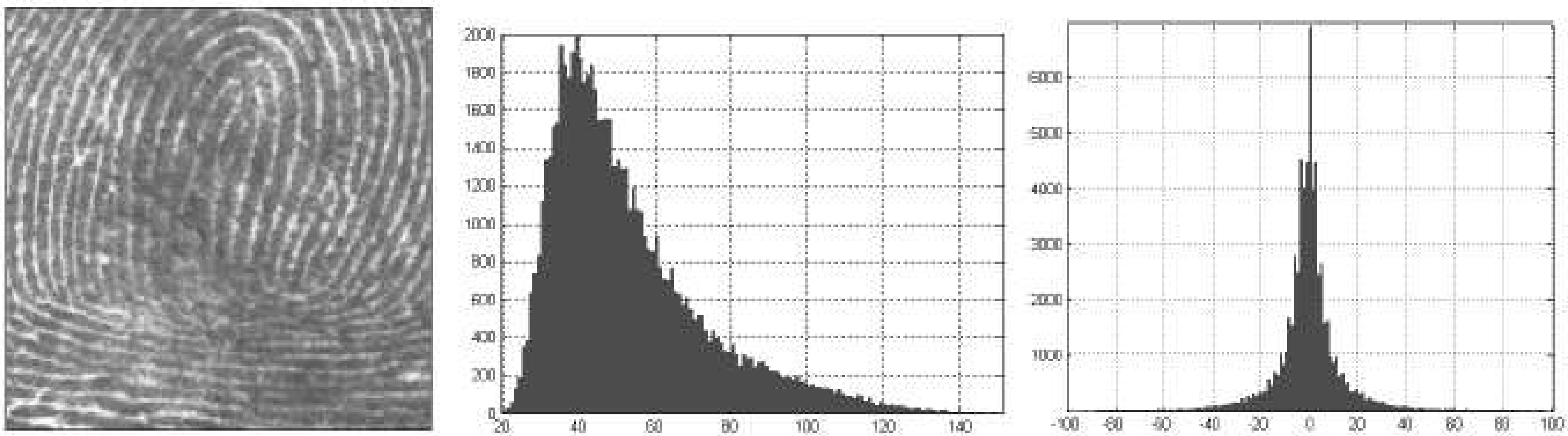


FIG. 6 – Histogram of values and histogram of wavelet coefficients

Sparsity property of wavelets - 4

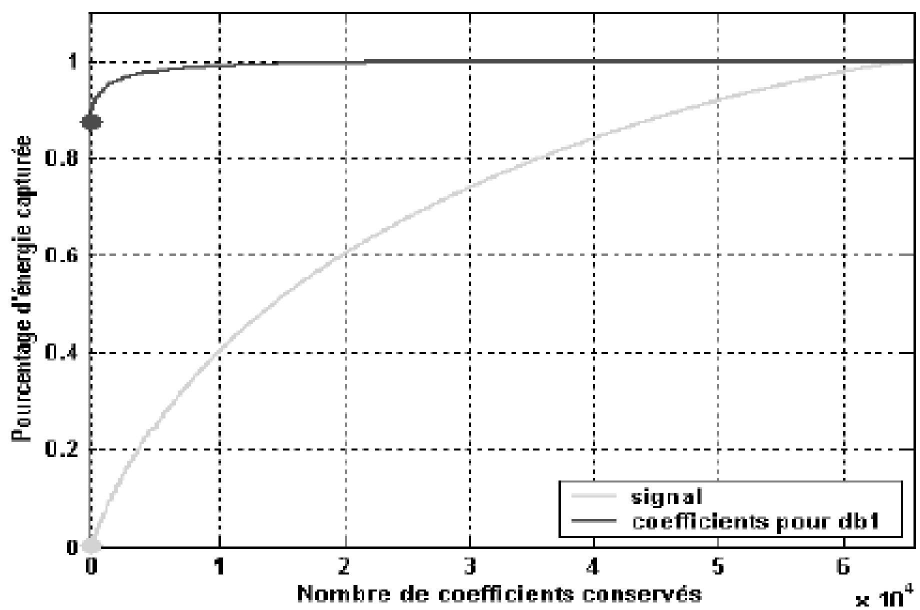


FIG. 7 – Comparison between energy percentages for the fingerprint

The image is decomposed on the canonical basis. Wavelet coefficients are analyzed at level 8 with the db1 wavelet

Image compression - Example 1

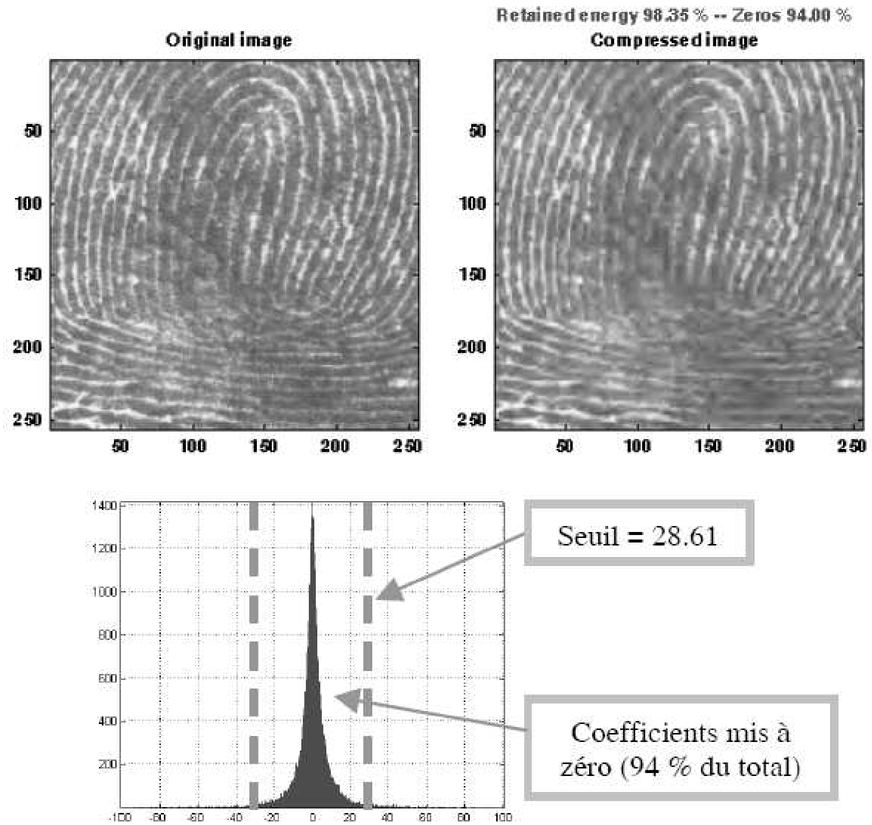
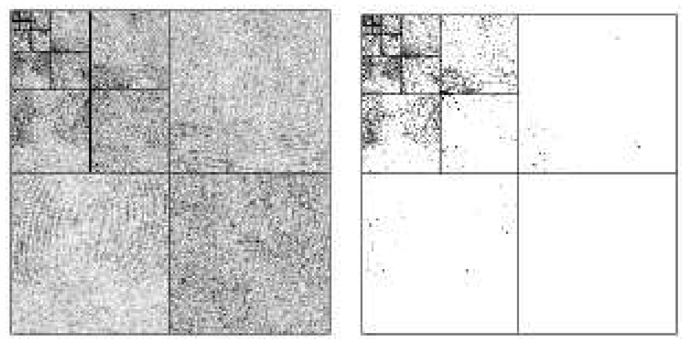


FIG. 8 – Compression of a fingerprint - Level = 5 - Sym4 wavelet

94% of wavelet coefficients are thresholded (universal thresholding).

Image compression - Example 1



	Effectif	Gardés	Rapport gardés / effectifs	Rapport gardés / total gardés
A5	64	64	100.00 %	1.63 %
D5	192	157	81.77 %	3.99 %
D4	768	420	54.69 %	10.68 %
D3	3 072	1 460	47.53 %	37.13 %
D2	12 288	1 704	13.87 %	43.34 %
D1	49 152	127	0.26 %	3.23 %
Total	65 536	3 932	6.00 %	100.00 %

FIG. 9 – Compression of a fingerprint - Level = 5 - Sym4 wavelet

On the left, the complete decomposition, in the middle, the “thresholded” decomposition and on the right, the coefficients distribution for the fingerprint

Image compression - Example 2

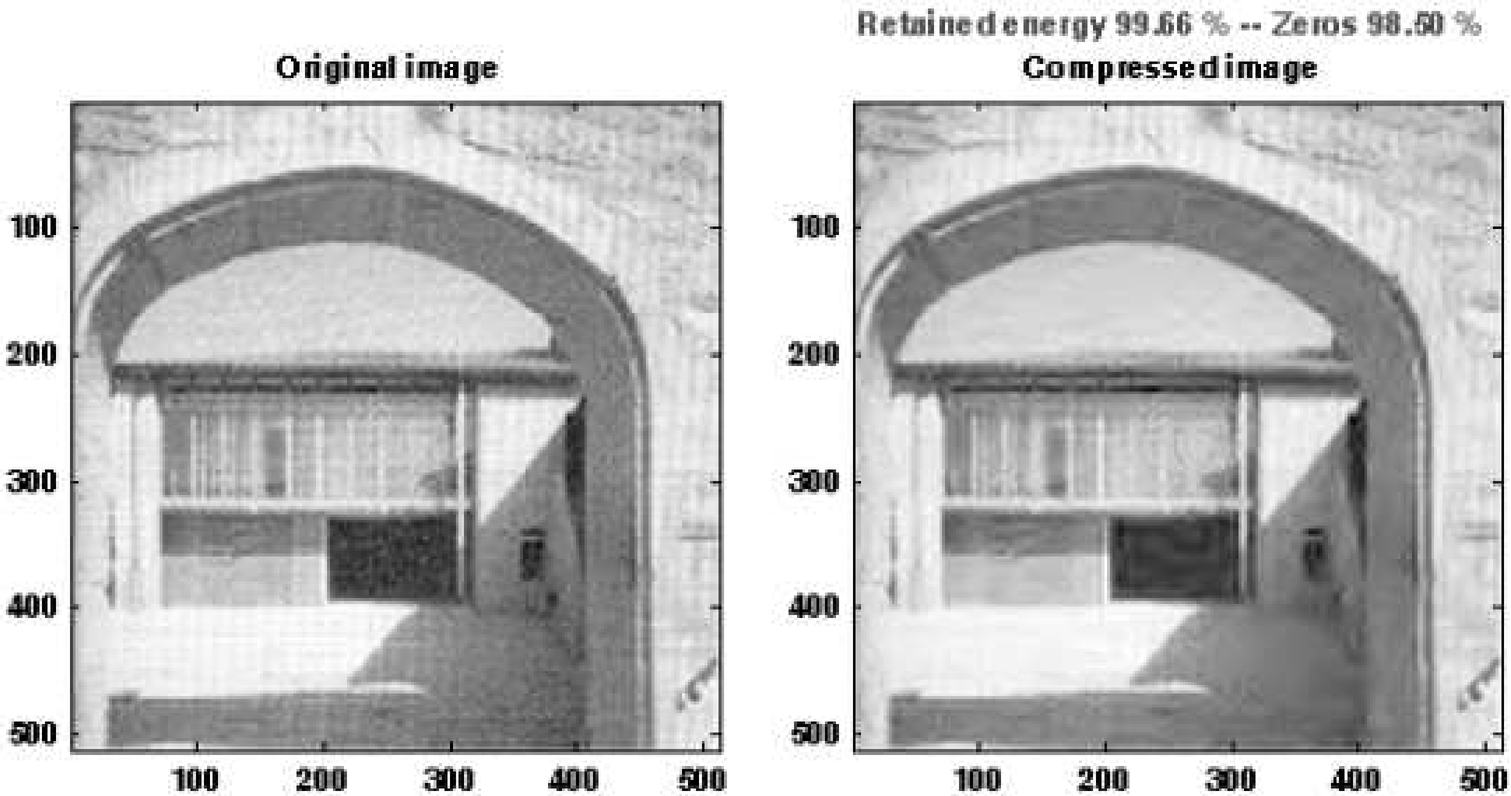
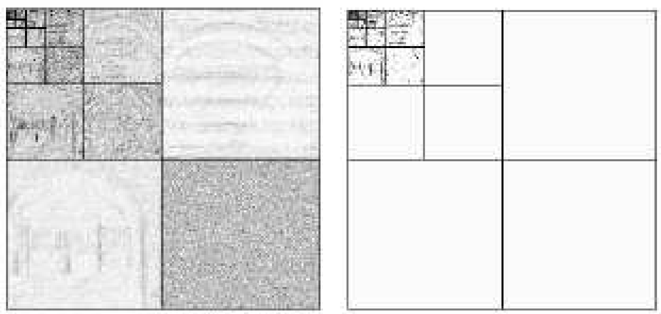


FIG. 10 – Compression of a porch- Level = 5 - Sym8 wavelet

98,5% of wavelet coefficients are thresholded

Image compression - Example 2



	Effectif	Gardés	Rapport gardés / effectifs	Rapport gardés / total gardés
A5	256	256	100.00 %	6.51 %
D5	768	480	62.50 %	12.21 %
D4	3 072	1 127	36.69 %	28.66 %
D3	12 288	1 881	15.31 %	47.84 %
D2	49 152	133	0.27 %	3.38 %
D1	196 608	55	0.03 %	1.40 %
Total	262 144	3 932	1.50 %	100.00 %

FIG. 11 – Compression of a porch

On the left, the complete decomposition, in the middle, the “thresholded” decomposition and on the right, the coefficients distribution for the porch

Edge detection - the rule

Wavelet decomposition of the image X :

$$X = A_1 + D_1$$

Representation :
- A_1 remains unchanged
- For D_1, we only use black and white : black if $|D_1| > 0$, white if $|D_1| = 0$.

More generally, the representation is as follows :
- A_1 remains unchanged
- For D_1, we only use black and white : black if $|D_1| >$ threshold, white if $|D_1| \leq$ threshold.

Edge detection - Example 1

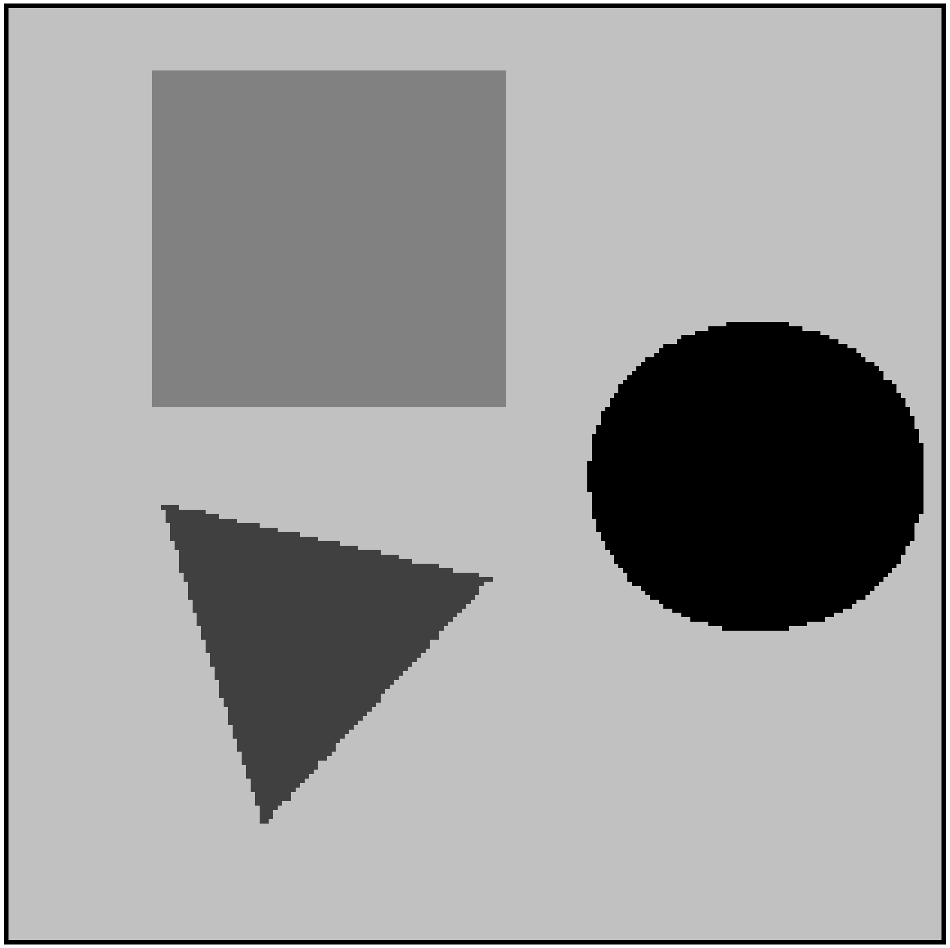


FIG. 12 – Original image : three simple geometric forms

Edge detection - Example 1

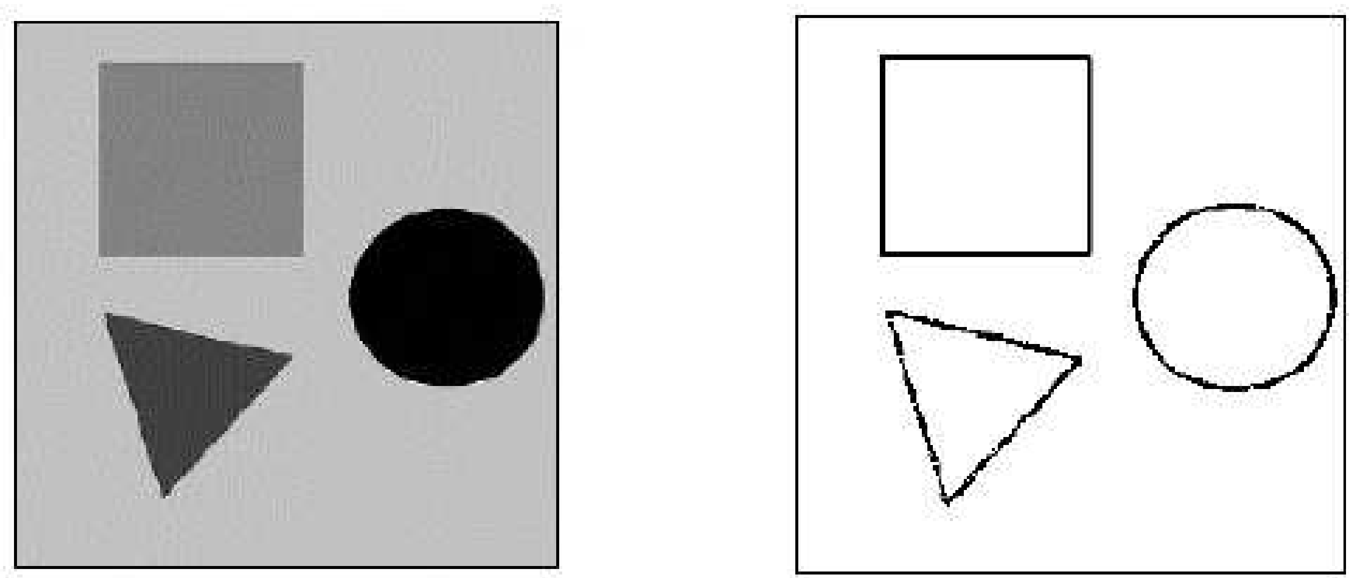


FIG. 13 – Approximation A_1 and detail D_1

Edge detection - Example 2

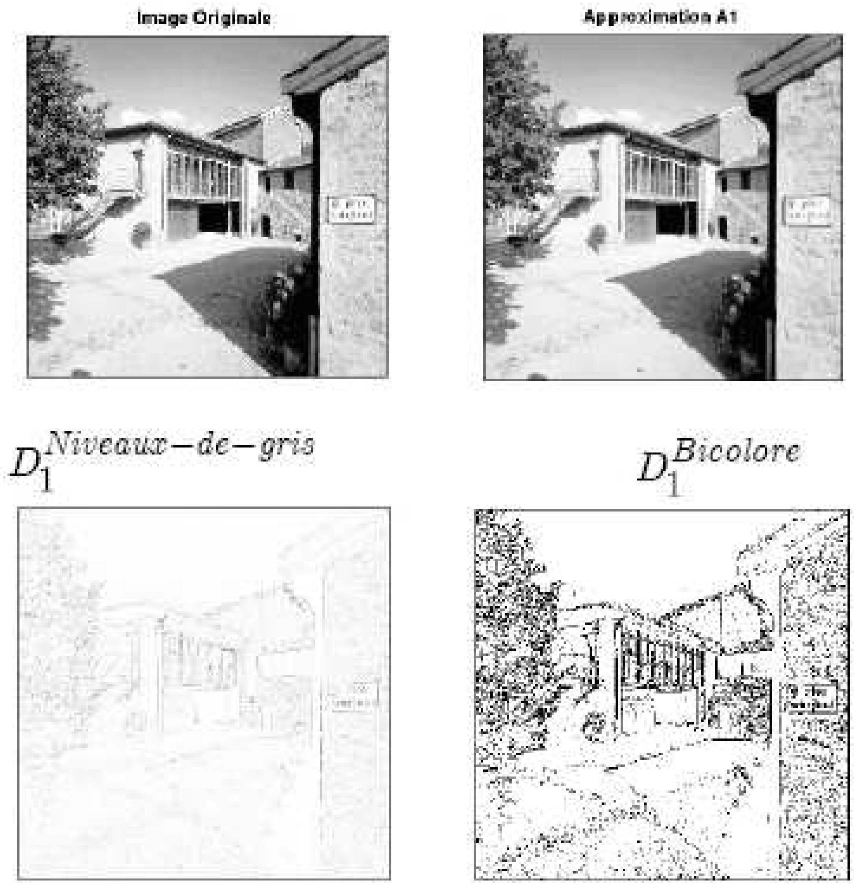


FIG. 14 – Edge detection (a building)

Edge detection - Example 3

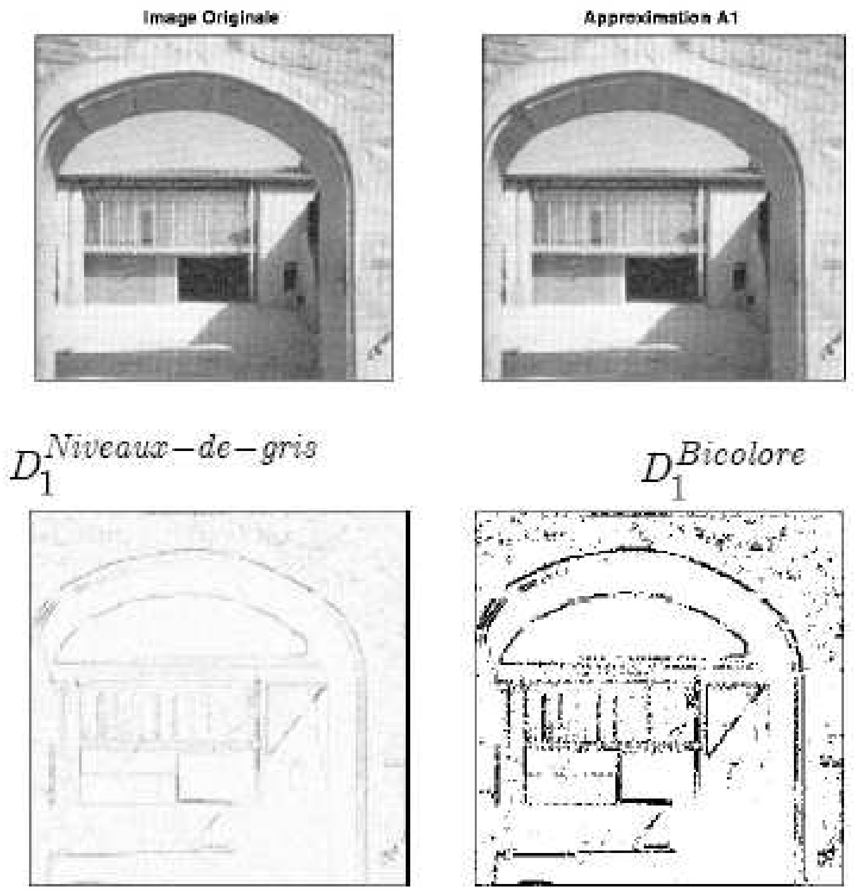


FIG. 15 – Edge detection (a porch)

Image denoising - the rule

Let us consider the model :

$$Y(i,j) = f(i,j) + \epsilon(i,j),$$

where ϵ is a Gaussian white noise whose covariance matrice is σ^2Id, Z is the image to be denoised and f is the image to be restored.

To restore f, the algorithm is the following :

1. Wavelet decomposition at the level J of the image to be denoised
2. Thresholding, in three directions, of the detail coefficients with absolute value smaller than a threshold depending on σ
3. Reconstruction of the image from the approximation coefficients of level J and from the modified detail coefficients of levels J, $J-1$,...,1

Noisy Image wavelet decomposition

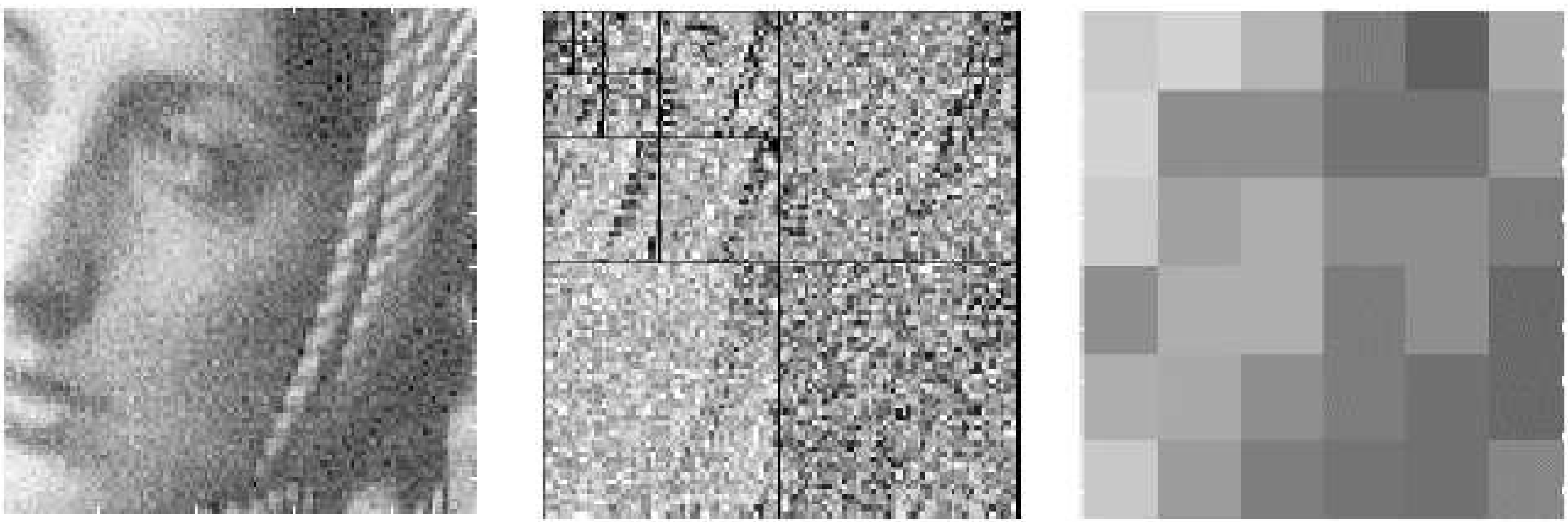


FIG. 16 – Noisy image, level 4 decomposition and level 4 approximation

Noisy Image wavelet decomposition

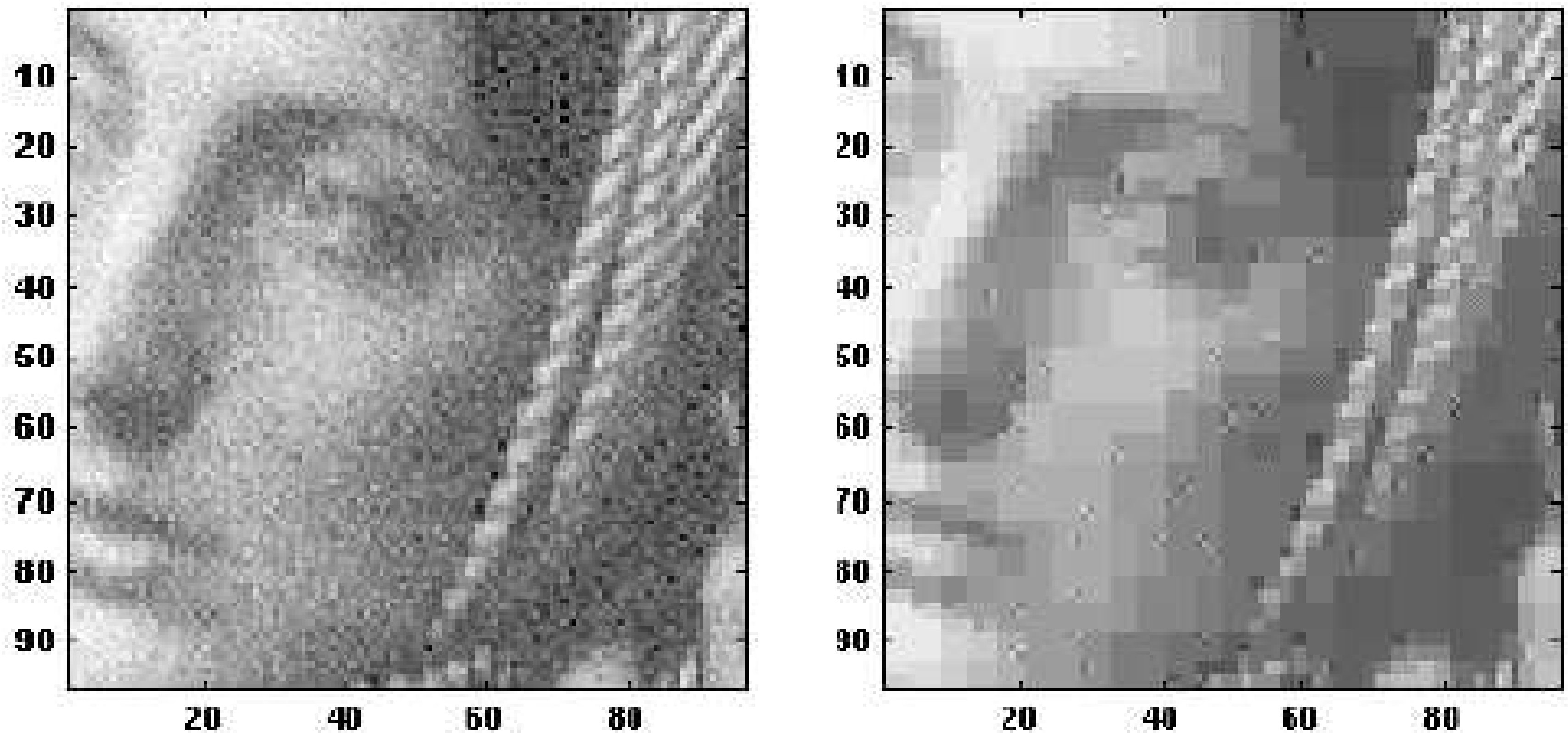


FIG. 17 – Noisy image and reconstruction by using the Haar system

Noisy Image wavelet decomposition

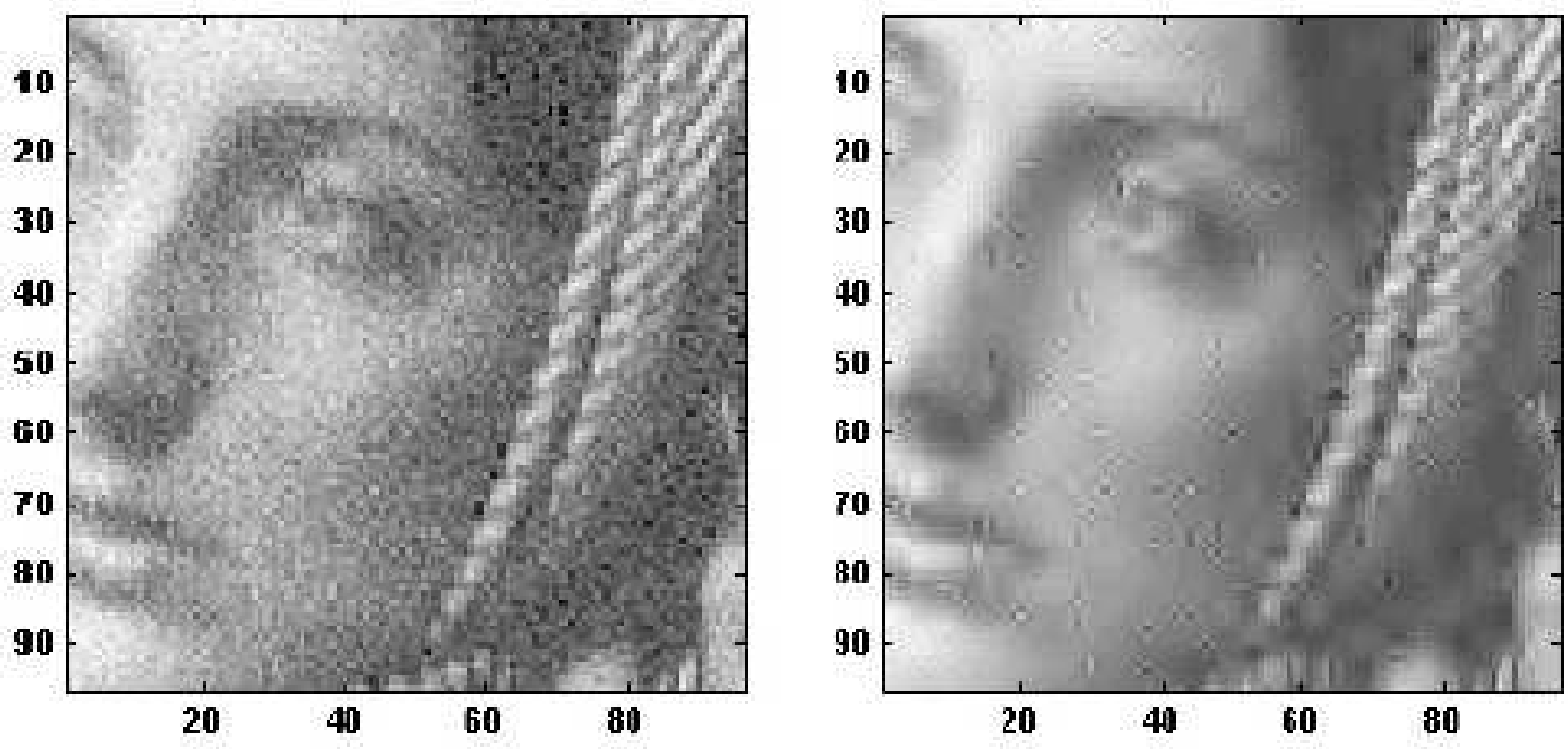


FIG. 18 – Noisy image and reconstruction by using the Sym6 system

Other examples of applications (not detailed)

Wavelets are a powerful tool for following applications

1. Denoising of biomedical signals and medical images
2. Video coding, coding of animated images
3. Compression of photographs, of video images
4. Classification (for instance star spectra, acoustic signals or eating behaviors)
5. Detection by signature recognition (boats, seismic jolts)
6. Numerical approximation of linear operators

Conclusions

Some points should be emphasized :

1. Wavelets bases that allow decomposition and reconstruction of signals are a new competitive tool that is easy to use. The wavelet transform is a fast algorithm.
2. Unlike the Fourier theory, wavelets allow local analysis and adapt to local properties of signals. Regularization by thresholding is possible.
3. Wavelets concentrate most of the information on few coefficients (wavelet decomposed signals are sparse). Modifications of only few coefficients affect only a small part of the signal.
4. Wavelets are related with multiresolution analyzes that allow to zoom in or zoom out on some parts of signals. So, wavelets perform as a mathematical and numerical microscope.
5. Wavelets offer easily tractable characterizations of some functional spaces such as Hölder spaces or Besov spaces.
6. The wave shapes of wavelets are adapted for the decomposition of many signals. Besides, wavelets are appropriate to represent self-similar signals such as fractional Brownian motion or fractals.

Bibliography

– Antoniadis (1994)
– Cohen, Daubechies and Vial (1993)
– Daubechies (1988, 1992)
– Donoho and Johnstone (1994)
– Donoho, Johnstone, Kerkyacharian and Picard (1995, 1996, 1997)
– Härdle, Kerkyacharian, Picard and Tsybakov (1998)
– Kerkyacharian and Picard (1992, 1993, 2000)
– Le Pennec (2002)
– Mallat (1989, 1998)
– Meyer (1992)
– Misiti, Misiti, Oppenheim and Poggi (2007)
– Reynaud-Bouret and Rivoirard (2007)
– ...