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High-dimensional statistics

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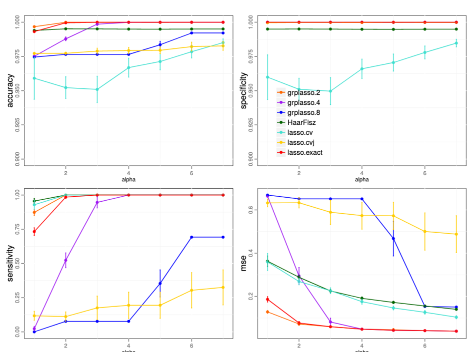

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Contents

The introduction of the course can be found on
`https://www.ceremade.dauphine.fr/~rivoirar/Intro-HDS.pdf`

Chapter 1

Variable selection

We study the problem of variable selection in the linear regression setting.

1.1 Introduction

Why linear regression?

1. It models various concrete situations
2. Is is simple to use from the mathematical point of view
3. It allows to introduce and to present new methodologies

Definition 1.1. *Let* $Y = (Y_1, \dots, Y_n)^T \in \mathbb{R}^n$ *a vector of observations. We say that* Y *obeys a linear regression model when*

$$Y = X\beta^* + \varepsilon,$$

where

- X *is a known* $n \times p$*-matrix*
- $\varepsilon \in \mathbb{R}^n$ *such that its components* ε_i *are centered and i.i.d.*
- $\beta^* \in \mathbb{R}^p$ *is unknown*

We say that the linear model is gaussian when $Y \sim \mathcal{N}(X\beta^*, \sigma^2 I_n)$*, where* $\sigma^2 := var(\varepsilon_1)$*.*

The terminology is the following:

- X_j, the jth column of X, is an explanatory variable or a predictor
- Y is the response variable

- ε is the error vector

We can consider 3 statistical problems:

- the estimation problem: Estimate β^*
- the prediction problem: Estimate $X\beta^*$
- Selection problem: Determine non-zero coordinates of β^*

1.2 Classical estimation

We still denote $\|\cdot\|$ the classical euclidian norm. We denote P_X the orthogonal projection on $\mathcal{I}m(X)$.

Definition 1.2. *We denote $\hat{\beta}$ the ordinary least squares estimate of β^*: $\hat{\beta}$ is the vector of $\mathbb{R}^p$ such that*

$$\hat{\beta} := \arg\min_{\beta\in\mathbb{R}^p} \|Y - X\beta\|^2.$$

Proposition 1.1. *If X is one to one (injective) then we have*

$$\hat{\beta} = (X^T X)^{-1} X^T Y.$$

Proof. Since X is one to one, $X^T X$ is invertible. When β describes $\mathbb{R}^p$, $X\beta$ describes $\mathcal{I}m(X)$. So, $X\hat{\beta}$ is the orthogonal projection of Y on $\mathcal{I}m(X)$ and

$$X\hat{\beta} = P_X Y = X(X^T X)^{-1} X^T Y.$$

Since X is one to one, we get the result. □

Remark 1.1. *Since $\mathbb{E}[\varepsilon] = 0$, $\mathbb{E}[\hat{\beta}] = \beta^*$ and since $var(\varepsilon) = \sigma^2 I_n$, $var(\hat{\beta}) = \sigma^2 (X^T X)^{-1}$.*

Definition 1.3. *The vector of residuals is given by*

$$\hat{\varepsilon} = Y - \hat{Y}, \quad \hat{Y} = X\hat{\beta} = P_X Y.$$

If $P_{X^\perp} = I_n - P_X$ is the projection matrix on $\mathcal{I}m(X)^\perp$, then $\hat{\varepsilon} = P_{X^\perp} Y$.

Remark 1.2. *If X is one to one, $\mathbb{E}[\hat{\varepsilon}] = 0$ and*

$$var(\hat{\varepsilon}) = \sigma^2 P_{X^\perp}, \quad cov(\hat{\varepsilon}, \hat{Y}) = 0.$$

Indeed,

$$cov(\hat{\varepsilon}, \hat{Y}) = \mathbb{E}[(P_{X^\perp}(Y - E[Y]))^T (P_X(Y - \mathbb{E}[Y])]$$

and we use $(I_n - P_X)P_X = 0$

We shall use the following lemma.

Lemma 1.1. *For any deterministic matrix A with n columns,*

$$\mathbb{E}[\|A\varepsilon\|^2] = \sigma^2 Tr(AA^T).$$

Proof.

$$\mathbb{E}[\|A\varepsilon\|^2] = \mathbb{E}[(A\varepsilon)^T(A\varepsilon)] = \mathbb{E}[Tr((A\varepsilon)^T(A\varepsilon))] = \mathbb{E}[Tr((A\varepsilon)(A\varepsilon)^T)] = \sigma^2 Tr(AA^T).$$

□

Definition 1.4. *The natural estimate of σ^2 is*

$$\hat{\sigma}^2 = \frac{\|\hat{\varepsilon}\|^2}{n-p} = \frac{\|Y - P_X Y\|^2}{n-p}.$$

Proposition 1.2. *If rank$(X) = p$, then $\mathbb{E}[\hat{\sigma}^2] = \sigma^2$.*

Proof. We have

$$\mathbb{E}[\|\hat{\varepsilon}\|^2] = \sigma^2 Tr(P_{X^\perp}) = \sigma^2(n-p).$$

□

1.3 Inference in the Gaussian case

In this section, we still assume that rank$(X) = p$, $\mathbb{E}[\varepsilon] = 0$ and var$(\varepsilon) = \sigma^2 I_n$ but we further assume that $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$. The likelihood of the observations is then available. We then obtain the following proposition.

Proposition 1.3. *The maximum likelihood estimate of (β^*, σ^2) is $(\hat{\beta}, (n-p)/n \times \hat{\sigma}^2)$.*

Proof. The proof is very classical. □

Note that most of the time, we prefer to use, in practice, $\hat{\sigma}^2$, which is unbiased, to estimate σ^2. Of course, asymptotically, there is no difference between $\hat{\sigma}^2$ and $(n-p)/n \times \hat{\sigma}^2)$. To establish the properties of $(\hat{\beta}, (n-p)/n \times \hat{\sigma}^2)$, we now recall Cochran's theorem.

Theorem 1.1. *Let $W \sim \mathcal{N}(m, I_d)$ a Gaussian vector of $\mathbb{R}^d$ and $E \oplus E^\perp = \mathbb{R}^d$ a decomposition of $\mathbb{R}^d$ in two orthogonal vector spaces. Then, the vectors W_E and $W_{E^\perp}$, orthogonal projections of W on E and $E^\perp$ respectively, are independent. Furthermore, the random variables $\|W_E\|^2$ and $\|W_{E^\perp}\|^2$ are independent and*

$$\|W_E\|^2 \sim \chi^2(dim(E), \|m_E\|^2), \quad \|W_{E^\perp}\|^2 \sim \chi^2(d - dim(E), \|m_{E^\perp}\|^2),$$

where m_E and $m_{E^\perp}$ are projections of m on E and $E^\perp$ respectively.

Remark 1.3. *Cochran's theorem can be extended to decompositions in more than 2 spaces.*

From this theorem, we deduce:

Proposition 1.4. *We have:*

1. $\hat{\beta} \sim \mathcal{N}(\beta^*, \sigma^2(X^TX)^{-1})$
2. $\frac{(n-p)\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-p)$
3. $\hat{\beta}$ *and* $\hat{\sigma}^2$ *are independent*

Proof. The first point is obvious. For the second point, we write

$$\frac{(n-p)\hat{\sigma}^2}{\sigma^2} = \frac{\|P_{X^\perp}Y\|^2}{\sigma^2} = \left\|P_{X^\perp}(\sigma^{-1}\varepsilon)\right\|^2.$$

Finally, for the last point, we show:

$$\hat{\beta} = \beta^* + (X^TX)^{-1}X^TP_X\varepsilon.$$

□

Theorem 1.2. *We fix two vector spaces V and W where W is a vector subspace of V. We assume $q = dim(W) < p = dim(V)$. We set*

$$F = \frac{\|P_WY - P_VY\|^2/(p-q)}{\|Y - P_VY\|^2/(n-p)}$$

where P_VY is the orthogonal projection of Y on V and P_WY is the orthogonal projection of Y on W. When $X\beta^* \in W \subset V$, then

$$F \sim \mathcal{F}(p-q, n-p).$$

We deduce the following corollary.

Corollary 1.1. *We fix two vector spaces V and W where W is a vector subspace of V. We assume $q = dim(W) < p = dim(V)$. We set*

$$F = \frac{\|P_WY - P_VY\|^2/(p-q)}{\|Y - P_VY\|^2/(n-p)}$$

where P_VY is the orthogonal projection of Y on V and P_WY is the orthogonal projection of Y on W. Then

$$\phi(Y) = 1_{\{F > f_{p-q,n-p,1-\alpha}\}},$$

where $f_{p-q,n-p,1-\alpha}$ is the quantile of order $1-\alpha$ of the Fisher distribution with $(p-q, n-p)$ degrees of freedom, is a test of size α for

$$H_0: \; X\beta^* \in W \quad \textit{versus} \quad H_1: \; X\beta^* \in V \setminus W.$$

1.4 Choosing a good model

We still consider the linear regression model

$$Y = X\beta^* + \varepsilon.$$

We wish to select a good model, namely a good set of predictors to explain and predict the response variable. We assume that $p < n$ with n and p large. However, we are not sure that all predictors X_j are necessary to predict Y. We wish to select only relevant predictors. We assume that the first column of X is the vector with only 1 in each row (i.e. X_1 is the intercept).

Remark 1.4. *Sometimes, predictors are called variables.*

In the sequel, we describe several methods to select a set of variables, called model.

Definition 1.5. *A model m will denote in the sequel a subset of the set $\{1, \dots, p\}$. With a slight abuse of notations, a model m may also denote the variables X_j for $j \in m$.*

It is often easy to choose between two given models, but the general question of choosing a model is more intricate because, most of the time there is no natural order between variables. Furthermore, when p is large the number of models is huge (2^p in full generality).

Notation: For any model m, we denote P_m the projection matrix on $\text{span}(X_j,\ j \in m)$. We also denote

$$\text{RSS}(m) = \|Y - P_m Y\|^2$$

the *residual sum of squares associated with the model m.*

Observe that if X_m is the matrix with the columns $(X_j)_{j\in m}$, then

$$P_m = X_m(X_m^T X_m)^{-1} X_m^T.$$

If X is one to one, X_m is also one to one. We now describe main methodologies to choose a model. In the sequel, all models will contain the first column of X (ie the intercept), which represents the mean response value when all predictors are set to 0.

1.4.1 Tests between models

We assume that $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$. Let m_0 a model, which contains the predictor $\mathbb{I}$, with

$$\mathbb{I} = (1, \dots, 1) \in \mathbb{R}^p.$$

Let m_1 such that $m_0 \subset m_1$ and $\text{card}(m_1) = \text{card}(m_0) + 1$. We denote $k = \text{card}(m_0)$. Therefore, $\text{card}(m_1) = k + 1$. We use the Fisher test to choose between m_0 and m_1. We can use

$$F = \frac{\text{RSS}(m_0) - \text{RSS}(m_1)}{\text{RSS}(m_1)} \times (n - k - 1)$$

or

$$\tilde{F} = \frac{\mathrm{RSS}(m_0) - \mathrm{RSS}(m_1)}{\hat{\sigma}^2},$$

with $\hat{\sigma}^2$, which is independent of the numerator. Observe that

$$\|Y - P_{m_0}Y\|^2 = \|P_{m_1}Y - P_{m_0}Y\|^2 + \|Y - P_{m_1}Y\|^2,$$

which is equivalent to

$$\mathrm{RSS}(m_0) = \|P_{m_1}Y - P_{m_0}Y\|^2 + \mathrm{RSS}(m_1).$$

Using Corollary 1.1, we have:

Corollary 1.2. *Since $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$, statistics F and $\tilde{F}$ allow to test*

$$H_0 : m_0 = \textit{true model} \quad vs \quad H_1 : m_1 = \textit{true model}.$$

- *If $F > f_{1,n-k-1,1-\alpha}$ then m_1 is chosen with respect to m_0 at risk α.*
- *If $\tilde{F} > f_{1,n-p,1-\alpha}$ then m_1 is chosen with respect to m_0 at risk α.*

It's hard to choose between F and $\tilde{F}$. Mose of the time, we use F. Note that assumption $m_0 \subset m_1$ is crucial.

1.4.2 R squared (R^2)

We recall that the R^2 (the R squared), or the coefficient of determination, is defined by

$$R^2 = \frac{\|\hat{Y} - \bar{Y}\mathbb{I}\|^2}{\|Y - \bar{Y}\mathbb{I}\|^2}, \quad \hat{Y} = P_X Y = X\hat{\beta}, \quad \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

Definition 1.6. *For any model m, we define*

$$R^2(m) = \frac{\|P_m Y - \bar{Y}\mathbb{I}\|^2}{\|Y - \bar{Y}\mathbb{I}\|^2} = 1 - \frac{\|Y - P_m Y\|^2}{\|Y - \bar{Y}\mathbb{I}\|^2} = 1 - \frac{RSS(m)}{RSS_p},$$

with

$$RSS_p = \|Y - \bar{Y}\mathbb{I}\|^2.$$

We observe that, if $m_0 \subset m_1$,

$$R^2(m_1) - R^2(m_0) = \frac{\|P_{m_0}Y - P_{m_1}Y\|^2}{\mathrm{RSS}_p} \geq 0$$

and

$$R^2(m_1) \geq R^2(m_0).$$

Most of the time, we don't use the R^2 as criterion since it will always increase with the size of the model. However, when the R^2 does not increase any longer, it can be useful. It can be used for two different models with the same number of variables.

1.4.3 The adjusted R^2

To take into account the number of variables, we use the adjusted R^2 defined by

$$R_a(m) := 1 - \frac{\mathrm{RSS}(m)}{(n-|m|)} \times \frac{(n-1)}{\mathrm{RSS}_p},$$

where we have denoted $|m|$ the cardinal of m. The selected model is the model which maximizes $m \longmapsto R_a(m)$.

1.4.4 Mallows' C_p

In this paragraph, we denote for any model m, $\hat{Y}_m = P_m Y$. We're going to use $\hat{Y}_m$ to estimate $X\beta^*$ for some model m which has to be selected. For this purpose, we use the Mallows' C_p criterion.

Remark 1.5. *In Section 2.1, we provide more arguments for the use of $\hat{Y}_m$, in particular in the Gaussian setting.*

Definition 1.7. *For any model m, the Mallows' C_p associated with m is defined by*

$$C_p(m) = \frac{RSS(m)}{\hat{\sigma}^2} - n + 2|m|,$$

where $|m|$ still denotes the cardinal of m.

The selected model is the model which minimizes $m \longmapsto C_p(m)$. We have:

Theorem 1.3. *An unbiased estimate of the risk of $\hat{Y}_m$ for estimating $X\beta^*$ is given by $C_p(m) \times \hat{\sigma}^2$. Indeed, we have:*

$$\mathbb{E}[C_p(m) \times \hat{\sigma}^2] = \mathbb{E}[\|\hat{Y}_m - X\beta^*\|^2].$$

Proof. First observe that

$$\mathbb{E}[\|P_m\varepsilon\|^2] = \sigma^2|m|.$$

Then,

$$\begin{aligned}
\hat{\sigma}^2 C_p(m) &= \mathrm{RSS}(m) + (2|m| - n)\hat{\sigma}^2 \\
&= \|Y - \hat{Y}_m\|^2 + (2|m| - n)\hat{\sigma}^2.
\end{aligned}$$

And on the one hand, we have

$$\begin{aligned}
\mathbb{E}[C_p(m) \times \hat{\sigma}^2] &= \mathbb{E}[\|(I_n - P_m)(X\beta^* + \varepsilon)\|^2] + (2|m| - n)\mathbb{E}[\hat{\sigma}^2] \\
&= \|(I_n - P_m)X\beta^*\|^2 + \sigma^2(n - |m|) + \sigma^2(2|m| - n) \\
&= \|(I_n - P_m)X\beta^*\|^2 + |m|\sigma^2.
\end{aligned}$$

and on the other hand,

$$\begin{aligned}\mathbb{E}[\|\hat{Y}_m - X\beta^*\|^2] &= \mathbb{E}[\|P_m Y - X\beta^*\|^2] \\ &= \mathbb{E}[\|P_m\varepsilon + (P_m - I_n)X\beta^*\|^2 \\ &= \|(I_n - P_m)X\beta^*\|^2 + \mathbb{E}[\|P_m\varepsilon\|^2] \\ &= \|(I_n - P_m)X\beta^*\|^2 + |m|\sigma^2\end{aligned}$$

□

The previous result shows that when m is fixed, $C_p(m) \times \hat{\sigma}^2$ is an unbiased estimate of the mean squared error of $\hat{Y}_m$. So, if we wish to minimize

$$m \longmapsto \mathbb{E}[\|\hat{Y}_m - X\beta^*\|^2],$$

then it is natural to minimize

$$m \longmapsto C_p(m) \times \hat{\sigma}^2,$$

which is equivalent to minimize

$$m \longmapsto C_p(m).$$

This justifies the introduction of the Mallows' C_p criterion. When we studied the classical R^2, we observed that when we add variables the RSS decreases. Therefore, adding the term $2|m|\hat{\sigma}^2$ is an alternative to the adjusted R^2 to face with this problem.

Remark 1.6. *The proof shows that for any m,*

$$\mathbb{E}[RSS(m)] = (n - |m|)\sigma^2 + \|(I_n - P_m)X\beta^*\|^2.$$

So, if the true model is included into m_0, we have

$$X\beta^* = P_{m_0}X\beta^*.$$

and

$$RSS(m_0) \approx \mathbb{E}[RSS(m_0)] = (n - |m_0|)\sigma^2 \approx (n - |m_0|)\hat{\sigma}^2.$$

And in this case,

$$C_p(m_0) \approx |m_0|.$$

So, if we add useless variables to m_0, $C_p(m_0)$ will increase. Furthermore, if we have forgotten important variables

$$RSS(m_0) \approx \mathbb{E}[RSS(m_0)] = (n - |m_0|)\sigma^2 + C \approx (n - |m_0|)\hat{\sigma}^2 + C$$

with $C > 0$ and

$$C_p(m_0) > |m_0|.$$

So, we are naturally interested in models m_0 such that $C_p(m_0) \leq |m_0|$.

1.4.5 AIC and BIC criteria

In the previous paragraph, we have only assumed that

$$\mathrm{rank}(X) = p, \quad \mathbb{E}[\varepsilon] = 0, \quad \mathrm{var}(\varepsilon) = \sigma^2 I_n.$$

In addition, we assume now that $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$. Therefore, we can compute the likelihood. The log-likelihood is equal to

$$L(\beta, \sigma^2) = -\frac{n}{2}\log(\sigma^2) - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\|Y - X\beta\|^2.$$

Given a model m, the estimate $\hat{\beta}^{(m)}$ maximizing $L(\beta, \sigma^2)$ such that $\hat{\beta}_j^{(m)} = 0$ if $j \notin m$ is such that $\|Y - X\beta\|^2$ is minimum. Therefore,

$$X\hat{\beta}^{(m)} = P_m Y$$

and

$$L(\hat{\beta}^{(m)}, \sigma^2) = -\frac{n}{2}\log(\sigma^2) - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\mathrm{RSS}(m).$$

To maximize $\sigma^2 \longmapsto L(\hat{\beta}^{(m)}, \sigma^2)$, we derive the last expression and $\sigma^2 = \frac{1}{n}\mathrm{RSS}(m)$ is the maximizer. Then, the maximum of L given a model m is

$$L(m) := \max_{\sigma^2 \in \mathbb{R}_+^*} L(\hat{\beta}^{(m)}, \sigma^2) = -\frac{n}{2}\log\left(\frac{\mathrm{RSS}(m)}{n}\right) - \frac{n}{2}\log(2\pi) - \frac{n}{2}.$$

Therefore, maximizing the likelihood is equivalent to minimizing the RSS. But minimizing $m \longmapsto \mathrm{RSS}(m)$ is not a good idea. So, we add a penalty and we minimize

$$m \longmapsto -L(m) + \mathrm{penalty}(m) = \frac{n}{2}\log\left(\frac{\mathrm{RSS}(m)}{n}\right) + \mathrm{penalty}(m) + Const.$$

For AIC, we take $\mathrm{penalty}(m) = |m|$. For BIC, we take $\mathrm{penalty}(m) = \frac{\log(n)}{2}|m|$.
Finally, the AIC procedure consists in minimizing

$$m \longmapsto \frac{n}{2}\log(\mathrm{RSS}(m)) + |m|.$$

The BIC procedure consists in minimizing

$$m \longmapsto n\log(\mathrm{RSS}(m)) + \log(n)|m|.$$

Note that if $n > 7$, then $\log(n) > 2$. Therefore, models selected by BIC are smaller than for AIC.

1.4.6 Comparisons between criteria

We compare criteria in the case $m_0 \subset m_1$ with $|m_1| = |m_0| + 1$. We study the case where m_0 is chosen instead of m_1.

1. With the F-statistics ($\tilde{F}$ is less used), we approximate $f_{1,n-|m_0|-1,1-\alpha}$ by 4, which is valid if $\alpha = 0.05$ and $n - |m_0| - 1 \geq 16$. Therefore, m_0 is chosen if
$$\frac{\mathrm{RSS}(m_0) - \mathrm{RSS}(m_1)}{\mathrm{RSS}(m_1)} \times (n - |m_0| - 1) \leq 4.$$

2. With the R^2, m_0 is never chosen.

3. With the adjusted R^2,
$$\begin{aligned} R_a^2(m_0) \geq R_a^2(m_1) &\iff \frac{\mathrm{RSS}(m_0)}{n - |m_0|} \leq \frac{\mathrm{RSS}(m_1)}{n - |m_0| - 1} \\ &\iff \frac{\mathrm{RSS}(m_0) - \mathrm{RSS}(m_1)}{\mathrm{RSS}(m_1)} \times (n - |m_0| - 1) \leq 1. \end{aligned}$$

4. With the Mallows' C_p,
$$\begin{aligned} C_p(m_0) \leq C_p(m_1) &\iff \frac{\mathrm{RSS}(m_0)}{\hat{\sigma}^2} \leq \frac{\mathrm{RSS}(m_1)}{\hat{\sigma}^2} + 2 \\ &\iff \frac{\mathrm{RSS}(m_0) - \mathrm{RSS}(m_1)}{\hat{\sigma}^2} \leq 2 \\ &\iff \frac{\mathrm{RSS}(m_0) - \mathrm{RSS}(m_1)}{\mathrm{RSS}(m_1)} \times (n - |m_0| - 1) \leq 2, \end{aligned}$$
if we can replace $\hat{\sigma}^2$ with $\mathrm{RSS}(m_1)/(n - |m_1|)$.

5. With AIC and BIC, then, setting $f(n) = 2/n$ for AIC $f(n) = \log(n)/n$ for BIC, m_0 will be selected
$$\begin{aligned} &\iff \log(\mathrm{RSS}(m_0)) - \log(\mathrm{RSS}(m_1)) \leq f(n) \\ &\iff \frac{\mathrm{RSS}(m_0) - \mathrm{RSS}(m_1)}{\mathrm{RSS}(m_1)} \times (n - |m_0| - 1) \leq (\exp(f(n)) - 1) \times (n - |m_0| - 1) \end{aligned}$$
Asymptotically, it gives for AIC
$$\frac{\mathrm{RSS}(m_0) - \mathrm{RSS}(m_1)}{\mathrm{RSS}(m_1)} \times (n - |m_0| - 1) \leq \frac{2}{n} \times (n - |m_0| - 1)$$
and for BIC
$$\frac{\mathrm{RSS}(m_0) - \mathrm{RSS}(m_1)}{\mathrm{RSS}(m_1)} \times (n - |m_0| - 1) \leq \frac{\log n}{n} \times (n - |m_0| - 1).$$

So, roughly speaking, each criterion is equivalent to

$$\frac{\mathrm{RSS}(m_0) - \mathrm{RSS}(m_1)}{\mathrm{RSS}(m_1)} \times (n - |m_0| - 1) \leq q,$$

with

1. $q = 4$ for the Fisher test
2. $q = -\infty$ for the R^2
3. $q = 1$ for the adjusted R^2
4. $q = 2$ for the Mallows' C_p
5. $q = \frac{2}{n} \times (n - |m_0| - 1)$ for AIC
6. $q = \frac{\log(n)}{n} \times (n - |m_0| - 1)$ for BIC

Then, BIC is the most favorable for m_0 and the R^2 is the most favorable for m_1.

1.5 Step by step procedures

Minimizing or maximizing a criterion may be a difficult task when the number of variables is large. Indeed, if we have p variables, we have 2^{p-1} different models (if each model contains the intercept). When the exhaustive research is not possible (either because we wish to use the Fisher test or because p is too large), we can use a step by step procedure combined with one of the 6 previous procedures. The drawback is that we don't test all possible combinations. So, we are not sure to obtain a global extremum. We can use one of the following methods.

1. *Forward selection:* At each step, a variable is added (the variable which has the strongest impact (if we use a test, it corresponds to the smallest p-value)).
2. *Backward selection:* At each step, a variable is removed (the variable which has the strongest impact (if we use a test, it corresponds to the largest p-value)).
3. *Stepwise selection:* Similar to forward selection, but at each step, we can question each variable of the model according to the backward selection.

The intercept is always one of the variables. For each of these algorithms, we stop at a p-value previously given or when the impact is not significative.

i	Type de véhicule	Prix (Frs) x_1	Cylindrée (cm3) x_2	Puissance (kW) x_3	Poids (kg) x_4	Consommation (l/100km) y_i
1	Daihatsu Cuore	11600	846	32	650	5,7
2	Suzuki Swift 1.0 GLS	12490	993	39	790	5,8
3	Fiat Panda Mambo L	10450	899	29	730	6,1
4	VW Polo 1.4 60	17140	1390	44	955	6,5
5	Opel Corsa 1.2i Eco	14825	1195	33	895	6,8
6	Subaru Vivio 4WD	13730	658	32	740	6,8
7	Toyota Corolla	19490	1331	55	1010	7,1
8	Ferrari 456 GT	285000	5474	325	1690	21,3
9	Mercedes S 600	183900	5987	300	2250	18,7
10	Maserati Ghibli GT	92500	2789	209	1485	14,5
11	Opel Astra 1.6i 16V	25000	1597	74	1080	7,4
12	Peugeot 306 XS 108	22350	1761	74	1100	9
13	Renault Safrane 2.2. V	36600	2165	101	1500	11,7
14	Seat Ibiza 2.0 GTI	22500	1983	85	1075	9,5
15	VW Golt 2.0 GTI	31580	1984	85	1155	9,5
16	Citroen ZX Volcane	28750	1998	89	1140	8,8
17	Fiat Tempra 1.6 Liberty	22600	1580	65	1080	9,3
18	Fort Escort 1.4i PT	20300	1390	54	1110	8,6
19	Honda Civic Joker 1.4	19900	1396	66	1140	7,7
20	Volvo 850 2.5	39800	2435	106	1370	10,8
21	Ford Fiesta 1.2 Zetec	19740	1242	55	940	6,6
22	Hyundai Sonata 3000	38990	2972	107	1400	11,7
23	Lancia K 3.0 LS	50800	2958	150	1550	11,9
24	Mazda Hachtback V	36200	2497	122	1330	10,8
25	Mitsubishi Galant	31990	1998	66	1300	7,6
26	Opel Omega 2.5i V6	47700	2496	125	1670	11,3
27	Peugeot 806 2.0	36950	1998	89	1560	10,8
28	Nissan Primera 2.0	26950	1997	92	1240	9,2
29	Seat Alhambra 2.0	36400	1984	85	1635	11,6
30	Toyota Previa salon	50900	2438	97	1800	12,8
31	Volvo 960 Kombi aut	49300	2473	125	1570	12,7

Figure 1.1: 31 types of cars with their price (prix), their engine capacity (cylindrée), their power (puissance), their weight (poids) and their consumption (consommation)

1.6 Illustrative example with R

We use the data file provided in Table 1.1.

1.6.1 Exhaustive selection of models

In this paragraph, we consider the problem of the model choice by using one of the following methodologies: The R^2, the adjusted R^2, the Mallows' C_p and the BIC. We use the following lines.

```
# Model choice (exhaustive method)
library(leaps)
choix_modele=regsubsets(Consommation~Prix+Cylindree+Puissance+Poids,int=T,
                        nbest=1,nvmax=4,method="exhaustive",data=conso_voit)
resume=summary(choix_modele)
```

```
print(resume)

quartz()
par(mfrow=c(2,2))
plot(choix_modele,scale="r2")
plot(choix_modele,scale="adjr2")
plot(choix_modele,scale="Cp")
plot(choix_modele,scale="bic")
par(mfrow=c(1,1))
```

We specify that `nbest` gives the number of models selected by dimension, `nvmax` the maximum number of the selected variables (without intercept). Imposing `int=T` allows to ensure that the intercept will be selected. We obtain following outputs.

```
Subset selection object
Call: regsubsets.formula(Consommation ~ Prix + Cylindree + Puissance +
    Poids, int = T, nbest = 1, nvmax = 4, method = "exhaustive",
    data = conso_voit)
4 Variables  (and intercept)
          Forced in Forced out
Prix          FALSE      FALSE
Cylindree     FALSE      FALSE
Puissance     FALSE      FALSE
Poids         FALSE      FALSE
1 subsets of each size up to 4
Selection Algorithm: exhaustive
         Prix Cylindree Puissance Poids
1  ( 1 ) " "  " "       "*"       " "
2  ( 1 ) " "  " "       "*"       "*"
3  ( 1 ) "*"  " "       "*"       "*"
4  ( 1 ) "*"  "*"       "*"       "*"
```

We also obtain Figure 1.2 : All the methodologies, except R^2, select a model with 3 variables (plus the intercept): `Prix`, `Puissance` and `Poids`. Note that except for Mallows' C_p, the variable `Prix` is not strictly excluded from the best model.

1.6.2 Step by step approaches

We now illustrate the step by step approaches with the AIC criterion. We start with stepwise selection.

```
# Model selection (step by step)
library(MASS)
```

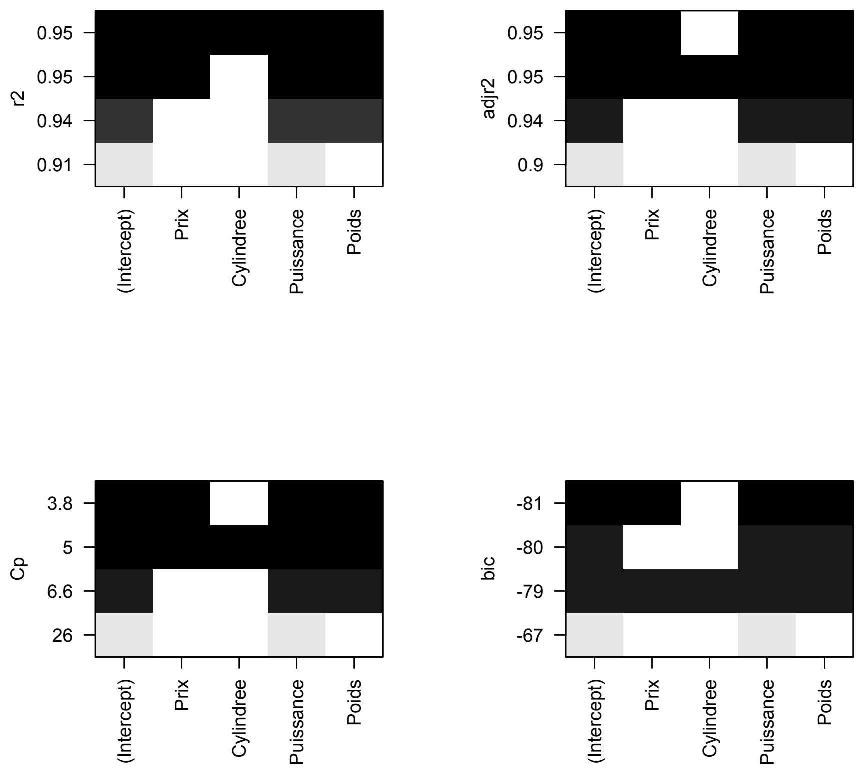

Figure 1.2: Models choice with R^2, adjusted R^2, Mallows' C_p and BIC criterion for Example 1.1.

```
step(lm(Consommation~1,data=conso_voit), Consommation~Prix+Cylindree+Puissance+Poids,
     data=conso_voit, direction="both")
```

We obtain the following output.

```
Start:  AIC=79.87
Consommation ~ 1

            Df Sum of Sq    RSS    AIC
+ Puissance  1    346.79  35.35  8.071
+ Cylindree  1    338.37  43.77 14.692
+ Prix       1    303.45  78.69 32.878
```

```
+ Poids      1    285.17  96.96 39.351
<none>                   382.14 79.866

Step:  AIC=8.07
Consommation ~ Puissance

            Df Sum of Sq    RSS    AIC
+ Poids      1     14.27  21.08 -5.961
+ Cylindree  1      3.01  32.34  7.310
<none>                    35.35  8.071
+ Prix       1      0.00  35.35 10.070
- Puissance  1    346.79 382.14 79.866

Step:  AIC=-5.96
Consommation ~ Puissance + Poids

            Df Sum of Sq    RSS    AIC
+ Prix       1     3.205 17.871 -9.074
<none>                   21.077 -5.961
+ Cylindree  1     0.058 21.019 -4.046
- Poids      1    14.273 35.350  8.071
- Puissance  1    75.888 96.964 39.351

Step:  AIC=-9.07
Consommation ~ Puissance + Poids + Prix

            Df Sum of Sq    RSS     AIC
<none>                   17.871 -9.0744
+ Cylindree  1    0.5065 17.365 -7.9657
- Prix       1    3.2053 21.077 -5.9605
- Puissance  1    3.9434 21.815 -4.8934
- Poids      1   17.4783 35.350 10.0704

Call:
lm(formula = Consommation ~ Puissance + Poids + Prix, data = conso_voit)

Coefficients:
(Intercept)    Puissance        Poids         Prix
  2.499e+00    2.013e-02    3.735e-03    1.852e-05
```

The alternative based on forward selection is the following.

```
step(lm(Consommation~1,data=conso_voit), Consommation~Prix+Cylindree+Puissance+Poids,
    data=conso_voit, direction="forward")
```

The output is then:

```
 Start:  AIC=79.87
Consommation ~ 1

            Df Sum of Sq    RSS    AIC
+ Puissance  1    346.79  35.35  8.071
+ Cylindree  1    338.37  43.77 14.692
+ Prix       1    303.45  78.69 32.878
+ Poids      1    285.17  96.96 39.351
<none>                   382.14 79.866

Step:  AIC=8.07
Consommation ~ Puissance

            Df Sum of Sq    RSS     AIC
+ Poids      1   14.2733 21.077 -5.9605
+ Cylindree  1    3.0114 32.339  7.3104
<none>                   35.350  8.0706
+ Prix       1    0.0002 35.350 10.0704

Step:  AIC=-5.96
Consommation ~ Puissance + Poids

            Df Sum of Sq    RSS     AIC
+ Prix       1    3.2053 17.871 -9.0744
<none>                   21.077 -5.9605
+ Cylindree  1    0.0580 21.019 -4.0460

Step:  AIC=-9.07
Consommation ~ Puissance + Poids + Prix

            Df Sum of Sq    RSS     AIC
<none>                   17.871 -9.0744
+ Cylindree  1   0.50652 17.365 -7.9657

Call:
lm(formula = Consommation ~ Puissance + Poids + Prix, data = conso_voit)
```

```
Coefficients:
(Intercept)    Puissance        Poids         Prix
  2.499e+00    2.013e-02    3.735e-03    1.852e-05
```

The alternative based on backward selection is the following.

```
 step(reg,direction='backward')
```

The output is then:

```
 Start:  AIC=-7.97
Consommation ~ Prix + Cylindree + Puissance + Poids

            Df Sum of Sq    RSS     AIC
- Cylindree  1    0.5065 17.871 -9.0744
<none>                   17.365 -7.9657
- Prix       1    3.6537 21.019 -4.0460
- Puissance  1    4.1792 21.544 -3.2805
- Poids      1   14.9706 32.335  9.3075

Step:  AIC=-9.07
Consommation ~ Prix + Puissance + Poids

            Df Sum of Sq    RSS     AIC
<none>                   17.871 -9.0744
- Prix       1    3.2053 21.077 -5.9605
- Puissance  1    3.9434 21.815 -4.8934
- Poids      1   17.4783 35.350 10.0704

Call:
lm(formula = Consommation ~ Prix + Puissance + Poids, data = conso_voit)

Coefficients:
(Intercept)         Prix    Puissance        Poids
  2.499e+00    1.852e-05    2.013e-02    3.735e-03
```

For backward selection we can also use the function `drop1`. All the step by step approaches give the same results, which is coherent with the exhaustive approach. We conclude that 3 variables have to be considered to explain the variable `Consommation`: `Prix`, `Puissance` and `Poids`.

Chapter 2

Model selection

2.1 Models and oracle

We still consider the problem of linear regression

$$Y = X\beta^* + \varepsilon,$$

with $Y = (Y_1, \ldots, Y_n)^T$, $X = [X_1, \ldots, X_p]$. We denote $f^* = X\beta^*$ and we assume that $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$, with σ^2 assumed to be known. We also assume that there exists a (small) subset m^* of $\{1, 2, \ldots, p\}$ such that $\forall\ j \notin m^*$, $\beta_j^* = 0$. If m^* were known, then we would estimate $X\beta^*$ by $\hat{f}_{m^*}$ where $\hat{f}_{m^*} = P_{S^*}Y$, and $P_{S^*} : \mathbb{R}^n \longmapsto \mathbb{R}^n$ is the orthogonal projection on $S^* = \text{span}(X_j,\ j \in m^*)$. Indeed, the log-likelihood with respect to an estimate candidate $\hat{f}$ is given by

$$\hat{f} \longmapsto -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\|Y - \hat{f}\|^2,$$

where $\|\cdot\|$ denotes the ℓ_2-norm. But the space S^* (or the subset m^*) is unknown, so $\hat{f}_{m^*}$ cannot be used. So, given $\mathcal{M}$ a collection of models m ($m \subset \{1, \ldots, p\}$), we wish to

1. Consider a collection $(S_m)_{m\in\mathcal{M}}$ of linear subspaces of $\mathbb{R}^n$, also denoted (with a slight abuse of notations) models.

2. Associate to each subspace S_m the constrained maximum likelihood estimates $\hat{f}_m = P_{S_m}Y$.

3. Finally select the best estimate among the collection $(\hat{f}_m)_{m\in\mathcal{M}}$.

To give a meaning to the terminology "best estimate", we need a criterion to quantify the quality of an estimator. In the sequel, we will measure the quality of an estimate $\hat{f}$ of f^* by its ℓ_2-risk, defined as follows.

$$R(\hat{f}) := \mathbb{E}[\|\hat{f} - f^*\|^2].$$

We also set

$$r_m := R(\hat{f}_m) = \mathbb{E}[\|\hat{f}_m - f^*\|^2].$$

Now, the best estimate in terms of the ℓ_2-risk is $\hat{f}_{m_0}$ with

$$m_0 := \arg\min_{m \in \mathcal{M}} r_m$$

and $\hat{f}_{m_0}$ is called the oracle estimate.

Remark 2.1. *f^* may not belong to S_{m_0}. It may even not belong to $\cup_{m \in \mathcal{M}} S_m$.*

Remark 2.2. *Even if $X\beta^* \in S_m$ with $m = \{1, \ldots, p\}$, this last model may be far from the oracle model which is our benchmark for comparison.*

We cannot use $\hat{f}_{m_0}$ since it depends on the unknown true vector β^* (via the expectation). A natural idea to circumvent this issue is to replace r_m by some estimate $\hat{r}_m$. Then we estimate f^* by $\hat{f}_{\hat{m}}$ with

$$\hat{m} := \arg\min_{m \in \mathcal{M}} \hat{r}_m.$$

The goal of this chapter is to provide some suitable $\hat{r}_m$ for which we can guarantee that the selected estimate $\hat{f}_{\hat{m}}$ performs almost as well as the oracle $\hat{f}_{m_0}$.

<u>Collections of models.</u> For this purpose, we denote

$$S_m := \text{span}(X_j,\ j \in m).$$

- We set $\mathcal{M} = \mathcal{P}(\{1, \ldots, p\})$ where $\mathcal{P}(\{1, \ldots, p\})$ denotes the set of all the subsets of $\{1, \ldots, p\}$. We have $\text{card}(\mathcal{M}) = 2^p$ and

$$S_m := \{X\beta :\ \beta \in \mathbb{R}^p \text{ with } \beta_j = 0 \text{ if } j \notin m\}.$$

- We set $\mathcal{M} = \{\{1, \ldots, J\},\ 1 \leq J \leq p\}$. In this case, $\text{card}(\mathcal{M}) = p$.

2.2 Model selection procedures

We first compute $r_m = R(\hat{f}_m)$. We denote d_m the dimension of S_m:

$$d_m := \dim(S_m).$$

Remark 2.3. *If X is one-to-one, by using the rank-nullity theorem, $d_m = |m|$.*

Lemma 2.1. *We have*

$$r_m = \|(I_n - P_{S_m})f^*\|^2 + \sigma^2 d_m.$$

Proof. The proof of the lemma is similar to the proof of Theorem 1.3. □

The risk involves two terms. The first one decreases when m increases whereas the second one increases. The first term is an approximation term. The second one is a variance term. The oracle model S_{m_0} is the model which achieves the best trade-off between these two terms. The Mallows' procedure studied in Section 1.4.4 is based on unbiased estimate of the risk and if we set

$$\hat{r}_m := \|Y - \hat{f}_m\|^2 + \sigma^2(2d_m - n)$$

then $\hat{r}_m$ is an unbiased estimate of the risk (see Theorem 1.3). The Mallows' procedure can produce very poor results since it does not take into account the variability of $\hat{r}_m$ around r_m. This is a problem when the number of models per dimension is large. Indeed, we have many estimators $\hat{r}_m$ and some of them deviate seriously from their expected value r_m. In particular, some $\hat{r}_m$ are very small, much smaller than $\hat{r}_{m_0}$. This leads to select a model $S_{\hat{m}}$ much bigger than S_{m_0} (overfitting). See Problem 1 of Exam 2016-2017 for more details.

Penalized estimator of the risk: To avoid the previous problem, we replace the term $2\sigma^2 d_m$ with something larger. We focus on a selection criterion of the form

$$\hat{m} := \arg\min_{m \in \mathcal{M}} \left\{ \|Y - \hat{f}_m\|^2 + \sigma^2 \text{pen}(m) \right\}, \tag{2.1}$$

where pen : $\mathcal{M} \mapsto \mathbb{R}_+$ is called the penalty function. To define pen(m), we associate to the collection of models $(S_m)_{m \in \mathcal{M}}$ a probability distribution $\Pi = (\Pi_m)_{m \in \mathcal{M}}$. Then, we set:

Definition 2.1. *Let*

$$\textit{pen}(m) = K\left(\sqrt{d_m} + \sqrt{2\log(1/\Pi_m)}\right)^2,$$

with $K > 1$. Then, we estimate f^* with $\hat{f} := \hat{f}_{\hat{m}}$ such that $\hat{m}$ is defined in (2.1).

We have the following risk bound on $R(\hat{f})$.

Theorem 2.1. *There exists a constant $C_K > 1$ depending only on $K > 1$ such that*

$$\begin{aligned}
\mathbb{E}\left[\|\hat{f} - f^*\|^2\right] &\le C_K \min_{m \in \mathcal{M}} \left\{ \mathbb{E}\left[\|\hat{f}_m - f^*\|^2\right] + \sigma^2 \log(\Pi_m^{-1}) + \sigma^2 \right\} \\
&\le C_K \min_{m \in \mathcal{M}} \left\{ \|(I_n - P_{S_m}) f^*\|^2 + \sigma^2 d_m + \sigma^2 \log(\Pi_m^{-1}) + \sigma^2 \right\}.
\end{aligned}$$

Remark 2.4. *Remember that our benchmark is $\hat{f}_{m_0}$ whose risk is*

$$R(\hat{f}_{m_0}) = r_{m_0} = \|(I_n - P_{S_{m_0}}) f^*\|^2 + \sigma^2 d_{m_0}.$$

Observe that if for some constant $L(p) > 0$ (that may depend on p but not on n), $\log(\Pi_m^{-1}) \leq L(p)d_m$, then $\hat{f}$ achieves the same risk as $\hat{f}_{m_0}$ up to a constant depending on $L(p)$. Indeed, in this case,

$$\log(\Pi_m^{-1}) \leq L(p)d_m \leq \sigma^{-2}L(p)r_m,$$

and

$$\mathbb{E}\left[\|\hat{f} - f^*\|^2\right] \leq C_K\Big(\big(1 + L(p)\big)\min_{m\in\mathcal{M}} r_m + \sigma^2\Big)$$

and $\min_{m\in\mathcal{M}} r_m = r_{m_0}$.

Remark 2.5. *The upper bound of Theorem 2.1 can be proved to be optimal. We can also prove that we cannot take $K < 1$.*

<u>Choice of $\Pi = (\Pi_m)_{m\in\mathcal{M}}$</u>: Π has to be a probability measure and in view of Theorem 2.1, The Π_m's have to be as small as possible. In the sequel, for the sake of simplicity, X is assumed to be one-to-one.

1. <u>$\mathcal{M} = \mathcal{P}(\{1, \ldots, p\})$</u>: We take, with $d_m = |m|$,

$$\Pi_m = C \times \binom{p}{|m|}^{-1} \times e^{-|m|}, \quad C = \frac{e-1}{1-e^{-p}}.$$

We have

$$\begin{aligned}\sum_{m\in\mathcal{M}} \Pi_m &= C \sum_{m\in\mathcal{M}} \binom{p}{|m|}^{-1} \times e^{-|m|} \\ &= C \sum_{d=1}^{p} \sum_{m\in\mathcal{M}\ |m|=d} \binom{p}{|m|}^{-1} \times e^{-|m|} \\ &= C \sum_{d=1}^{p} e^{-d} = 1.\end{aligned}$$

Lemma 2.2. *For $1 \leq d \leq p$, we have the upper bound*

$$\log\binom{p}{d} \leq d\left(1 + \log\left(\frac{p}{d}\right)\right).$$

Proof. We prove the result by induction on d. The result of the lemma is obvious for $d=1$. For $d\geq 2$, we use

$$\begin{aligned}\binom{p}{d} &= \binom{p}{d-1}\times\frac{p-d+1}{d}\\ &\leq \exp\left((d-1)\left(1+\log\left(\frac{p}{d-1}\right)\right)\right)\times\frac{p}{d}\\ &\leq \left(\frac{ep}{d-1}\right)^{d-1}\times\frac{p}{d}\\ &\leq \left(\frac{ep}{d}\right)^{d-1}\times\left(1+\frac{1}{d-1}\right)^{d-1}\times\frac{p}{d}\leq\left(\frac{ep}{d}\right)^{d}.\end{aligned}$$

We have used for $x>0$, $(1+x^{-1})^x\leq e$. □

So, we have

$$\begin{aligned}\log(\Pi_m^{-1}) &= |m|+\log\binom{p}{|m|}+\log(C^{-1})\\ &\leq 2|m|+|m|\log\left(\frac{p}{|m|}\right)+\log(C^{-1})\\ &\lesssim |m|\log(p),\end{aligned}$$

and in this case, we can take $L(p)$, introduced in Remark 2.4, proportional to $\log(p)$. Therefore, applying Theorem 2.1,

$$R(\hat{f})\lesssim\log p\times R(\hat{f}_{m_0}).$$

it can be proved that term $\log p$ is unavoidable.

2. $\underline{\mathcal{M}=\{\{1,\dots,J\},\ 1\leq J\leq p\}}$: We take

$$\Pi_m=|m|^{-2}\times\left(\sum_{d=1}^{p}d^{-2}\right)^{-1}.$$

So, we have

$$\log(\Pi_m^{-1})\leq 2\log(|m|)+const$$

and applying Theorem 2.1,

$$R(\hat{f})\lesssim R(\hat{f}_{m_0}).$$

meaning that the Mallows heuristics works. Indeed, in this case, $L(p)$, the constant introduced in Remark 2.4, does not depend on p. An alternative consists in taking Π_m proportional to $e^{-|m|}$.

Proof of Theorem 2.1: By definition, for any $m \in \mathcal{M}$,

$$\|Y - \hat{f}_{\hat{m}}\|^2 + \sigma^2 \text{pen}(\hat{m}) \le \|Y - \hat{f}_m\|^2 + \sigma^2 \text{pen}(m).$$

Since $Y = f^* + \varepsilon$, we obtain

$$\|f^* - \hat{f}\|^2 \le \|f^* - \hat{f}_m\|^2 + 2\langle \varepsilon, \hat{f} - f^* \rangle + 2\langle \varepsilon, f^* - \hat{f}_m \rangle + \sigma^2(\text{pen}(m) - \text{pen}(\hat{m})).$$

We have

$$\begin{aligned}\mathbb{E}\left[\langle \varepsilon, f^* - \hat{f}_m \rangle\right] &= \mathbb{E}\left[\langle \varepsilon, f^* - P_{S_m}(f^* + \varepsilon)\rangle\right] \\ &= 0 - \mathbb{E}\left[\|P_{S_m}\varepsilon\|^2\right] = -\sigma^2 d_m \le 0\end{aligned}$$

and

$$\begin{aligned}\sigma^2 \text{pen}(m) &\le 2K\sigma^2 \left(d_m + 2\log(\Pi_m^{-1})\right) \\ &\le 2K r_m + 4K\sigma^2 \log(\Pi_m^{-1}) \\ &\le 2K\mathbb{E}[\|f^* - \hat{f}_m\|^2] + 4K\sigma^2 \log(\Pi_m^{-1}).\end{aligned}$$

Finally,

$$\begin{aligned}\mathbb{E}[\|f^* - \hat{f}\|^2] \le (1 + 2K)&\mathbb{E}[\|f^* - \hat{f}_m\|^2] + 4K\sigma^2 \log(\Pi_m^{-1}) \\ &+ \mathbb{E}\left[2\langle \varepsilon, \hat{f} - f^* \rangle - \sigma^2 \text{pen}(\hat{m})\right].\end{aligned}$$

Lemma 2.3. *There exists a random variable Z such that*

$$2\langle \varepsilon, \hat{f} - f^* \rangle - \sigma^2 \textit{pen}(\hat{m}) \le a^{-1}\|\hat{f} - f^*\|^2 + Z,$$

with $\mathbb{E}[Z] \le c\sigma^2$ for some constants $a > 1$ and $c \ge 0$ depending on K.

Using the lemma, we have

$$\begin{aligned}\mathbb{E}\left[2\langle \varepsilon, \hat{f} - f^* \rangle - \sigma^2 \text{pen}(\hat{m})\right] &\le a^{-1}\mathbb{E}[\|\hat{f} - f^*\|^2] + \mathbb{E}[Z] \\ &\le a^{-1}\mathbb{E}[\|\hat{f} - f^*\|^2] + c\sigma^2.\end{aligned}$$

We obtain

$$(1 - a^{-1})\mathbb{E}[\|f^* - \hat{f}\|^2] \le (1 + 2K)\mathbb{E}[\|f^* - \hat{f}_m\|^2] + 4K\sigma^2 \log(\Pi_m^{-1}) + c\sigma^2.$$

The theorem is proved.

2.3 Appendix: Proof of Lemma 2.3

We have, with $\bar{S}_{\hat{m}} = \text{span}(S_{\hat{m}}, f^*)$ and with $P_{\bar{S}_{\hat{m}}}$ the projection on $\bar{S}_{\hat{m}}$

$$\begin{aligned} 2\langle \varepsilon, \hat{f} - f^* \rangle &= 2\langle P_{\bar{S}_{\hat{m}}}\varepsilon, \hat{f} - f^* \rangle \\ &\leq a\|P_{\bar{S}_{\hat{m}}}\varepsilon\|^2 + a^{-1}\|\hat{f} - f^*\|^2 \\ &\leq a\sigma^2(N^2 + U_{\hat{m}}) + a^{-1}\|\hat{f} - f^*\|^2, \end{aligned}$$

where $N^2 = \|P_{\text{span}(f^*)}\varepsilon\|^2/\sigma^2 \sim \chi^2(1)$ and for any $m \in \mathcal{M}$, we define U_m and $\tilde{S}_m$ such that

$$U_m = \frac{\|P_{\tilde{S}_m}\varepsilon\|^2}{\sigma^2}, \quad \tilde{S}_m \oplus \text{span}(f^*) = \bar{S}_m.$$

We have $U_m \sim \chi^2(d_m)$ if $f^* \notin S_m$ and $U_m \sim \chi^2(d_m - 1)$ if $f^* \in S_m$. So, we take

$$Z = a\sigma^2(N^2 + U_{\hat{m}}) - \sigma^2 \text{pen}(\hat{m}).$$

Note that $\hat{m}$ depends on the data so $U_{\hat{m}}$ is not a χ^2-variable. We set

$$a = \frac{K+1}{2} > 1.$$

To prove the lemma, we just have to prove that $\mathbb{E}[aU_{\hat{m}} - \text{pen}(\hat{m})]$ is bounded. Then,

$$\begin{aligned} \mathbb{E}[aU_{\hat{m}} - \text{pen}(\hat{m})] &\leq \frac{K+1}{2}\mathbb{E}\left[\max_{m\in\mathcal{M}}\left(U_m - \frac{2}{K+1}\text{pen}(m)\right)\right] \\ &\leq \frac{K+1}{2}\sum_{m\in\mathcal{M}}\mathbb{E}\left[\left(U_m - \frac{2}{K+1}\text{pen}(m)\right)_+\right] \\ &\leq \frac{K+1}{2}\sum_{m\in\mathcal{M}}\mathbb{E}\left[\left(U_m - \frac{2K}{K+1}\left(\sqrt{d_m} + \sqrt{2\log(\Pi_m^{-1})}\right)^2\right)_+\right] \end{aligned}$$

The following lemma is used, the proof of which is accepted.

Lemma 2.4. *Assume that $F : \mathbb{R}^d \longmapsto \mathbb{R}$ is 1-Lipschitz and Z has a Gaussian $\mathcal{N}(0, \sigma^2 I_d)$-distribution. Then, there exists a variable $\zeta \sim \exp(1)$ such that*

$$F(Z) \leq \mathbb{E}[F(Z)] + \sigma\sqrt{2\zeta}.$$

Now, observe that $\varepsilon \longmapsto \|P_{\tilde{S}_m}\varepsilon\|$ is 1-Lipschitz. Therefore, there exists a variable $\zeta_m \sim \exp(1)$ such that

$$\|P_{\tilde{S}_m}\varepsilon\| \leq \mathbb{E}[\|P_{\tilde{S}_m}\varepsilon\|] + \sigma\sqrt{2\zeta_m}.$$

It implies

$$\begin{aligned} U_m &= \frac{\|P_{\tilde{S}_m}\varepsilon\|^2}{\sigma^2} \\ &\leq \left(\mathbb{E}[\|P_{\tilde{S}_m}\sigma^{-1}\varepsilon\|] + \sqrt{2\zeta_m}\right)^2 \\ &\leq \left((\mathbb{E}[\|P_{\tilde{S}_m}\sigma^{-1}\varepsilon\|^2])^{1/2} + \sqrt{2\zeta_m}\right)^2. \end{aligned}$$

Since $\frac{\|P_{\tilde{S}_m}\varepsilon\|^2}{\sigma^2} \sim \chi^2(d_m)$ or $\frac{\|P_{\tilde{S}_m}\varepsilon\|^2}{\sigma^2} \sim \chi^2(d_m - 1)$, we have

$$\begin{aligned} U_m &\leq \left(\sqrt{d_m} + \sqrt{2\zeta_m}\right)^2 \\ &\leq \left(\sqrt{d_m} + \sqrt{2\log(\Pi_m^{-1})} + \sqrt{2(\zeta_m - \log(\Pi_m^{-1}))_+}\right)^2 \\ &\leq (1+\alpha)\left(\sqrt{d_m} + \sqrt{2\log(\Pi_m^{-1})}\right)^2 + (1+\alpha^{-1}) \times 2(\zeta_m - \log(\Pi_m^{-1}))_+, \end{aligned}$$

with $\alpha = \frac{K-1}{K+1}$. Then $1+\alpha = \frac{2K}{K+1}$ and $2(1+\alpha^{-1}) = \frac{4K}{K-1}$. Finally,

$$\mathbb{E}\left[\left(U_m - \frac{2K}{K+1}\left(\sqrt{d_m} + \sqrt{2\log(\Pi_m^{-1})}\right)^2\right)_+\right] \leq \frac{4K}{K-1}\mathbb{E}[(\zeta_m - \log(\Pi_m^{-1}))_+] = \frac{4K}{K-1}\Pi_m$$

and

$$\mathbb{E}[aU_{\hat{m}} - \mathrm{pen}(\hat{m})] \leq \frac{K+1}{2}\sum_{m\in\mathcal{M}} \frac{4K}{K-1}\Pi_m = \frac{2K(K+1)}{K-1}.$$

Chapter 3

From Bridge estimates to Lasso estimates

The methodological material of this chapter can be found on
`https://www.ceremade.dauphine.fr/~rivoirar/Cours-HD2023.pdf`

3.1 Characterization of the Lasso estimate

We still consider the regression model

$$Y = X\beta^* + \varepsilon,$$

where

- X is a known $n \times p$-matrix
- $\varepsilon \in \mathbb{R}^n$ such that its components ε_i are centered and i.i.d.
- $\beta^* \in \mathbb{R}^p$ is unknown.

The Lasso estimate of β^*, proposed by Tibshirani (1996), is the bridge estimate with $\gamma = 1$:

$$\hat{\beta}_\lambda^{lasso} := \underset{\beta \in \mathbb{R}^p}{\arg\min} \left\{ \|Y - X\beta\|^2 + \lambda \|\beta\|_1 \right\}$$

This estimate has no closed form in full generality but we can characterize it by using the following result.

Theorem 3.1 (Characterization of the Lasso). *A vector $\hat{\beta} \in \mathbb{R}^p$ is a global minimizer of the criterion $C_{\lambda,1}$ defined for $\beta \in \mathbb{R}^p$, by*

$$C_{\lambda,1}(\beta) := \|Y - X\beta\|^2 + \lambda \|\beta\|_1$$

if and only if $\hat{\beta}$ satisfies following conditions: For any $j \in \{1, \dots, p\}$,

- *if* $\hat{\beta}_j \neq 0$, $2X_j^T(Y - X\hat{\beta}) = \lambda sign(\hat{\beta}_j)$
- *if* $\hat{\beta}_j = 0$, $|2X_j^T(Y - X\hat{\beta})| \leq \lambda$

Furthermore, $\hat{\beta}$ *is the unique minimizer if* $X_{\mathcal{E}}$ *is one to one with*

$$\mathcal{E} := \left\{ j : \ |2X_j^T(Y - X\hat{\beta})| = \lambda \right\}$$

Proof. Let $f : \mathbb{R}^p \longmapsto \mathbb{R}$ a convex function. We define the subdifferential of f at $x \in \mathbb{R}^p$ by

$$\partial f(x) := \left\{ w \in \mathbb{R}^p : \quad f(y) \geq f(x) + w^T(y - x), \ \forall\, y \in \mathbb{R}^p \right\}.$$

A vector $w \in \partial f(x)$ is called a subgradient of f in x. We recall the following classical facts:

1. If f is differentiable at $x \in \mathbb{R}^p$, $\partial f(x) = \{\nabla f(x)\}$
2. If f and g are two convex functions on $\mathbb{R}^p$ with f differentiable on $\mathbb{R}^p$ then

$$\partial(f + g)(x) = \nabla f(x) + \partial g(x), \quad x \in \mathbb{R}^p$$

3. For a convex function f, $\hat{\beta}$ is a minimizer of f if and only if $0 \in \partial f(\hat{\beta})$.

The next lemma determines the subdifferential of the ℓ_1-norm.

Lemma 3.1. *We define* $f : \mathbb{R}^p \longmapsto \mathbb{R}$ *by* $f(x) = \|x\|_1 = \sum_{j=1}^p |x_j|$ *for any* $x \in \mathbb{R}^p$. *In this case, we have, for* $x \in \mathbb{R}^p$*:*

$$\partial f(x) := \left\{ w \in \mathbb{R}^p : \quad \|w\|_\infty \leq 1, \ w^T x = \|x\|_1 \right\}.$$

Remark 3.1. *Let* $x \in \mathbb{R}^p$. *If* w *is such* $\|w\|_\infty \leq 1$ *then* $w^T x = \|x\|_1$ *if and only if for* $j \in \{1, \ldots, p\}$ *such that* $x_j \neq 0$, *we have* $w_j = sign(x_j)$.

Proof of the lemma. Let $w \in \partial f(x)$. By taking successively $y = 0$ and $y = 2x$, we obtain

$$0 \geq \|x\|_1 + w^T(0 - x), \quad 2\|x\|_1 \geq \|x\|_1 + w^T(2x - x).$$

It yields $w^T x = \|x\|_1$. We take a vector $s \in \mathbb{R}^p$ such that $\|s\|_1 \leq 1$ and $\|w\|_\infty = s^T w$. We have:

$$\|x\|_1 + 1 \geq \|x\|_1 + \|s\|_1 \geq \|x + s\|_1 \geq \|x\|_1 + s^T w = \|x\|_1 + \|w\|_\infty.$$

We obtain $\|w\|_\infty \leq 1$.
Conversely, we take $w \in \mathbb{R}^p$ such that $\|w\|_\infty \leq 1$ and $w^T x = \|x\|_1$. For any $y \in \mathbb{R}^p$, we have:

$$\|x\|_1 + w^T(y - x) = w^T y \leq \|y\|_1$$

and $w \in \partial f(x)$. □

We now prove the theorem. The criterion $C_{\lambda,1}$ is convex. Therefore, a vector $\hat{\beta} \in \mathbb{R}^p$ is a global minimizer of the criterion $C_{\lambda,1}$ if and only if $0 \in \partial C_{\lambda,1}(\hat{\beta})$. Using previous facts, it means that there exists $w \in \mathbb{R}^p$ such that $\|w\|_\infty \leq 1$ and $w^T\hat{\beta} = \|\hat{\beta}\|_1$ such that

$$0 = -2X^T(Y - X\hat{\beta}) + \lambda w \iff 2X^T(Y - X\hat{\beta}) = \lambda w.$$

This gives the result. □

3.2 Theoretical properties of the Lasso estimate for linear regression

We still consider the regression model

$$Y = X\beta^* + \varepsilon,$$

where

- X is a known $n \times p$-matrix
- $\varepsilon \in \mathbb{R}^n$ such that its components ε_i are centered and i.i.d.
- $\beta^* \in \mathbb{R}^p$ is unknown

and we estimate β^* by using the Lasso estimate

$$\widehat{\beta} \in \arg\min_{\beta \in \mathbb{R}^p} \left\{ \|Y - X\beta\|^2 + \lambda\|\beta\|_1 \right\},$$

for $\lambda > 0$. We have the following result.

Theorem 3.2. *We assume that*

$$\|X^T\varepsilon\|_\infty := \max_{j \in \{1,\dots,p\}} |X_j^T\varepsilon| \leq \frac{\lambda}{4}.$$

Then,

$$\|X\widehat{\beta} - X\beta^*\|^2 \leq 2\lambda\|\beta^*\|_1.$$

If we further assume that all eigenvalues of the symmetric matrix X^TX are larger than a constant κ assumed to be positive, then, by denoting for any $\beta \in \mathbb{R}^p$,

$$S(\beta) := \{j : \beta_j \neq 0\}$$

and $|S(\beta)|$ the cardinal of $S(\beta)$, we have

$$\|X\widehat{\beta} - X\beta^*\|^2 \leq \min_{\beta \in \mathbb{R}^p} \left\{ 3\|X\beta - X\beta^*\|^2 + \frac{8\lambda^2}{\kappa}|S(\beta)| \right\} \tag{3.1}$$

and

$$\|\widehat{\beta} - \beta^*\|_1 \leq \frac{4\lambda}{\kappa}|S(\beta^*)|.$$

Proof. Let $\beta \in \mathbb{R}^p$. We have

$$\|Y - X\widehat{\beta}\|^2 + \lambda\|\widehat{\beta}\|_1 \leq \|Y - X\beta\|^2 + \lambda\|\beta\|_1,$$

which is equivalent to

$$\begin{aligned}\|X\widehat{\beta} - X\beta^*\|^2 &\leq \|X\beta - X\beta^*\|^2 + 2\varepsilon^T(X\widehat{\beta} - X\beta) + \lambda\sum_{j=1}^{p}\left(|\beta_j| - |\widehat{\beta}_j|\right)\\ &\leq \|X\beta - X\beta^*\|^2 + 2(X^T\varepsilon)^T(\widehat{\beta} - \beta) + \lambda\sum_{j=1}^{p}\left(|\beta_j| - |\widehat{\beta}_j|\right).\end{aligned}$$

Since $\|X^T\varepsilon\|_\infty \leq \frac{\lambda}{4}$,

$$\|X\widehat{\beta} - X\beta^*\|^2 \leq \|X\beta - X\beta^*\|^2 + \frac{\lambda}{2}\|\widehat{\beta} - \beta\|_1 + \lambda\sum_{j=1}^{p}\left(|\beta_j| - |\widehat{\beta}_j|\right)$$

and

$$\|X\widehat{\beta} - X\beta^*\|^2 + \frac{\lambda}{2}\|\widehat{\beta} - \beta\|_1 \leq \|X\beta - X\beta^*\|^2 + \lambda\|\widehat{\beta} - \beta\|_1 + \lambda\sum_{j=1}^{p}\left(|\beta_j| - |\widehat{\beta}_j|\right).$$

Now, we study the right hand side. We have

$$\begin{aligned}\lambda\|\widehat{\beta} - \beta\|_1 + \lambda\sum_{j=1}^{p}\left(|\beta_j| - |\widehat{\beta}_j|\right) &= \lambda\sum_{j\in S(\beta)}\left(|\beta_j - \widehat{\beta}_j| + |\beta_j| - |\widehat{\beta}_j|\right) + \lambda\sum_{j\notin S(\beta)}\left(|\widehat{\beta}_j| - |\widehat{\beta}_j|\right)\\ &\leq \lambda\sum_{j\in S(\beta)}\min\left(2|\beta_j|; 2|\beta_j - \widehat{\beta}_j|\right)\\ &\leq 2\lambda\min\left(\sum_{j\in S(\beta)}|\beta_j|; \sum_{j\in S(\beta)}|\widehat{\beta}_j - \beta_j|\right).\end{aligned}$$

Finally,

$$\|X\widehat{\beta} - X\beta^*\|^2 + \frac{\lambda}{2}\|\widehat{\beta} - \beta\|_1 \leq \|X\beta - X\beta^*\|^2 + 2\lambda\min\left(\sum_{j\in S(\beta)}|\beta_j|; \sum_{j\in S(\beta)}|\widehat{\beta}_j - \beta_j|\right).$$

By taking $\beta = \beta^*$, we obtain the first inequality:

$$\|X\widehat{\beta} - X\beta^*\|^2 \leq 2\lambda\|\beta^*\|_1.$$

Now, we further assume that all eigenvalues of the symmetric matrix X^TX are larger than a constant κ assumed to be positive. Then, by diagonalizing the matrix X^TX, we have

$$\|X\widehat{\beta} - X\beta\|^2 \geq \kappa\|\widehat{\beta} - \beta\|^2.$$

Then, for any $\alpha > 0$,

$$\begin{aligned} 2\lambda \sum_{j\in S(\beta)} |\widehat{\beta}_j - \beta_j| &\leq 2\lambda \times \sqrt{|S(\beta)|} \times \|\widehat{\beta} - \beta\| \\ &\leq 2 \times \kappa^{-1/2}\lambda\sqrt{|S(\beta)|} \times \|X\widehat{\beta} - X\beta\| \\ &\leq \alpha^{-1}\kappa^{-1}\lambda^2|S(\beta)| + \alpha\|X\widehat{\beta} - X\beta\|^2. \end{aligned}$$

This yields, for $0 < \alpha < 1/2$,

$$\begin{aligned} \|X\widehat{\beta} - X\beta^*\|^2 + \frac{\lambda}{2}\|\widehat{\beta} - \beta\|_1 &\leq \|X\beta - X\beta^*\|^2 + \alpha^{-1}\kappa^{-1}\lambda^2|S(\beta)| + \alpha\|X\widehat{\beta} - X\beta\|^2 \\ &\leq (1 + 2\alpha)\|X\beta - X\beta^*\|^2 + \alpha^{-1}\kappa^{-1}\lambda^2|S(\beta)| \\ &\qquad +2\alpha\|X\widehat{\beta} - X\beta^*\|^2. \end{aligned}$$

Therefore,

$$(1 - 2\alpha)\|X\widehat{\beta} - X\beta^*\|^2 + \frac{\lambda}{2}\|\widehat{\beta} - \beta\|_1 \leq (1 + 2\alpha)\|X\beta - X\beta^*\|^2 + \alpha^{-1}\kappa^{-1}\lambda^2|S(\beta)|.$$

Now, we take $\alpha = 1/4$ and

$$\|X\widehat{\beta} - X\beta^*\|^2 \leq 3\|X\beta - X\beta^*\|^2 + \frac{8\lambda^2}{\kappa}|S(\beta)|,$$

which yields the second inequality 3.1. We also obtain with $\alpha = 1/2$,

$$\frac{\lambda}{2}\|\widehat{\beta} - \beta\|_1 \leq 2\|X\beta - X\beta^*\|^2 + \frac{2\lambda^2}{\kappa}|S(\beta)|$$

and taking $\beta = \beta^*$ gives the third inequality. □

Remark 3.2. *Inequality* (3.1) *of Theorem 3.2 is an oracle inequality. The assumption on eigenvalues of X^TX are quite strong. These assumptions can be relaxed by considering Restricted Eigenvalues Conditions (see slides).*

Remark 3.3. *By taking $\beta = \beta^*$ of Inequality* (3.1) *of Theorem 3.2, we obtain, under the assumptions of the theorem,*

$$\|X\widehat{\beta} - X\beta^*\|^2 \leq \frac{8\lambda^2}{\kappa}|S(\beta^*)|.$$

Under assumptions on ε, we can prove oracle inequalities in the same spirit as Theorem 2.1.

Proposition 3.1. *Assume that for any $j \in \{1, \dots, p\}$, $\|X_j\| = 1$ and $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$. Then, for any $\beta > 0$, by taking*

$$\lambda = 4\sigma\sqrt{2\log p + 2\beta},$$

we have

$$\mathbb{P}\left(\|X^T\varepsilon\|_\infty \leq \frac{\lambda}{4}\right) \geq 1 - e^{-\beta}.$$

Proof. We have

$$\begin{aligned}\mathbb{P}\left(\|X^T\varepsilon\|_\infty > \frac{\lambda}{4}\right) &\leq \sum_{j=1}^{p} \mathbb{P}\left(|X_j^T\varepsilon| > \frac{\lambda}{4}\right) \\ &= \sum_{j=1}^{p} \mathbb{P}\left(|X_j^T\varepsilon| > \sigma\sqrt{2\log p + 2\beta}\right).\end{aligned}$$

But, for any j,

$$X_j^T\varepsilon \sim \mathcal{N}(0, X_j^T\sigma^2 I_n X_j) \sim \mathcal{N}(0, \sigma^2).$$

If $Z \sim \mathcal{N}(0,1)$, for any $t \geq 0$,

$$\begin{aligned}\mathbb{P}(|Z| > t) &= \frac{2}{\sqrt{2\pi}}\int_t^{+\infty} \exp(-x^2/2)dx \\ &= \frac{2}{\sqrt{2\pi}}\int_0^{+\infty} \exp(-(y+t)^2/2)dy \\ &\leq \exp(-t^2/2)\frac{2}{\sqrt{2\pi}}\int_0^{+\infty} \exp(-y^2/2)dy \\ &\leq \exp(-t^2/2).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}\left(\|X^T\varepsilon\|_\infty > \frac{\lambda}{4}\right) &\leq p\exp\left(-\frac{1}{2} \times (2\log p + 2\beta)\right) \\ &\leq \exp(-\beta).\end{aligned}$$

This yields the result. □

Using the result of the proposition, by taking $\lambda = c\sigma\sqrt{\log p}$ (i.e. β proportional to $\log p$) for c large enough, we have that inequalities of Theorem 3.2 hold with large probability. For instance, with $\beta = \gamma\log p$, taking $\lambda = 4\sigma\sqrt{2\log p + 2\gamma\log p} = 4\sigma\sqrt{2(1+\gamma)\log p}$, we obtain, with probability larger than $1 - p^{-\gamma}$,

$$\|X\widehat{\beta} - X\beta^*\|^2 \leq 8\sigma\sqrt{2(1+\gamma)\log p}\|\beta^*\|_1.$$

If all eigenvalues of $X^T X$ are larger than κ

$$\|X\widehat{\beta} - X\beta^*\|^2 \leq \min_{\beta \in \mathbb{R}^p} \left\{ 3\|X\beta - X\beta^*\|^2 + \frac{256\sigma^2(1+\gamma)\log p}{\kappa}|S(\beta)| \right\},$$

$$\|\widehat{\beta} - \beta^*\|_1 \leq \frac{16\sigma\sqrt{2(1+\gamma)\log p}}{\kappa}|S(\beta^*)|.$$

Chapter 4

Multiple testing

We explain some possible ways to handle the impact of high-dimensionality for testing. More precisely, we focus on the problem of performing simultaneously a large number of tests.

4.1 Introduction

Assume that we have n measurements for the expression of a gene g in two different conditions A and B (corresponding, for instance, to some normal cells and some cancerous cells). We want to know if there is a difference in the expression of this gene between these two conditions A and B. We can formulate the problem as follows. We denote:
- $X_1^A, \dots, X_n^A$: measurements under condition A,
- $X_1^B, \dots, X_n^B$: measurements under condition B.
We assume that the X_i^A's are i.i.d. and that the X_i^B's are i.i.d. with respective mean μ^A and μ^B. We want to test

$$H_0 : \mu^A = \mu^B \quad \text{vs} \quad H_1 : \mu^A \neq \mu^B.$$

The classical test statistic associated with this problem is the following. We set $Z_i = X_i^A - X_i^B$. We reject H_0 when $\hat{S} \geq s$, where

$$\hat{S} := \frac{|\bar{Z}|}{\sqrt{\frac{\hat{\sigma}^2}{n}}},$$

where

$$\bar{Z} := \frac{1}{n}\sum_{i=1}^{n} Z_i, \quad \hat{\sigma}^2 := \frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})^2$$

and s is a threshold. Give $\alpha > 0$, we choose $s := s_\alpha$ such that the probability to wrongly reject H_0 is not larger than α. It is common in the scientific literature to display the

p-palue of a test rather than the outcome of the test. In our case, the p-value of the test is simply the value $\hat{\alpha}$ such that $\hat{S} = s_{\hat{\alpha}}$. Since the function $\alpha \mapsto s_\alpha$ is decreasing, we have:
- If $\hat{\alpha} > \alpha$, then $s_\alpha > \hat{S}$ and H_0 is not rejected.
- If $\hat{\alpha} < \alpha$, then $s_\alpha < \hat{S}$ and H_0 is rejected.

DNA microarrays and NGS technologies allow us to measure the expression level of thousands of genes simultaneously. Our statistical objective is then to test simultaneously for all genes $g \in \{1, \ldots, p\}$:

H_{0g} : "*the mean expression levels of the gene g in conditions A and B are the same*"

versus

H_{1g} : "*the mean expression levels of the gene g in conditions A and B are different*".

Assume that we are given a test $T_g := 1_{\{\hat{S}_g \geq s_\alpha\}}$ for any g of size α. If we consider the p genes simultaneously, the number of hypotheses H_{0g} wrongly rejected (false positives) can be high. The mean number of false positives is:

$$\mathbb{E}[\text{number of False Positives}] = \sum_{g:\ H_{0g}\ \text{true}} \mathbb{P}(\hat{S}_g \geq s_\alpha) = \text{card}\{g : H_{0g} \text{ true}\} \times \alpha.$$

For instance, if $\text{card}\{g : H_{0g} \text{ true}\} = 10000$ and $\alpha = 5\%$, then

$$\mathbb{E}[\text{number of False Positives}] = 500.$$

To study 500 genes may be expensive. The biologists ask for powerful tests but with as few false positives as possible.

4.2 Statistical setting

We assume that we have m families of probability distribution $\{\mathbb{P}_\theta : \theta \in \Theta_i\}$ with $i \in \{1, \ldots, m\}$ and we consider simultaneously the m tests

$$H_{0i} : \theta \in \Theta_{0i} \quad \text{vs} \quad H_{1i} : \theta \in \Theta_{1i}$$

for $i \in \{1, \ldots, m\}$, where Θ_{0i} and Θ_{1i} are 2 disjoint subsets of Θ_i. We assume that for each $i \in \{1, \ldots, m\}$, we have a test of the form $1_{\{\hat{S}_i \geq s_i\}}$, where $\hat{S}_i$ is some observed statistic and s_i some threshold value. For $\theta \in \Theta_i$, we denote $T_\theta(s) = \mathbb{P}_\theta(\hat{S}_i \geq s)$. The p-value associated to the statistic $\hat{S}_i$ for the test i is

$$\hat{p}_i = \sup_{\theta \in \Theta_{0i}} T_\theta(\hat{S}_i).$$

The p-values are distributed as follows.

Proposition 4.1. *The p-values $\hat{p}_i$ fulfill the distributional property*

$$\sup_{\theta\in\Theta_{0i}} \mathbb{P}_\theta(\hat{p}_i \le u) \le u, \quad \forall u \in [0,1].$$

We say that the p-values are ***stochastically larger*** *than a uniform random variable.*

Proof. For any $\theta \in \Theta_{0i}$ and $u \in [0,1]$, we have

$$\begin{aligned}\mathbb{P}_\theta(\hat{p}_i \le u) &= \mathbb{P}_\theta\Big(\sup_{\theta'\in\Theta_{0i}} T_{\theta'}(\hat{S}_i) \le u\Big)\\ &\le \mathbb{P}_\theta(T_\theta(\hat{S}_i) \le u).\end{aligned}$$

For $u \in [0,1]$, we define

$$T_\theta^{-1}(u) := \inf\left\{s \in \mathbb{R} : \quad T_\theta(s) \le u\right\}.$$

Since T_θ is decreasing, we have:

$$(T_\theta^{-1}(u), +\infty) \subset \{s \in \mathbb{R} : \quad T_\theta(s) \le u\} \subset [T_\theta^{-1}(u), +\infty).$$

- If $T_\theta(T_\theta^{-1}(u)) \le u$, we have

$$\{s \in \mathbb{R} : \quad T_\theta(s) \le u\} = [T_\theta^{-1}(u), +\infty)$$

and

$$\mathbb{P}_\theta(T_\theta(\hat{S}_i) \le u) = \mathbb{P}_\theta(\hat{S}_i \ge T_\theta^{-1}(u)) = T_\theta(T_\theta^{-1}(u)) \le u.$$

- If $T_\theta(T_\theta^{-1}(u)) > u$, we have

$$(T_\theta^{-1}(u), +\infty) = \{s \in \mathbb{R} : \quad T_\theta(s) \le u\}$$

and therefore,

$$\begin{aligned}\mathbb{P}_\theta(T_\theta(\hat{S}_i) \le u) &= \mathbb{P}_\theta(\hat{S}_i > T_\theta^{-1}(u))\\ &= 1 - \mathbb{P}_\theta(\hat{S}_i \le T_\theta^{-1}(u)).\end{aligned}$$

Since $x \mapsto \mathbb{P}_\theta(\hat{S}_i \le x)$ is right-continuous,

$$\begin{aligned}\mathbb{P}_\theta(T_\theta(\hat{S}_i) \le u) &= 1 - \lim_{\varepsilon\searrow 0} \mathbb{P}_\theta(\hat{S}_i \le T_\theta^{-1}(u) + \varepsilon)\\ &= \lim_{\varepsilon\searrow 0} \mathbb{P}_\theta(\hat{S}_i > T_\theta^{-1}(u) + \varepsilon)\\ &\le \lim_{\varepsilon\searrow 0} T_\theta(T_\theta^{-1}(u) + \varepsilon)\\ &\le u.\end{aligned}$$

In both cases, $\mathbb{P}_\theta(\hat{p}_i \le u) \le u$. □

4.3 Multiple testing setting

A multiple testing procedure is a procedure that takes as imput the vector of p-values $(\hat{p}_1, \ldots, \hat{p}_m)$ corresponding to the m tests and returns a set of indices

$$\hat{R} \subset I = \{1, \ldots, m\}$$

which gives the set of the null hypotheses $\{H_{0i},\ i \in \hat{R}\}$ that are rejected. Writing I_0 for the set

$$I_0 := \{i \in I :\ H_{0i} \text{ is true}\},$$

we call:
- $\hat{R}$: indices of positives
- $\hat{R} \cap I_0$: indices of false positives
- $\hat{R} \setminus I_0$: indices of true positives.

We denote:

$$FP = \text{card}(\hat{R} \cap I_0), \quad TP = \text{card}(\hat{R} \setminus I_0).$$

Ideally, we would like a procedure that selects $\hat{R}$ in such a way that
- FP is small
- TP is large.

The tradeoff between these two goals is sensitive.

Bonferroni correction: The Bonferroni correction provides a severe control of FP. It is designed to control $\mathbb{P}(FP > 0)$. Its rejection region is defined by

$$\hat{R}_{bonf} := \left\{i :\ \hat{p}_i \leq \frac{\alpha}{m}\right\}.$$

We set $m_0 = \text{card}(I_0)$. We have:

$$\begin{aligned}
\mathbb{P}(FP > 0) &= \mathbb{P}\left(\exists\, i \in I_0 :\ \hat{p}_i \leq \frac{\alpha}{m}\right) \\
&\leq \sum_{i \in I_0} \sup_{\theta \in \Theta_{0i}} \mathbb{P}_\theta\left(\hat{p}_i \leq \frac{\alpha}{m}\right) \\
&\leq \sum_{i \in I_0} \frac{\alpha}{m} = \alpha \frac{m_0}{m} \leq \alpha.
\end{aligned}$$

The Bonferroni correction avoids false positives but produces only a few true positives in general (when m is large).

4.4 False Discovery Rate

The **False Discovery Proportion (FDP)** corresponds to

$$FDP = \frac{FP}{FP + TP},$$

with the convention $0/0 = 0$. The **False Discovery Rate (FDR)** is defined as the mean of the False Discovery Proportion:

$$FDR = \mathbb{E}\left[\frac{FP}{FP + TP} 1_{\{FP+TP \geq 1\}}\right].$$

Let $\beta : I := \{1, \dots, m\} \longmapsto \mathbb{R}_+$ and let us set

$$\hat{R} = \left\{ i \in I : \; \hat{p}_i \leq \frac{\alpha \beta(\hat{k})}{m} \right\}, \tag{4.1}$$

with

$$\hat{k} = \max \left\{ k \in I : \; \hat{p}_{(k)} \leq \frac{\alpha \beta(k)}{m} \right\},$$

and

$$\hat{p}_{(1)} \leq p_{(2)} \leq \cdots \leq p_{(m)}$$

are the p-values ranked in non-decreasing order. When $\left\{ k \in I : \; \hat{p}_{(k)} \leq \frac{\alpha\beta(k)}{m} \right\} = \emptyset$, we set $\hat{R} = \emptyset$.

Theorem 4.1. *Let $\beta : I := \{1, \dots, m\} \longmapsto \mathbb{R}_+$ be a non-decreasing function and for $\alpha > 0$, we define $\hat{R}$ as in (4.1). Then,*

$$FDR(\hat{R}) \leq \frac{\alpha m_0}{m} \sum_{j=1}^{+\infty} \frac{\beta(j \wedge m)}{j(j+1)}.$$

Proof. We first prove that $\text{card}(\hat{R}) = \hat{k}$.
- If $k > \hat{k}$, then $\hat{p}_{(k)} > \frac{\alpha\beta(k)}{m}$, by definition of $\hat{k}$. And since β is non-decreasing, $\hat{p}_{(k)} > \frac{\alpha\beta(\hat{k})}{m}$.
- If $k \leq \hat{k}$, then, by definition of $\hat{k}$,

$$\hat{p}_{(k)} \leq \hat{p}_{(\hat{k})} \leq \frac{\alpha\beta(\hat{k})}{m}.$$

We have exactly $\hat{k}$ indices i such that

$$\hat{p}_i \leq \frac{\alpha\beta(\hat{k})}{m}$$

and $\text{card}(\hat{R}) = \hat{k}$. Now,

$$\begin{aligned} FDR(\hat{R}) &= \mathbb{E}\left[\frac{\text{card}\left\{i \in I_0 : \ \hat{p}_i \leq \frac{\alpha\beta(\hat{k})}{m}\right\}}{\hat{k}} 1_{\{\hat{k}\geq 1\}}\right] \\ &= \sum_{i\in I_0} \mathbb{E}\left[1_{\left\{\hat{p}_i \leq \frac{\alpha\beta(\hat{k})}{m}\right\}} \frac{1}{\hat{k}} 1_{\{\hat{k}\geq 1\}}\right]. \end{aligned}$$

For $\hat{k} \geq 1$, we have

$$\frac{1}{\hat{k}} = \sum_{j=1}^{+\infty} \frac{1_{\{j\geq \hat{k}\}}}{j(j+1)}.$$

So,

$$\begin{aligned} FDR(\hat{R}) &= \sum_{i\in I_0} \mathbb{E}\left[1_{\left\{\hat{p}_i \leq \frac{\alpha\beta(\hat{k})}{m}\right\}} \sum_{j=1}^{+\infty} \frac{1_{\{j\geq \hat{k}\}}}{j(j+1)} 1_{\{\hat{k}\geq 1\}}\right] \\ &= \sum_{j=1}^{+\infty} \frac{1}{j(j+1)} \sum_{i\in I_0} \mathbb{E}\left[1_{\left\{\hat{p}_i \leq \frac{\alpha\beta(\hat{k})}{m}\right\}} 1_{\{j\geq \hat{k}\}} 1_{\{\hat{k}\geq 1\}}\right] \\ &\leq \sum_{j=1}^{+\infty} \frac{1}{j(j+1)} \sum_{i\in I_0} \mathbb{P}\left(\hat{p}_i \leq \frac{\alpha\beta(j\wedge m)}{m}\right) \\ &\leq \frac{\alpha m_0}{m} \sum_{j=1}^{+\infty} \frac{\beta(j\wedge m)}{j(j+1)}, \end{aligned}$$

where we have used Proposition 4.1 for the last inequality. □

Remark 4.1. *It can be proved that the upper bound of the theorem cannot be improved.*

Now, we choose β non-decreasing and such that

$$\sum_{j=1}^{+\infty} \frac{\beta(j\wedge m)}{j(j+1)} \leq 1.$$

In this case $FDR(\hat{R}) \leq \alpha$. We set

$$\beta(k) = \frac{k}{H_m}, \quad H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \overset{m\to+\infty}{\sim} \log(m).$$

We have

$$\begin{aligned}\sum_{j=1}^{+\infty}\frac{\beta(j\wedge m)}{j(j+1)} &= \sum_{j=1}^{m-1}\frac{\beta(j)}{j(j+1)}+\sum_{j=m}^{+\infty}\frac{\beta(m)}{j(j+1)}\\ &= \frac{1}{H_m}\sum_{j=1}^{m-1}\frac{1}{j+1}+\frac{m}{H_m}\sum_{j=m}^{+\infty}\left(\frac{1}{j}-\frac{1}{j+1}\right)\\ &= \frac{1}{H_m}\left(\sum_{j=1}^{m-1}\frac{1}{j+1}+1\right)\\ &= 1.\end{aligned}$$

It yields the Benjamini-Yekuteli procedure which is associated with the following rejection region:

$$\hat{R}=\left\{i\in I:\ \hat{p}_i\leq\frac{\alpha\hat{k}}{mH_m}\right\}.$$

4.5 Benjamini-Hochberg procedure

We would like to enlarge the rejection region provided by the Benjamini-Yekuteli procedure. Most of the time, it is not possible. However, in the case where we have some independence properties, we can remove the term H_m of $\hat{R}$.

Definition 4.1. *The distribution of the p-values $(\hat{p}_1,\dots,\hat{p}_m)$ is said to fulfill the Weak Positive Regression Dependence Property (WPRDP) if for any measurable bounded non-decreasing function $g:[0;1]^m\longmapsto\mathbb{R}_+$ and for all $i\in I_0$, the function*

$$u\longmapsto\mathbb{E}[g(\hat{p}_1,\dots,\hat{p}_m)\mid\hat{p}_i\leq u]$$

is non-decreasing on the interval $\mathcal{J}_i:=\{u\in[0;1]:\ \mathbb{P}(\hat{p}_i\leq u)>0\}$.

The set of distributions fulfilling the WPRDP includes the independent distributions.

Lemma 4.1. *Assume that the $(\hat{p}_i)_{i\in I_0}$'s are independent random variables and that the $(\hat{p}_i)_{i\in I\setminus I_0}$'s are independent from the $(\hat{p}_i)_{i\in I_0}$'s. The distribution of $(\hat{p}_1,\dots,\hat{p}_m)$ fulfills the WPRDP.*

Proof. We consider g a measurable non-negative bounded non-decreasing function. We consider $i\in I_0$. Without loss of generality, we take $i=1$. The variable $\hat{p}_1$ is independent of $(\hat{p}_2,\dots,\hat{p}_m)$. So, for $u\in[0;1]$ such that $\mathbb{P}(\hat{p}_1\leq u)>0$, we have:

$$\begin{aligned}\mathbb{E}[g(\hat{p}_1,\dots,\hat{p}_m)\mid\hat{p}_1\leq u]=\int_{(x_2,\dots,x_m)\in[0;1]^{m-1}}&\mathbb{E}[g(\hat{p}_1,x_2,\dots,x_m)\mid\hat{p}_1\leq u]\\ &\times\mathbb{P}(\hat{p}_2\in dx_2,\dots,\hat{p}_m\in dx_m).\end{aligned}$$

Since g is non-decreasing, then the function $g_1 : x_1 \longmapsto g(x_1, x_2, \ldots, x_m)$ is also non-decreasing. We denote for $t \in \mathbb{R}_+$,

$$g_1^{(-1)}(t) := \inf\left\{x \in [0;1] :\ g_1(x) \geq t\right\}.$$

Let $t \in \mathbb{R}_+$ and $x \in [0;1]$.
- If $g_1(x) \geq t$ then $g_1^{(-1)}(t) \leq x$.
- If $g_1^{(-1)}(t) \leq x$, then $g_1(g_1^{(-1)}(t)) \leq g_1(x)$ and if g_1 is continuous at $g_1^{(-1)}(t)$, then by considering a decreasing sequence $(u_n)_n$ belonging to $\{x \in [0;1] :\ g_1(x) \geq t\}$ and going to $g_1^{(-1)}(t)$, we have $g_1(g_1^{(-1)}(t)) \geq t$. Therefore $g_1(x) \geq t$.

We have proved that $g_1(x) \geq t \iff g_1^{(-1)}(t) \leq x$ as soon as g_1 is continuous at $g_1^{(-1)}(t)$. Since g_1 is non-decreasing, there exists at most a countable set of discontinuities for g_1. So there exists at most a countable set of reals numbers t such that g_1 is not continuous at $g_1^{(-1)}(t)$. The Lebesgue measure of this set of points, denoted $\mathcal{N}$, is null. Now, using that g_1 is non-negative, we can write

$$\begin{aligned}\mathbb{E}[g(\hat{p}_1, x_2 \ldots, x_m) \mid \hat{p}_1 \leq u] &= \mathbb{E}[g_1(\hat{p}_1) \mid \hat{p}_1 \leq u] \\ &= \int_0^{+\infty} \mathbb{P}(g_1(\hat{p}_1) \geq t \mid \hat{p}_1 \leq u)dt \\ &\int_{\mathbb{R}_+ \setminus \mathcal{N}} \mathbb{P}(\hat{p}_1 \geq g_1^{(-1)}(t) \mid \hat{p}_1 \leq u)dt\end{aligned}$$

To conclude, we simply notice that

$$u \longmapsto \mathbb{P}(\hat{p}_1 \geq g_1^{(-1)}(t) \mid \hat{p}_1 \leq u) = \max\left(0, 1 - \frac{\mathbb{P}(\hat{p}_1 < g_1^{(-1)}(t))}{\mathbb{P}(\hat{p}_1 \leq u)}\right)$$

is non-decreasing for all $t \in \mathbb{R}_+$. □

Under the Weak Positive Regression Dependence Property, we can enlarge the rejection region $\hat{R}$ and we obtain the Benjamini-Hochberg procedure.

Theorem 4.2. *We denote*

$$\mathcal{S} := \left\{k \in I :\ \hat{p}_{(k)} \leq \frac{\alpha k}{m}\right\}.$$

When the distribution of the p-values fulfills the WPRDP, the Benjamini-Hochberg procedure defined by $\hat{R} = \emptyset$ *if*

$$\mathcal{S} = \emptyset$$

and

$$\hat{R} := \left\{i \in I :\ \hat{p}_i \leq \frac{\alpha \hat{k}}{m}\right\}$$

with $\hat{k} := \max \mathcal{S}$ *has a FDR bounded by* α*.*

Proof. When $\hat{R} = \emptyset$, $\hat{k} = 0$. We now assume that $\hat{R} \neq \emptyset$. Using the same arguments as in the proof of Theorem 4.1, we prove that $\hat{k} = \mathrm{card}(\hat{R})$. Then,

$$\begin{aligned} FDR(\hat{R}) &= \mathbb{E}\left[\sum_{i\in I_0} \mathbb{I}_{\left\{\hat{p}_i \leq \frac{\alpha \hat{k}}{m}\right\}} \frac{1}{\hat{k}} 1_{\{\hat{k}\geq 1\}}\right] \\ &= \sum_{i\in I_0}\sum_{k=1}^{m} \frac{1}{k}\mathbb{P}\left(\hat{k} = k, \hat{p}_i \leq \frac{\alpha k}{m}\right) \\ &= \sum_{i\in I_0}\sum_{k=k_i^*}^{m} \frac{1}{k}\mathbb{P}\left(\hat{k} = k \;\middle|\; \hat{p}_i \leq \frac{\alpha k}{m}\right)\mathbb{P}\left(\hat{p}_i \leq \frac{\alpha k}{m}\right), \end{aligned}$$

where $k_i^* = \inf\left\{k \in \mathbb{N} :\ \mathbb{P}\left(\hat{p}_i \leq \frac{\alpha k}{m}\right) > 0\right\}$. Then,

$$\begin{aligned} FDR(\hat{R}) &\leq \sum_{i\in I_0}\sum_{k=k_i^*}^{m} \frac{1}{k}\mathbb{P}\left(\hat{k} = k \;\middle|\; \hat{p}_i \leq \frac{\alpha k}{m}\right)\frac{\alpha k}{m} \\ &\leq \frac{\alpha}{m}\sum_{i\in I_0}\sum_{k=k_i^*}^{m}\left[\mathbb{P}\left(\hat{k} \leq k \;\middle|\; \hat{p}_i \leq \frac{\alpha k}{m}\right) - \mathbb{P}\left(\hat{k} \leq k-1 \;\middle|\; \hat{p}_i \leq \frac{\alpha k}{m}\right)\right]. \end{aligned}$$

The function

$$g(\hat{p}_1, \ldots, \hat{p}_m) := 1_{\{\hat{k}\leq k\}} = 1_{\left\{\max\left\{j\in I:\ \hat{p}_{(j)} \leq \frac{\alpha j}{m}\right\}\leq k\right\}}$$

is non-decreasing with respect to $(\hat{p}_1, \ldots, \hat{p}_m)$. So, the WPRDP ensures that

$$\mathbb{P}\left(\hat{k} \leq k \;\middle|\; \hat{p}_i \leq \frac{\alpha k}{m}\right) \leq \mathbb{P}\left(\hat{k} \leq k \;\middle|\; \hat{p}_i \leq \frac{\alpha(k+1)}{m}\right)$$

and

$$\begin{aligned} FDR(\hat{R}) &\leq \frac{\alpha}{m}\sum_{i\in I_0}\left[\mathbb{P}\left(\hat{k} \leq m \;\middle|\; \hat{p}_i \leq \frac{\alpha(m+1)}{m}\right) - \mathbb{P}\left(\hat{k} \leq k_i^* - 1 \;\middle|\; \hat{p}_i \leq \frac{\alpha k_i^*}{m}\right)\right] \\ &\leq \frac{\alpha m_0}{m} \leq \alpha. \end{aligned}$$

□

Chapter 5

Wavelets and statistics

5.1 Continuous wavelet transform

We consider a function $\psi : \mathbb{R} \mapsto \mathbb{R}$ such that $\psi \in \mathbb{L}_1(\mathbb{R}) \cap \mathbb{L}_2(\mathbb{R})$,

$$\int_{\mathbb{R}} \psi(t)dt = 0 \quad \text{and} \quad \|\psi\|_2 = 1. \tag{5.1}$$

In the sequel, the function ψ will be called **mother wavelet** or just **wavelet** or **analyzing function**. We build a family of time-frequency atoms by translating ψ at positions u and by dilating it as scale s, for any $u \in \mathbb{R}$ and $s > 0$. We set

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right), \quad t \in \mathbb{R}.$$

We can easily check that

$$\int_{\mathbb{R}} \psi_{u,s}(t)dt = 0 \quad \text{and} \quad \|\psi_{u,s}\|_2 = 1.$$

We set:

Definition 5.1. *The* **continuous wavelet transform** *of a signal $f \in \mathbb{L}_2(\mathbb{R})$ at time $u \in \mathbb{R}$ and $s > 0$ is defined by*

$$\mathcal{W}[f](u,s) = \int_{\mathbb{R}} f(t)\psi_{u,s}(t)dt.$$

The function $\mathcal{W}[f] : \mathbb{R}^2 \mapsto \mathbb{R}$ is called the continuous wavelet transform of f.

In the sequel, we denote for any $f \in \mathbb{L}_2(\mathbb{R})$, $\widehat{f}$ the **Fourier transform** of f:

$$\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-it\xi} f(t)dt, \quad \xi \in \mathbb{R}.$$

Then, observe that (5.1) implies $\widehat{\psi}(0) = 0$. Next remarks show basic elements for time and frequency analysis of a signal f.

Remark 5.1. *If ψ is well-localized around 0, it will be also the case for $\psi_{u,s}$ around u and*

$$\mathcal{W}[f](u,s) \neq 0 \Rightarrow f \neq 0 \text{ in the neighborhood of } u.$$

Remark 5.2. *If we set for any $t \in \mathbb{R}$, $\tilde{\psi}(t) = \psi(-t)$ and $\tilde{\psi}_{u,s}(t) = \psi_{u,s}(-t)$, then*

$$\mathcal{W}[f](u,s) = [f \star \tilde{\psi}_{0,s}](u),$$

where $\star$ denotes the standard convolution product. Then

$$\widehat{\mathcal{W}[f]}(\xi, s) := \int_{\mathbb{R}} \mathcal{W}[f](u,s) e^{-iu\xi} du = \widehat{f}(\xi)\sqrt{s}\widehat{\psi}(-s\xi).$$

Therefore,

$$\widehat{\mathcal{W}[f]}(\xi, s) \neq 0 \Rightarrow \widehat{f}(\xi) \neq 0.$$

We can reconstruct a signal from the knowledge of its continuous wavelet transform.

Theorem 5.1. *We assume that ψ satisfies*

$$C_\psi := \int_0^{+\infty} \frac{|\widehat{\psi}(w)|^2}{w} dw < \infty.$$

Then, any function $f \in \mathbb{L}_2(\mathbb{R})$ satisfies

$$f(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{\mathbb{R}} \mathcal{W}[f](u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du \frac{ds}{s^2}.$$

Proof. We denote

$$b(t) := \frac{1}{C_\psi} \int_0^{+\infty} \int_{\mathbb{R}} \mathcal{W}[f](u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du \frac{ds}{s^2} = \frac{1}{C_\psi} \int_0^{+\infty} [f \star \tilde{\psi}_{0,s} \star \psi_{0,s}](t) \frac{ds}{s^2}$$

and we show that for any $\xi \in \mathbb{R}$,

$$\widehat{f}(\xi) = \widehat{b}(\xi).$$

□

Remark 5.3. *The condition $C_\psi < \infty$ implies $\int \psi(t)dt = \widehat{\psi}(0) = 0$. So, (5.1) is a necessary condition for Theorem 5.1.*

Example 5.1. *If for $t \in \mathbb{R}$, $\psi(t) = \frac{1}{\sqrt{2\pi}}(1-t^2)e^{-t^2/2}$, then*

$$\widehat{\psi}(0) = 0 \quad \text{and} \quad \widehat{\psi}(\xi) = \xi^2 e^{-\xi^2/2}.$$

5.2 Orthogonal wavelets

The continuous wavelet transform is redundant since we can show that we don't need all $(\mathcal{W}[f](u,s))_{u\in\mathbb{R},s>0}$ to reconstruct f. In this paragraph, we explain how to subsample this redundant transformation in an optimal way. More precisely, we wish to keep the minimal number of wavelet coefficients to ensure the reconstruction of the signal. The idea is to determine ψ such that the family of functions $(\psi_{k2^j,2^j})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an orthonormal basis of $\mathbb{L}_2(\mathbb{R})$.

Change of notation: From now on, for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$, we denote

$$\psi_{jk}(t) := \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-k2^j}{2^j}\right) = 2^{-j/2}\psi(2^{-j}t-k).$$

The goal is to determine conditions on ψ such that $(\psi_{jk})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an orthonormal basis of $\mathbb{L}_2(\mathbb{R})$, ie

- $\forall\ (j,k) \in \mathbb{Z}^2$, $\forall\ (j',k') \in \mathbb{Z}^2$,

$$\langle \psi_{jk}, \psi_{j',k'}\rangle = \begin{cases} 1 & \text{if} \quad (j,k) = (j',k') \\ 0 & \text{if} \quad (j,k) \neq (j',k') \end{cases}$$

- $\forall\ f \in \mathbb{L}_2(\mathbb{R})$,

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle f, \psi_{jk}\rangle\psi_{jk}.$$

We have denoted $\langle\cdot,\cdot\rangle$ the scalar product associated with the $\mathbb{L}_2$-norm. We shall need the notion of multiresolution analysis.

5.2.1 Multiresolution analysis

We start with the definition of a multiresolution analysis.

Definition 5.2. *A multiresolution analysis is a sequence of nested vector spaces*

$$\{0\} \subset \cdots \subset V_{j+1} \subset V_j \subset V_{j-1} \subset \cdots \subset \mathbb{L}_2(\mathbb{R})$$

such that, for any $j \in \mathbb{Z}$, if P_{V_j} is the orthogonal projection on V_j, for any $f \in \mathbb{L}_2(\mathbb{R})$,

1. $\|P_{V_j}f - f\|_2 \overset{j\to-\infty}{\longrightarrow} 0$

2. $\|P_{V_j}f\|_2 \overset{j\to+\infty}{\longrightarrow} 0$

3. $f \in V_j \iff x \mapsto f(x/2) \in V_{j+1}$ *for any* $j \in \mathbb{Z}$

4. *$f \in V_j \iff x \mapsto f(x + 2^j k) \in V_j$ for any $k \in \mathbb{Z}$*

5. *$\exists\, \phi$ such that $(\phi_k)_{k\in\mathbb{Z}}$ is an orthonormal basis of V_0 with for any $x \in \mathbb{R}$, $\phi_k(x) = \phi(x-k)$.*

The previous definition can be interpreted as follows. In our setting, the **resolution** means the quantity of information which can be used to represent a signal. In some sense, the resolution is the inverse of the **scale** which describes the sharpness of visible details of a function for a given resolution. So, the space V_{-j} represents the set of functions of maximal resolution equal to 2^j (equivalently at the scale 2^{-j}). The larger j, the sharper the details for representing a signal.
The dilation of a function by the factor 2 allows to increase details by a factor 2, which can be expressed by the fact that we can represent it in a space of weaker resolution.
Finally, in the previous definition, ϕ is the **scaling function** (or **father wavelet**), which is the main brick to build a multiresolution analysis.

Example 5.2. *The first example is provided by piecewise constant functions, where for any $j \in \mathbb{Z}$,*

$$V_j = \left\{ f \in \mathbb{L}_2(\mathbb{R}) : \quad f(t) = f(2^j k) \text{ for } t \in [2^j k, 2^j(k+1)) \right\}.$$

If we take $\phi = 1_{[0,1)}$, $V = (V_j)_{j\in\mathbb{Z}}$ is a multiresolution analysis.

Example 5.3. *The second example is provided by*

$$V_j = \left\{ f \in \mathbb{L}_2(\mathbb{R}) : \quad supp(\widehat{f}) \subset [-2^{-j}\pi, 2^{-j}\pi) \right\}.$$

If we take for $t \in \mathbb{R}$, $\phi(t) = \sin(\pi t)/(\pi t)$, $V = (V_j)_{j\in\mathbb{Z}}$ is a multiresolution analysis. See Exam 2020-2021.

5.2.2 Study of the scaling function

Definition 5.3. *Let $\phi \in \mathbb{L}_1(\mathbb{R})$ a scaling function associated with a multiresolution analysis $V = (V_j)_{j\in\mathbb{Z}}$. We set $\forall\, j \in \mathbb{Z}$ and $\forall\, k \in \mathbb{Z}$,*

$$\phi_{jk}(t) = \frac{1}{\sqrt{2^j}}\phi\left(\frac{t - k2^j}{2^j}\right) = 2^{-j/2}\phi(2^{-j}t - k), \quad t \in \mathbb{R}.$$

Remark 5.4. *For any $k \in \mathbb{Z}$, $\phi_k = \phi_{0k}$.*

We then prove the following proposition.

Proposition 5.1. *Let $j \in \mathbb{Z}$ be fixed. Then, $(\phi_{jk})_{k\in\mathbb{Z}}$ is an orthonormal basis of V_j. The orthogonal projection on V_j is*

$$P_{V_j} f = \sum_{k\in\mathbb{Z}} \langle f, \phi_{jk}\rangle \phi_{jk}.$$

Proof. We successively prove:

1. $\langle \phi_{jk}, \phi_{jk'} \rangle = 1_{\{k=k'\}}$, for any $(k,k') \in \mathbb{Z}^2$
2. For any $f \in V_j$, $\exists\, (a_k)_{k\in\mathbb{Z}}$ such that
$$f(x) = \sum_{k\in\mathbb{Z}} a_k \phi_{jk}(x), \quad x \in \mathbb{R}.$$

□

Remark 5.5. *Remember that* $P_{V_j} f = \arg\min_{g\in V_j} \|f-g\|_2$.

Using orthonormality of the ϕ_{jk}'s, we prove the following result.

Proposition 5.2. *If* $\phi \in \mathbb{L}_1(\mathbb{R})$ *is a scaling function then its Fourier transform satisfies*

- $|\widehat{\phi}(0)| = 1$
- $\sum_{k\in\mathbb{Z}} |\widehat{\phi}(w+2k\pi)|^2 = 1$, $w \in \mathbb{R}$

Proof. For f a function chosen later, we set
$$g_j(t) := \sqrt{2^j} f(2^j t), \quad t \in \mathbb{R}.$$
Then, with $\tilde{\phi}(t) = \phi(-t)$ for $t \in \mathbb{R}$,
$$\langle f, \phi_{jk} \rangle = [g_j \star \tilde{\phi}](k)$$
and for $\xi \in \mathbb{R}$,
$$\widehat{P_{V_j} f}(\xi) = \sqrt{2^j}\,\widehat{\phi}(\xi 2^j) \sum_{k\in\mathbb{Z}} [g_j \star \tilde{\phi}](k) e^{-i\xi k 2^j}.$$
We then use the Poisson formula: If $h \in \mathbb{L}_1(\mathbb{R})$ and $\widehat{h}$ is compactly supported,
$$\sum_{k\in\mathbb{Z}} h(k) e^{-ikw} = \sum_{k\in\mathbb{Z}} \widehat{h}(w+2k\pi).$$
This implies
$$\widehat{P_{V_j} f}(\xi) = \sqrt{2^j}\,\widehat{\phi}(\xi 2^j) \sum_{k\in\mathbb{Z}} \widehat{g}_j(\xi 2^j + 2k\pi) \widehat{\tilde{\phi}}(\xi 2^j + 2k\pi).$$
Now, we take $\widehat{f}(w) = 1_{[-\pi,\pi]}(w)$. So, if $j < 0$ and $\xi \in [-\pi, \pi]$,
$$\widehat{P_{V_j} f}(\xi) = |\widehat{\phi}(\xi 2^j)|^2.$$
Using $\|P_{V_j} f - f\|_2 \overset{j\to-\infty}{\longrightarrow} 0$, we obtain
$$\int_{-\pi}^{\pi} \left| 1 - |\widehat{\phi}(\xi 2^j)|^2 \right|^2 d\xi \overset{j\to-\infty}{\longrightarrow} 0$$
and finally $|\widehat{\phi}(0)| = 1$.

For the second point, we apply the Poisson formula to $h = \phi\star\tilde{\phi}$ and use the orthonormality property of the functions ϕ_k's. □

5.2.3 Conjugate mirror filter

We first express the decomposition of $x \mapsto \phi(x/2)$ on the ϕ_k's. Indeed, we have $x \mapsto \phi(x/2) \in V_1 \subset V_0$. So, for any $x \in \mathbb{R}$,

$$\phi(x/2) = \sum_{k\in\mathbb{Z}} \langle \phi(\cdot/2), \phi_k \rangle \phi_k(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k \phi(x-k),$$

by setting

$$h_k = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \phi\left(\frac{x}{2}\right) \phi(x-k)dx. \tag{5.2}$$

Definition 5.4. *The function $k \in \mathbb{Z} \mapsto h_k$ is called the* **conjugate mirror filter** *associated with the function ϕ. The* **transfer function** *associated with h is*

$$m_0 : w \longmapsto \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} h_k e^{-ikw}.$$

The following proposition provides the connections between ϕ and m_0.

Proposition 5.3. *We have*

1. $\widehat{\phi}(2w) = \widehat{\phi}(w) m_0(w)$, $w \in \mathbb{R}$
2. $m_0(0) = 1$
3. $|m_0(w)|^2 + |m_0(w+\pi)|^2 = 1$, $w \in \mathbb{R}$

Proof. The first point is obtained by computing the Fourier transform of

$$\phi\left(\frac{x}{2}\right) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k \phi(x-k).$$

The second point is an easy consequence of the first one (since $\widehat{\phi}(0) \neq 0$). The third point is proved by using

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi}(w/2+k\pi)|^2 |m_0(w/2+k\pi)|^2 = \sum_{k\in\mathbb{Z}} |\widehat{\phi}(w+2k\pi)|^2 = 1.$$

□

The previous introduced notions are very important for the building of wavelets and for algorithmic aspects associated with wavelets.

5.2.4 Wavelets

In this paragraph, we describe a construction of wavelets ψ starting from ϕ. We first define W_j as the orthogonal complement of V_j in V_{j-1}:

$$V_j \oplus W_j = V_{j-1}.$$

We have the following theorem:

Theorem 5.2. *Let h the conjugate mirror filter associated with a function ϕ. Let m_0 is the transfer function associated with h. We define the function g as*

$$\widehat{g}(w) = e^{-iw}\overline{m_0(w+\pi)}, \quad w \in \mathbb{R}.$$

We set

$$\widehat{\psi}(w) = \widehat{g}\left(\frac{w}{2}\right)\widehat{\phi}\left(\frac{w}{2}\right) \quad w \in \mathbb{R}.$$

We set for any $j \in \mathbb{Z}$ and any $k \in \mathbb{Z}$

$$\psi_{jk}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-k2^j}{2^j}\right) = 2^{-j/2}\psi(2^{-j}t-k), \quad t \in \mathbb{R}.$$

Then, for any $j \in \mathbb{Z}$, $(\psi_{jk})_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j. Furthermore, $(\psi_{jk})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an orthonormal basis of $\mathbb{L}_2(\mathbb{R})$.

Proof. The proof is quite technical and very long. See Exam 2018-2019. □

This result shows that the goal specified at the beginning of this section is achieved.

Remark 5.6. *Observe that*

$$\int \psi(t)dt = \widehat{\psi}(0) = \widehat{g}(0)\widehat{\phi}(0) = \overline{m_0(\pi)}\widehat{\phi}(0) = 0.$$

Remark 5.7. *The function $x \mapsto \psi(x/2) \in W_1 \subset V_0$, so there exists $(\lambda_k)_{k\in\mathbb{Z}}$ such that we can write for any $x \in \mathbb{R}$,*

$$\psi(x/2) = \sqrt{2}\sum_{k\in\mathbb{Z}} \lambda_k \phi(x-k).$$

It can be proved that for any $k \in \mathbb{Z}$,

$$\lambda_k = (-1)^{k+1}h_{1-k}.$$

See Exercise 1 of Exam 2022-2023. The sequence $(\lambda_k)_{k\in\mathbb{Z}}$ is the ***conjugate mirror filter associated with*** *ψ.*

5.2.5 Representation of a signal on a wavelet basis

We have built W_j such that $V_j = V_{j+1} \oplus W_{j+1}$. So, for any $j_0 > j$,

$$V_j = V_{j_0} \oplus_{j'=j+1}^{j_0} W_{j'}.$$

Therefore, for any $f \in \mathbb{L}_2(\mathbb{R})$,

$$\begin{aligned} P_{V_j} f &= P_{V_{j_0}} f + \sum_{j'=j+1}^{j_0} P_{W_{j'}} f \\ &= \sum_{k\in\mathbb{Z}} \langle \phi_{j_0 k}, f\rangle \phi_{j_0 k} + \sum_{j'=j+1}^{j_0} \sum_{k\in\mathbb{Z}} \langle \psi_{j'k}, f\rangle \psi_{j'k}. \end{aligned}$$

We denote

$$\alpha_{j_0 k} = \langle \phi_{j_0 k}, f\rangle \quad \text{and} \quad \beta_{j'k} = \langle \psi_{j'k}, f\rangle.$$

Then, with $j \to -\infty$, since $P_{V_j} f \to f$, we have:

$$f = \sum_{k\in\mathbb{Z}} \alpha_{j_0 k}\phi_{j_0 k} + \sum_{j'=-\infty}^{j_0} \sum_{k\in\mathbb{Z}} \beta_{j'k}\psi_{j'k}.$$

This representation is the wavelet decomposition of f. The spaces V_j are called **approximation spaces**. The spaces W_j are called **detail spaces**.

Daubechies' theory allows to take ϕ and ψ with some nice properties:

- regular (i.e. belonging to some Hölder spaces H^s, $s > 0$)
- compactly supported
- with vanishing moments: for some $N \in \mathbb{N}^*$,
  - $\int \phi(t)t^\ell dt = 0, \quad \ell = 1, \ldots, N$
  - $\int \psi(t)t^\ell dt = 0, \quad \ell = 0, \ldots, N$

The size of the support, s and N are connected.

Observe that once ϕ is fixed, the theory is complete. The building of ϕ relies on the relations satisfied for any $w \in \mathbb{R}$:

$$\widehat{\phi}(2w) = \widehat{\phi}(w)m_0(w), \quad m_0(w) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} h_k e^{-ikw}.$$

By iterating,

$$\widehat{\phi}(w) = \widehat{\phi}(0) \prod_{\ell=1}^{+\infty} m_0\left(\frac{w}{2^\ell}\right) = \prod_{\ell=1}^{+\infty} m_0\left(\frac{w}{2^\ell}\right).$$

So, once the conjugate mirror filter is fixed, the theory is complete. Figure 5.1 provides some very famous examples of father and mother wavelets with their associated filters.

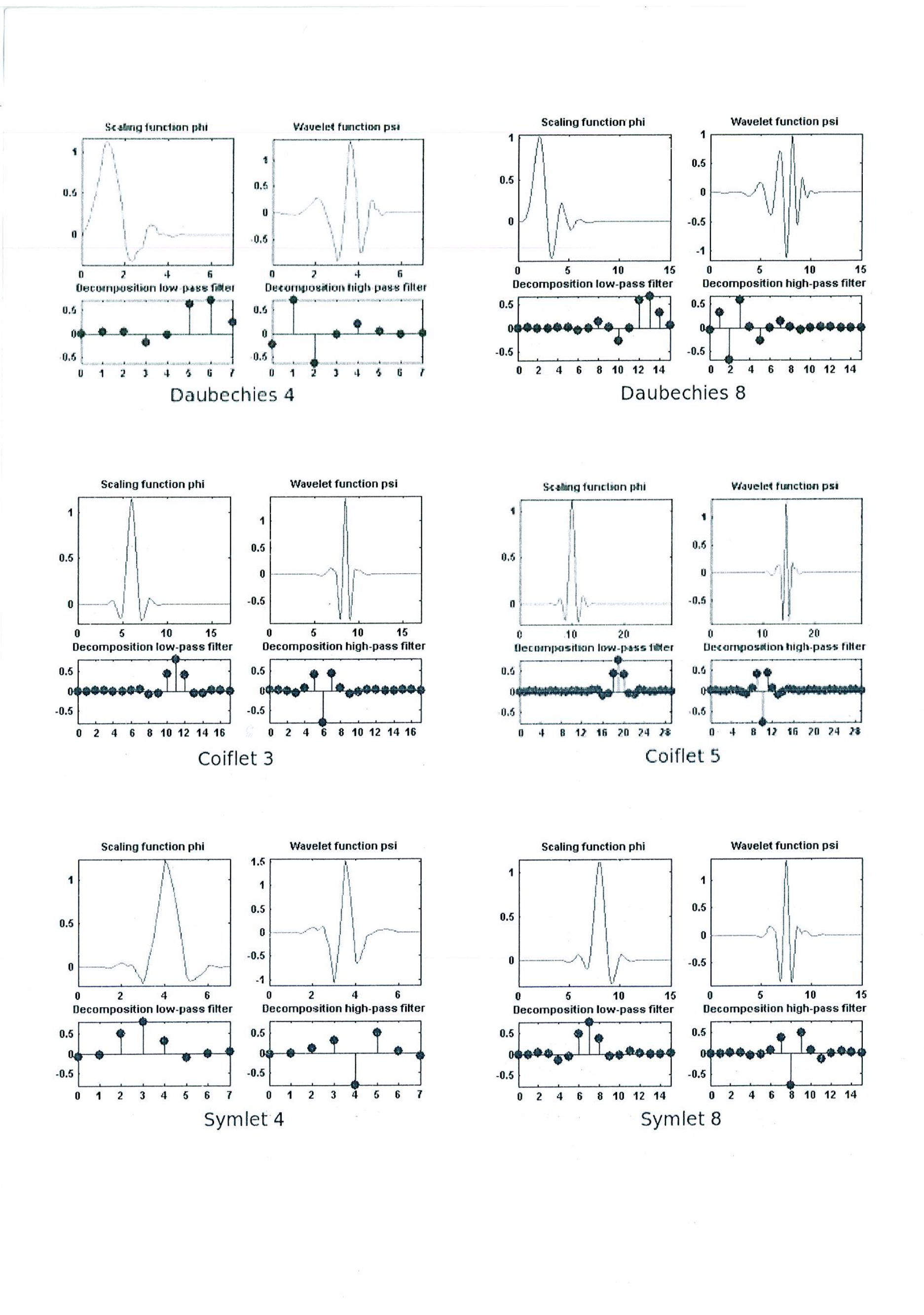

Figure 5.1: Examples of father and mother wavelets with their associated filters.

5.2.6 Fast wavelet transform

We now present an algorithm that allows to compute all wavelet coefficients once we have the approximation wavelets at a given level j_0, namely the $(\alpha_{j_0,k})_k$'s. We use the discrete convolution, denoted $\star_d$, between two sequence of real number $a = (a_k)_{k\in\mathbb{Z}}$ and $b = (b_k)_{k\in\mathbb{Z}}$:

$$[a \star_d b](k) := \sum_{\ell\in\mathbb{Z}} a_\ell b_{k-\ell}, \quad k \in \mathbb{Z}.$$

We have the following result.

Theorem 5.3. *We consider h the conjugate mirror filter associated with ϕ and we set for $k \in \mathbb{Z}$, $\tilde{h}_k = h_{-k}$. We then have for any $j \in \mathbb{Z}$ and any $k \in \mathbb{Z}$,*

$$\alpha_{j+1k} = [\alpha_{j\cdot} \star_d \tilde{h}](2k).$$

Similarly,

$$\beta_{j+1k} = [\alpha_{j\cdot} \star_d \tilde{\lambda}](2k),$$

where $\tilde{\lambda}_k = \lambda_{-k}$ and $(\lambda_k)_k$ is defined in Remark 5.7.

Proof. Let $j \in \mathbb{Z}$. Since for any $k \in \mathbb{Z}$, $\phi_{j+1k} \in V_{j+1} \subset V_j$, we can decompose ϕ_{j+1k} on the $\phi_{j\ell}$'s:

$$\phi_{j+1k} = \sum_{\ell\in\mathbb{Z}} \langle \phi_{j+1k}, \phi_{j\ell}\rangle \phi_{j\ell}$$

with

$$\begin{aligned}
\langle \phi_{j+1k}, \phi_{j\ell}\rangle &:= \int_{\mathbb{R}} \frac{1}{\sqrt{2^{j+1}}}\phi(2^{-(j+1)}t - k)\frac{1}{\sqrt{2^j}}\phi(2^{-j}t - \ell)dt \\
&= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \phi\left(\frac{t-2k}{2}\right)\phi(t-\ell)dt \\
&= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \phi\left(\frac{t}{2}\right)\phi(t-(\ell-2k))dt \\
&= h_{\ell-2k},
\end{aligned}$$

see (5.2). Therefore,

$$\phi_{j+1k} = \sum_{\ell\in\mathbb{Z}} h_{\ell-2k}\phi_{j\ell}$$

and

$$\alpha_{j+1k} = \langle f, \phi_{j+1k}\rangle = \sum_{\ell\in\mathbb{Z}} h_{\ell-2k}\langle f, \phi_{j\ell}\rangle = \sum_{\ell\in\mathbb{Z}} h_{\ell-2k}\alpha_{j\ell} = [\alpha_{j\cdot} \star_d \tilde{h}](2k).$$

The second point is proved by using similar arguments and by replacing ϕ_{j+1k} with ψ_{j+1k}.

□

We obtain a converse result.

Theorem 5.4. *We use the same notations as for Theorem 5.3. Then, we have for any $j \in \mathbb{Z}$ and any $k \in \mathbb{Z}$,*

$$\alpha_{jk} = \sum_{\ell \in \mathbb{Z}} h_{k-2\ell} \alpha_{j+1\ell} + \sum_{\ell \in \mathbb{Z}} \lambda_{k-2\ell} \beta_{j+1\ell}.$$

Proof. To prove the result, we decompose $\phi_{jk} \in V_j = V_{j+1} \oplus W_{j+1}$ and we obtain

$$\begin{aligned}\phi_{jk} &= \sum_{\ell \in \mathbb{Z}} \langle \phi_{jk}, \phi_{j+1\ell} \rangle \phi_{j+1\ell} + \sum_{\ell \in \mathbb{Z}} \langle \phi_{jk}, \psi_{j+1\ell} \rangle \psi_{j+1\ell} \\ &= \sum_{\ell \in \mathbb{Z}} h_{k-2\ell} \phi_{j+1\ell} + \sum_{\ell \in \mathbb{Z}} \lambda_{k-2\ell} \psi_{j+1\ell},\end{aligned}$$

see the proof of Theorem 5.3. Then, taking the scalar product with f, we obtain the result. □

Discrete convolutions can be computed very quickly. This is another reason for the popularity of wavelets.

5.3 Numerical illustrations of wavelets

Some illustrations of what wavelets can offer in practice can be found on
`https://www.ceremade.dauphine.fr/~rivoirar/Cours-Ondelettes.pdf`

5.4 Estimation of a signal decomposed on a wavelet basis

In the sequel, we modify the notation and replace j with $-j$.

5.4.1 Nonparametric regression model

We assume that we observe $y = (Y_1, \ldots, Y_n)^T$ such that

$$Y_i = f(t_i) + \varepsilon_i, \quad i = 1, \ldots, n, \tag{5.3}$$

where f is the signal to be estimated, the ε_i's are i.i.d. such that $\mathbb{E}[\varepsilon_1] = 0$, $\mathrm{var}(\varepsilon_1) = \sigma^2$, with $\sigma^2 > 0$ assumed to be known. We assume that f is compactly supported and, without loss of generality, we assume that $\mathrm{supp}(f) \subset [0, 1]$. In the previous **nonparametric**

regression model, we assume that $t_i = i/n$, meaning that observations are equispaced. We decompose f on a wavelet basis (we have set $j_0 = 0$):

$$f = \sum_{k\in\mathbb{Z}} \alpha_k\phi_k + \sum_{j=0}^{+\infty}\sum_{k\in\mathbb{Z}} \beta_{jk}\psi_{jk},$$

with

$$\alpha_k = \langle f, \phi_k\rangle, \quad \beta_{jk} = \langle f, \psi_{jk}\rangle.$$

With the change of notation, for any $x \in \mathbb{R}$,

$$\phi_k(x) = \phi(x-k), \quad \psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^j x - k).$$

Estimating f is equivalent to estimating the α_k's and the β_{jk}'s.

Remark 5.8. *If ϕ and ψ are compactly supported with support included into $[-A; A]$, with $A > 0$, then ϕ_k is supported by $[k-A; k+A]$ and ψ_{jk} is supported by $I_{jk} := [2^{-j}(k-A); 2^{-j}(k+A)]$. Therefore if $k < -A$ or $k > 2^j + A$, then $\beta_{jk} = 0$. We have no more than $2^j + 2A + 1$ non-zero wavelet coefficients to estimate.*

The regression model is "equivalent" to a white noise model where we observe

$$\begin{cases} X_{-1k} &= \alpha_k + \frac{\sigma}{\sqrt{n}} z_{-1k} \quad k \in \mathbb{Z} \\ X_{jk} &= \beta_{jk} + \frac{\sigma}{\sqrt{n}} z_{jk} \quad j \geq 0,\ k \in \mathbb{Z} \end{cases} \tag{5.4}$$

where the z_{jk}'s are i.i.d. $\mathcal{N}(0,1)$. Indeed, if $n = 2^{J+1}$ one may construct an n-by-n orthogonal matrix $\mathcal{W}$, the *discrete wavelet transform* matrix. This matrix yields a vector w of the discrete wavelet coefficients of y via

$$w = \mathcal{W}y$$

and because the matrix is orthogonal we have the inversion formula $y = \mathcal{W}^T w$. The vector w has $n = 2^{J+1}$ elements. It is convenient to index dyadically the rows of $\mathcal{W}$. By denoting $\mathcal{W}_{jk}(i)$ the element of row (j,k) and column i of $\mathcal{W}$, we can prove the following approximation:

$$\sqrt{n} \times \mathcal{W}_{jk}(i) \approx 2^{\frac{j}{2}}\psi(2^j t_i - k), \quad t_i = \frac{i}{n}.$$

Since $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_n)^T \sim \mathcal{N}(0, \sigma^2 I_n)$ and $\mathcal{W}$ is orthogonal,

$$z := \sigma^{-1}\mathcal{W}\varepsilon \sim \mathcal{N}(0, I_n).$$

We finally use

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n f(t_i)W_{jk}(i) \approx \frac{1}{n}\sum_{i=1}^n f(t_i)2^{\frac{j}{2}}\psi(2^j t_i - k) \approx \int f(t)2^{\frac{j}{2}}\psi(2^j t - k)dt = \langle f, \psi_{jk}\rangle.$$

Setting

$$X_{jk} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i W_{jk}(i),$$

using (5.3), we obtain

$$X_{jk} \approx \langle f, \psi_{jk} \rangle + \frac{\sigma}{\sqrt{n}} z_{jk},$$

which approximately justifies (5.4).

5.4.2 Thresholding

We denote

$$\mathcal{K}_j := \{k : \quad I_{jk} \cap [0,1] \neq \emptyset\}.$$

We only have to deal with the coefficients $(\alpha_k)_{k \in \mathcal{K}_0}$ and $(\beta_{jk})_{j \geq 0, k \in \mathcal{K}_j}$. We work with the model

$$X_{jk} = \beta_{jk} + \frac{\sigma}{\sqrt{n}} z_{jk}, \quad j \geq -1, \quad k \in \mathcal{K}_j,$$

with $\beta_{-1k} = \alpha_k$ and $\mathcal{K}_{-1} = \mathcal{K}_0$. For practical reasons, we only estimate a finite set of wavelet coefficients. This set will have the form

$$\Gamma = \{(j,k) : \quad -1 \leq j \leq J,\ k \in \mathcal{K}_j\}$$

with J to be chosen. Considering that most of signals are sparse (ie. most of wavelet coefficients are zero or negligible), the procedure is the following:

- If $|X_{jk}|$ is small (namely $|X_{jk}|$ smaller than a threshold), we estimate β_{jk} by 0.
- If $|X_{jk}|$ is large (namely $|X_{jk}|$ larger than the threshold), we estimate β_{jk} by X_{jk}.

From the mathematical point of view, we use the following procedure. We consider η_{jk} a threshold (to be chosen later) and we set for any j and any k,

$$\hat{\beta}_{jk} = X_{jk} 1_{\{|X_{jk}| > \eta_{jk}\}}.$$

The estimate of f is then

$$\hat{f} = \sum_{j=-1}^{J} \sum_{k \in \mathcal{K}_j} \hat{\beta}_{jk} \psi_{jk}.$$

We set $\eta_{-1k} = 0$. So, it remains to choose J and the η_{jk}'s for $0 \leq j \leq J$ and $k \in \mathcal{K}_j$. To study the theoretical performance of $\hat{f}$, we shall use the oracle approach. We set

$$\tilde{\beta}^0_{jk} = c_{jk} X_{jk}, \quad c_{jk} \in \{0,1\}$$

and c_{jk} non-random. The oracle approach will give us the ideal value for c_{jk}. It may depend on the signal. The ideal value for c_{jk} will be the value that minimizes the ℓ_2-risk of $\tilde{\beta}^0_{jk}$. The latter can be computed and we obtain

$$\mathbb{E}\left[(\tilde{\beta}^0_{jk} - \beta_{jk})^2\right] = c^2_{jk}\frac{\sigma^2}{n} + (1 - c_{jk})^2\beta^2_{jk}.$$

The ideal value for $c_{jk} \in \{0, 1\}$ is then

$$c_{jk} = 1_{\left\{\beta^2_{jk} \geq \frac{\sigma^2}{n}\right\}}.$$

Indeed, with this value, the risk of $\tilde{\beta}^0_{jk}$, **called the oracle risk of** β_{jk}, is minimum and equal to $\min\left(\frac{\sigma^2}{n}, \beta^2_{jk}\right)$. The **"oracle estimator"** is then

$$\tilde{f}^o = \sum_{k\in\mathcal{K}_{-1}} X_{-1k}\phi_k + \sum_{j=0}^{J}\sum_{k\in\mathcal{K}_j} \tilde{\beta}^o_{jk}\psi_{jk}$$

and the oracle risk is

$$\mathbb{E}\left[\|\tilde{f}^o - f\|^2\right] = \frac{\sigma^2}{n}\mathrm{card}(\mathcal{K}_{-1}) + \sum_{j=0}^{J}\sum_{k\in\mathcal{K}_j}\min\left(\frac{\sigma^2}{n}, \beta^2_{jk}\right) + \sum_{j>J}\sum_{k\in\mathcal{K}_j}\beta^2_{jk}.$$

The goal is to find η_{jk} such that the true estimate $\hat{f}$ has (almost) the same risk. This is given by the next theorem.

Theorem 5.5. *We choose*

$$\eta_{jk} = \sigma\sqrt{\frac{2\gamma\log n}{n}},$$

with γ *a constant larger than* 1 *and such that*

$$card(\Gamma) \leq n^{\frac{\gamma}{8}}.$$

In this case,

$$\mathbb{E}\left[\|\hat{f} - f\|^2\right] \leq \frac{\sigma^2}{n}card(\mathcal{K}_{-1}) + C_1\sum_{j=-1}^{J}\sum_{k\in\mathcal{K}_j}\min\left(\frac{\sigma^2\log n}{n}, \beta^2_{jk}\right) + \sum_{j>J}\sum_{k\in\mathcal{K}_j}\beta^2_{jk} + \frac{C_2}{n},$$

where C_1 *and* C_2 *are two constants.*

Proof. We start with the following lemma.

Lemma 5.1. *If $Z \sim \mathcal{N}(0,1)$, then for any $x > 0$,*

$$\mathbb{P}(|Z| > x) \leq \frac{2}{x\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

Proof. We just use:

$$\begin{aligned}\mathbb{P}(|Z| > x) &= \frac{2}{\sqrt{2\pi}} \int_x^{+\infty} \exp\left(-\frac{t^2}{2}\right) dt \\ &\leq \frac{2}{\sqrt{2\pi}} \int_x^{+\infty} \frac{t}{x} \exp\left(-\frac{t^2}{2}\right) dt = \frac{2}{x\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).\end{aligned}$$

□

We can now prove the theorem:

$$\begin{aligned}\mathbb{E}\left[\|\hat{f} - f\|^2\right] &= \sum_{k \in \mathcal{K}_{-1}} \mathbb{E}\left[(X_{-1k} - \beta_{-1k})^2\right] + \sum_{j=0}^{J} \sum_{k \in \mathcal{K}_j} \mathbb{E}\left[(\hat{\beta}_{jk} - \beta_{jk})^2\right] + \sum_{j>J} \sum_{k \in \mathcal{K}_j} \beta_{jk}^2 \\ &= \frac{\sigma^2}{n} \mathrm{card}(\mathcal{K}_{-1}) + \sum_{j=0}^{J} \sum_{k \in \mathcal{K}_j} \mathbb{E}\left[(\hat{\beta}_{jk} - \beta_{jk})^2\right] + \sum_{j>J} \sum_{k \in \mathcal{K}_j} \beta_{jk}^2.\end{aligned}$$

So, it remains to study $\mathbb{E}\left[(\hat{\beta}_{jk} - \beta_{jk})^2\right]$. We can prove that

$$\mathbb{E}\left[(\hat{\beta}_{jk} - \beta_{jk})^2\right] \leq \min\left(\frac{\sigma^2 \log n}{n}, \beta_{jk}^2\right).$$

See Exam 2019-2020. □

Appendix A

Past exams

A.1 Exam 2016-2017

High-dimensional statistics

Correction of the EXAM (duration 2h30)

Documents, calculators, phones and smartphones are forbidden

Problem 1: Selection bias of the Mallows criterion.

We consider the multivariate linear regression model :

$$Y = X\beta + \varepsilon$$

with $Y = (Y_1, \ldots, Y_n)^T$ the vector of observations. The matrix X of size $n \times p$ is assumed to be known. The rank of X is p (with $p < n$) and $\beta \in \mathbb{R}^p$ is the vector to be estimated. Finally, the error vector is $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$ and satisfies $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$ with $\sigma^2 > 0$ unknown. We denote $(X_1, \ldots, X_p)$ the columns of X. The classical ℓ_2-norm is denoted $\|\cdot\|_2$. We denote for all m, a subset of indexes of $\{1, \ldots, p\}$,

$$\text{RSS}(m) = \|Y - P_m Y\|_2^2,$$

where P_m is the projection on $\text{span}(X_j : j \in m)$. We shall denote $P_m Y = 0$ if the model m is empty. Let m be some model, we recall that the Mallows criterion associated to m is defined by:

$$C_p(m) = \frac{RSS(m)}{\hat{\sigma}^2} - n + 2|m|,$$

where $|m|$ is the cardinality of m and $\hat{\sigma}^2$ is the estimator of σ^2 studied in course. We suppose that the columns of the matrix X are orthogonal and of unit norm (consequently $X^T X = I_p$).

1. Recall the expression of $\hat{\beta}$, the ordinary least squares estimator of β, and give its distribution.
   Correction : *We have:*
   $$\hat{\beta} = (X^T X)^{-1} X^T Y.$$
   Since
   $$Y \sim \mathcal{N}(X\beta, \sigma^2 I_n),$$
   then
   $$\hat{\beta} \sim \mathcal{N}(\beta, \sigma^2 (X^T X)^{-1}) = \mathcal{N}(\beta, \sigma^2 I_p)$$

2. We denote P_X the projection matrix on the image of X.

a) Prove that for all model m,

$$RSS(m) = \|Y - P_X Y\|_2^2 + \|P_X Y - P_m Y\|_2^2.$$

Correction : *We have:*

$$Y - P_X Y \perp \mathcal{I}m(X), \quad P_X Y - P_m Y \in \mathcal{I}m(X).$$

So, $Y - P_X Y \perp P_X Y - P_m Y$ *and*

$$RSS(m) = \|Y-P_mY\|_2^2 = \|Y-P_XY+P_XY-P_mY\|_2^2 = \|Y-P_XY\|_2^2+\|P_XY-P_mY\|_2^2$$

b) After having given for all $j \in \{1, \dots, p\}$ the expression of $\hat{\beta}_j$ in function of X_j et Y, deduce that

$$RSS(m) = (n-p)\hat{\sigma}^2 + \sum_{j \notin m} \hat{\beta}_j^2.$$

Correction : *We recall that*

$$\hat{\sigma}^2 = \frac{\|Y - P_X Y\|_2^2}{n-p}$$

so

$$\|Y - P_X Y\|_2^2 = (n-p)\hat{\sigma}^2.$$

If we denote U *the matrix whose columns are the columns of* X *without columns corresponding to the indices of* m*,*

$$P_X Y - P_m Y = U(U^T U)^{-1} U^T Y.$$

But, $U^T U = I_{p-|m|}$ *and* $P_X Y - P_m Y = UU^T Y$ *and*

$$\begin{aligned}
\|P_X Y - P_m Y\|_2^2 &= (UU^T Y)^T UU^T Y \\
&= Y^T UU^T UU^T Y \\
&= Y^T UU^T Y \\
&= \|U^T Y\|_2^2 \\
&= \|((X_j)_{j \notin m})^T Y\|_2^2 = \sum_{j \notin m} \hat{\beta}_j^2
\end{aligned}$$

c) Conclude that we have the following expression

$$\hat{\sigma}^2 C_p(m) = \sum_{j=1}^{p} (\hat{\beta}_j^2 - \hat{\sigma}^2) - \sum_{j \in m} (\hat{\beta}_j^2 - 2\hat{\sigma}^2).$$

Correction : *We have:*

$$\begin{aligned}
\hat{\sigma}^2 C_p(m) &= RSS(m) - n\hat{\sigma}^2 + 2|m|\hat{\sigma}^2 \\
&= (n-p)\hat{\sigma}^2 + \sum_{j \notin m} \hat{\beta}_j^2 - n\hat{\sigma}^2 + 2|m|\hat{\sigma}^2 \\
&= \sum_{j=1}^{p} \hat{\beta}_j^2 - p\hat{\sigma}^2 - \sum_{j \in m} \hat{\beta}_j^2 + 2|m|\hat{\sigma}^2 \\
&= \sum_{j=1}^{p} (\hat{\beta}_j^2 - \hat{\sigma}^2) - \sum_{j \in m} (\hat{\beta}_j^2 - 2\hat{\sigma}^2)
\end{aligned}$$

3. Using the last expression above, what is the one-variable-model which minimizes the Mallows criterion ?
   Correction : *The Mallows criterion selects the model m which minimizes $m \mapsto C_p(m)$. Using the previous question, if we force $|m| = 1$, we're going to choose $m = \{j\}$, with $\hat{\beta}_j^2$ as large as possible.*

4. Let k be a fixed non null and non random integer. According to the Mallows criterion, what is the model with k variables which will be chosen?
   Correction : *Using the same arguments, the selected model m will be the model with the k indices j corresponding to the k largest $\hat{\beta}_j^2$.*

5. By now, we do not suppose anymore that the number of selected variables is fixed in advance and we denote $\hat{m}$ the selected model by the Mallows criterion. Prove that $j \in \hat{m}$ if and only if $\hat{\sigma}^{-1}|\hat{\beta}_j| > \sqrt{2}$.
   Correction : *Obvious*

6. Compute for all $j \in \{1, \ldots, p\}$ the expectation of $\hat{\beta}_j^2 - \hat{\sigma}^2$.
   Correction : $\mathbb{E}[\hat{\beta}_j^2 - \hat{\sigma}^2] = var(\hat{\beta}_j) + (\mathbb{E}[\hat{\beta}_j])^2 - \mathbb{E}[\hat{\sigma}^2] = \sigma^2 + \beta_j^2 - \sigma^2 = \beta_j^2$

7. By now, we suppose that for all $j \in \{1, \ldots, p\}$, $\beta_j = 0$.

   a) Determine $\mathbb{E}[|\hat{m}|]$ the expectation of the cardinality of $\hat{m}$ in function of p and the cumulative distribution function of the Student distribution with $n-p$ degrees of freedom.
   Correction : *We have:* $\hat{m} = \{j : \ \hat{\beta}_j^2 > 2\hat{\sigma}^2\}$.

$$\begin{aligned}
\mathbb{E}[|\hat{m}|] &= \sum_{j=1}^{p} \mathbb{E}[1_{\{j:\ \hat{\beta}_j^2 > 2\hat{\sigma}^2\}}] \\
&= \sum_{j=1}^{p} \mathbb{P}\left(|\hat{\beta}_j|/\hat{\sigma} > \sqrt{2}\right).
\end{aligned}$$

If we suppose that for all $j \in \{1, \ldots, p\}$, $\beta_j = 0$ then

$$\hat{\beta} \sim \mathcal{N}(0, \sigma^2 I_p)$$

and $\hat{\beta}_j/\hat{\sigma} \sim t(n-p)$. With $T_{n,p} \sim t(n-p)$ and $\alpha_{n,p}$ the value of the cumulative distribution function of $T_{n,p}$ at $\sqrt{2}$, then

$$\mathbb{E}[|\hat{m}|] = p \times \mathbb{P}(|T_{n,p}| > \sqrt{2}) = 2p(1 - \alpha_{n,p}).$$

b) By using the fact that on $]0, +\infty[$ the cumulative distribution function of the Student distribution with $n-p$ degrees of freedom is smaller than the cumulative distribution function of the standard normal distribution, what is the limit of $\mathbb{E}[|\hat{m}|]$ when p tends to $+\infty$ with $n > p$?
Correction : *If $Z \sim \mathcal{N}(0,1)$*

$$\begin{aligned}
\mathbb{E}[|\hat{m}|] &= 2p(1 - \alpha_{n,p}) \\
&\geq 2p(1 - \mathbb{P}(Z \leq \sqrt{2})) \\
&\geq 2p\mathbb{P}(Z > \sqrt{2})
\end{aligned}$$

and

$$\lim_{p \to +\infty} \mathbb{E}[|\hat{m}|] = +\infty.$$

c) What is your conclusion ?
Correction : *The size of the selected model is too large in expectation.*

8. We still denote $\hat{m}$ the selected model by the Mallows criterion. For any model m, we define $\tilde{C}_p(m)$ by

$$\tilde{C}_p(m) = \frac{RSS(m)}{\hat{\sigma}^2} - n + 3|m|.$$

If $\tilde{m}$ minimizes $m \mapsto \tilde{C}_p(m)$, prove that $|\tilde{m}| \leq |\hat{m}|$.
Correction : *We take m such that $|m| > |\hat{m}|$. Then,*

$$\begin{aligned}
\tilde{C}_p(\hat{m}) &= \frac{RSS(\hat{m})}{\hat{\sigma}^2} - n + 3|\hat{m}| \\
&= \frac{RSS(\hat{m})}{\hat{\sigma}^2} - n + 2|\hat{m}| + |\hat{m}| \\
&= C_p(\hat{m}) + |\hat{m}| \\
&\leq C_p(m) + |\hat{m}| \\
&< C_p(m) + |m| \\
&< \tilde{C}_p(m)
\end{aligned}$$

Problem 2: Support properties of the Lasso estimator

We consider the following model of nonparametric regression :

$$Y_i = f(x_i) + \varepsilon_i, \quad i = 1, \dots, n,$$

where $f : [0,1] \longrightarrow \mathbb{R}$ is a function to be estimated thanks to observations $(Y_i)_{i=1,\dots,n}$. Points $(x_i)_{i=1,\dots,n}$ are known and non random. Finally, variables $(\varepsilon_i)_{i=1,\dots,n}$ are independent with common distribution a standard normal distribution of variance σ^2 supposed to be known. To estimate f, we rely on functions of a given dictionary $(\phi_j)_{j=1,\dots,p}$ ($p \geq 2$) and we denote for all $\beta = (\beta_j)_{j=1,\dots,p} \in \mathbb{R}^p$,

$$f_\beta = \sum_{j=1}^{p} \beta_j \phi_j.$$

We define for every function g,

$$\|g\|_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n} g^2(x_i)}.$$

If for any functions g and g', we denote

$$\langle g, g' \rangle_n = \frac{1}{n}\sum_{i=1}^{n} g(x_i) g'(x_i)$$

we have

$$\|g + g'\|_n^2 = \|g\|_n^2 + \|g'\|_n^2 + 2\langle g, g' \rangle_n.$$

For all $j \in \{1, \dots, p\}$, one is given $r_{n,j} > 0$. The Lasso estimator of f is then $f_{\hat{\beta}}$ where $\hat{\beta}$ is a minimizer of the function crit where for all $u \in \mathbb{R}^p$

$$\text{crit}(u) = \frac{1}{n}\sum_{i=1}^{n} (Y_i - f_u(x_i))^2 + 2\sum_{j=1}^{p} r_{n,j} |u_j|.$$

We suppose that f can be developed on the dictionary. Hence there exists β^* such that $f = f_{\beta^*}$. One studies the properties of the Lasso estimator to estimate S^*, the support of β^* :

$$S^* = \left\{ j : \quad \beta_j^* \neq 0 \right\}.$$

One denotes for all $x \in \mathbb{R}$,

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

One admits that u is a minimizer of the function crit if and only if for all $j \in \{1, \dots, p\}$,

$$\begin{cases} \frac{1}{n}\sum_{i=1}^{n} Y_i \phi_j(x_i) - \sum_{k=1}^{p} u_k \langle \phi_j, \phi_k \rangle_n = r_{n,j}\text{sign}(u_j) & \text{if } u_j \neq 0 \\ \left| \frac{1}{n}\sum_{i=1}^{n} Y_i \phi_j(x_i) - \sum_{k=1}^{p} u_k \langle \phi_j, \phi_k \rangle_n \right| \leq r_{n,j} & \text{if } u_j = 0. \end{cases}$$

The goal is to give the conditions insuring that the support of β^* actually contains the support of $\hat{\beta}$.

1. Prove that for all $u \in \mathbb{R}^p$,

$$\mathrm{crit}(\hat{\beta}+u)-\mathrm{crit}(\hat{\beta}) = \|f_u\|_n^2+2\sum_{j=1}^p r_{n,j}\left(|\hat{\beta}_j+u_j|-|\hat{\beta}_j|\right)+\frac{2}{n}\sum_{i=1}^n\sum_{k=1}^p u_k\phi_k(x_i)\left(\sum_{j=1}^p \hat{\beta}_j\phi_j(x_i)-Y_i\right).$$

<u>Correction :</u>

$$\begin{aligned}
crit(\hat{\beta}+u)-crit(\hat{\beta}) &= \frac{1}{n}\sum_{i=1}^n\left(Y_i-f_{\hat{\beta}+u}(x_i)\right)^2-\frac{1}{n}\sum_{i=1}^n\left(Y_i-f_{\hat{\beta}}(x_i)\right)^2+2\sum_{j=1}^p r_{n,j}(|u_j+\hat{\beta}_j|-|\hat{\beta}_j|)\\
&= -\frac{2}{n}\sum_{i=1}^n\left(Y_i-f_{\hat{\beta}}(x_i)\right)f_u(x_i)+\frac{1}{n}\sum_{i=1}^n f_u^2(x_i)+2\sum_{j=1}^p r_{n,j}(|u_j+\hat{\beta}_j|-|\hat{\beta}_j|)\\
&= \|f_u\|_n^2+2\sum_{j=1}^p r_{n,j}(|u_j+\hat{\beta}_j|-|\hat{\beta}_j|)+\frac{2}{n}\sum_{i=1}^n\sum_{k=1}^p u_k\phi_k(x_i)\left(\sum_{j=1}^p \hat{\beta}_j\phi_j(x_i)-Y_i\right)
\end{aligned}$$

2. Deduce that

$$\mathrm{crit}(\hat{\beta}+u)-\mathrm{crit}(\hat{\beta}) = \|f_u\|_n^2+2\sum_{j=1}^p r_{n,j}\left(|\hat{\beta}_j+u_j|-|\hat{\beta}_j|-u_js_j\right),$$

with $|s_j|\leq 1$ and $s_j=\mathrm{sign}(\hat{\beta}_j)$ if $\hat{\beta}_j\neq 0$.
<u>Correction :</u>

$$\frac{1}{n}\sum_{i=1}^n\phi_k(x_i)\left(\sum_{j=1}^p\hat{\beta}_j\phi_j(x_i)-Y_i\right)=\sum_{j=1}^p\hat{\beta}_j\langle\phi_j,\phi_k\rangle_n-\frac{1}{n}\sum_{i=1}^n\phi_k(x_i)Y_i.$$

Since $\hat{\beta}$ is a minimizer of crit,

$$\left|\sum_{j=1}^p\hat{\beta}_j\langle\phi_j,\phi_k\rangle_n-\frac{1}{n}\sum_{i=1}^n\phi_k(x_i)Y_i\right|\leq r_{n,k}.$$

We set

$$s_k=r_{n,k}^{-1}\left(\frac{1}{n}\sum_{i=1}^n\phi_k(x_i)Y_i-\sum_{j=1}^p\hat{\beta}_j\langle\phi_j,\phi_k\rangle_n\right)$$

therefore $s_k=sign(\hat{\beta}_k)$ if $\hat{\beta}_k\neq 0$ and $|s_k|\leq 1$ if $\hat{\beta}_k=0$ and

$$crit(\hat{\beta}+u)-crit(\hat{\beta}) = \|f_u\|_n^2+2\sum_{j=1}^p r_{n,j}\left(|\hat{\beta}_j+u_j|-|\hat{\beta}_j|-u_js_j\right).$$

3. Then prove that
$$\text{crit}(\hat\beta + u) - \text{crit}(\hat\beta) \geq \|f_u\|_n^2. \tag{A.1}$$

<u>Correction :</u> *We have:*

- *If* $\hat\beta_j > 0$, $|\hat\beta_j + u_j| - |\hat\beta_j| - u_j s_j = |\hat\beta_j + u_j| - \hat\beta_j - u_j \geq 0$.
- *If* $\hat\beta_j < 0$, $|\hat\beta_j + u_j| - |\hat\beta_j| - u_j s_j = |\hat\beta_j + u_j| + \hat\beta_j + u_j \geq 0$.
- *If* $\hat\beta_j = 0$, $|\hat\beta_j + u_j| - |\hat\beta_j| - u_j s_j = |u_j| - u_j s_j \geq 0$.

Then, $\textit{crit}(\hat\beta + u) - \textit{crit}(\hat\beta) \geq \|f_u\|_n^2$.

If $k^* = \text{card}(S^*)$, we now denote for all $u = (u_j)_{j\in S^*} \in \mathbb{R}^{k^*}$:
$$\text{critS}^*(u) = \frac{1}{n}\sum_{i=1}^n \left(Y_i - \sum_{j\in S^*} u_j \phi_j(x_i)\right)^2 + 2\sum_{j\in S^*} r_{n,j}|u_j|,$$
with $\tilde\mu$ a minimizer of critS* and we finally denote
$$\mathcal{T} = \bigcap_{j\notin S^*} \left\{ \left| \frac{1}{n}\sum_{i=1}^n Y_i\phi_j(x_i) - \sum_{k\in S^*} \tilde\mu_k \langle \phi_j, \phi_k\rangle_n \right| < r_{n,j} \right\}.$$
One can show that the probability of the event $\mathcal{T}$ is asymptotically close to 1.

4. Prove that on $\mathcal{T}$, the vector $\hat\mu = (\hat\mu_j)_{j=1,\dots,p}$ such that $\hat\mu_j = \tilde\mu_j$ if $j \in S^*$ and $\hat\mu_j = 0$ if $j \in \{1,\dots,p\} \setminus S^*$, is also a minimizer of the function crit.
<u>Correction :</u> *On* $\mathcal{T}$*, we have:*
$$\frac{1}{n}\sum_{i=1}^n Y_i\phi_j(x_i) - \sum_{k\in S^*} \tilde\mu_k \langle \phi_j, \phi_k\rangle_n = r_{n,j}\,\textit{sign}(\tilde\mu_j) \;\; \textit{if } j \in S^*,\; \tilde\mu_j \neq 0$$
$$\left| \frac{1}{n}\sum_{i=1}^n Y_i\phi_j(x_i) - \sum_{k\in S^*} \tilde\mu_k \langle \phi_j, \phi_k\rangle_n \right| \leq r_{n,j} \;\; \textit{if } j \in S^*,\; \tilde\mu_j = 0$$
$$\left| \frac{1}{n}\sum_{i=1}^n Y_i\phi_j(x_i) - \sum_{k\in S^*} \tilde\mu_k \langle \phi_j, \phi_k\rangle_n \right| \leq r_{n,j} \;\; \textit{if } j \notin S^*$$
Therefore, we have $\hat\mu$ *that satifies* ($\hat\mu_j \neq 0 \iff j \in S^*$ *and* $\tilde\mu_j \neq 0$)
$$\frac{1}{n}\sum_{i=1}^n Y_i\phi_j(x_i) - \sum_{k=1}^p \hat\mu_k \langle \phi_j, \phi_k\rangle_n = r_{n,j}\,\textit{sign}(\hat\mu_j) \;\; \textit{if } \hat\mu_j \neq 0$$
$$\left| \frac{1}{n}\sum_{i=1}^n Y_i\phi_j(x_i) - \sum_{k=1}^p \hat\mu_k \langle \phi_j, \phi_k\rangle_n \right| \leq r_{n,j} \;\; \textit{if } \hat\mu_j = 0$$

5. Using (A.1), prove that on $\mathcal{T}$, for all $1 \leq i \leq n$,

$$\sum_{k=1}^{p}(\hat{\beta}_k - \hat{\mu}_k)\phi_k(x_i) = 0.$$

Correction :

$$0 = crit(\hat{\mu}) - crit(\hat{\beta}) = \|f_{\hat{\mu}-\hat{\beta}}\|_n^2.$$

Therefore

$$\frac{1}{n}\sum_{i=1}^{n} f^2_{\hat{\mu}-\hat{\beta}}(x_i) = 0$$

and for any $i \in \{1, \ldots, n\}$

$$f_{\hat{\mu}-\hat{\beta}}(x_i) = 0$$

which means that for any $i \in \{1, \ldots, n\}$

$$\sum_{k=1}^{p}(\hat{\mu}_k - \hat{\beta}_k)\phi_k(x_i) = 0.$$

6. Deduce that on $\mathcal{T}$, for $j \notin S^*$,

$$\left|\frac{1}{n}\sum_{i=1}^{n} Y_i\phi_j(x_i) - \sum_{k=1}^{p}\hat{\beta}_k\langle\phi_j, \phi_k\rangle_n\right| < r_{n,j}.$$

Correction : *For* $j \notin S^*$,

$$\begin{aligned}
& \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\phi_j(x_i) - \sum_{k\in S^*}\tilde{\mu}_k\langle\phi_j, \phi_k\rangle_n\right| < r_{n,j} \\
\Rightarrow\ & \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\phi_j(x_i) - \sum_{k=1}^{p}\hat{\mu}_k\langle\phi_j, \phi_k\rangle_n\right| < r_{n,j} \\
\Rightarrow\ & \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\phi_j(x_i) - \sum_{k=1}^{p}\hat{\mu}_k\frac{1}{n}\sum_{i=1}^{n}\phi_j(x_i)\phi_k(x_i)\right| < r_{n,j} \\
\Rightarrow\ & \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\phi_j(x_i) - \frac{1}{n}\sum_{i=1}^{n}\phi_j(x_i)\sum_{k=1}^{p}\hat{\mu}_k\phi_k(x_i)\right| < r_{n,j} \\
\Rightarrow\ & \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\phi_j(x_i) - \frac{1}{n}\sum_{i=1}^{n}\phi_j(x_i)\sum_{k=1}^{p}\hat{\beta}_k\phi_k(x_i)\right| < r_{n,j} \\
\Rightarrow\ & \left|\frac{1}{n}\sum_{i=1}^{n} Y_i\phi_j(x_i) - \sum_{k=1}^{p}\hat{\beta}_k\langle\phi_j, \phi_k\rangle_n\right| < r_{n,j}
\end{aligned}$$

7. Conclude that on $\mathcal{T}$, the support of $\hat{\beta}$ is included in the support of β^*:

$$\left\{j : \quad \hat{\beta}_j \neq 0\right\} \subset \left\{j : \quad \beta_j^* \neq 0\right\}.$$

<u>Correction :</u> *If $j \notin S^*$ and if $\hat{\beta}_j \neq 0$, then*

$$\left| \frac{1}{n} \sum_{i=1}^{n} Y_i \phi_j(x_i) - \sum_{k=1}^{p} \hat{\beta}_k \langle \phi_j, \phi_k \rangle_n \right| < r_{n,j},$$

which is excluded. So, if $j \notin S^*$ $\hat{\beta}_j = 0$. Therefore

$$\left\{j : \quad \hat{\beta}_j \neq 0\right\} \subset \left\{j : \quad \beta_j^* \neq 0\right\}.$$

A.2 Exam 2017-2018

High-dimensional statistics

Correction of the EXAM (duration 2h30)

Documents, calculators, phones and smartphones are forbidden

Problem 1: Lasso for density estimation.

The goal of this problem is to estimate the unknown density f^* of a real variable X. For this purpose, we assume we are given an n-sample $\mathcal{X}_n := (X_i)_{1\leq i\leq n}$ with density f^*. We assume that $f^* \in \mathbb{L}_2(\mathbb{R})$ and assume we are given a dictionary of known functions $\Phi = (\phi_j)_{1\leq j\leq p}$ such that for any j, $\|\phi_j\| = 1$, where $\|\cdot\|$ denotes the $\mathbb{L}_2$-norm. We also assume that the functions ϕ_j are all bounded by a finite positive constant L: Denoting $\|\cdot\|_\infty$ the sup-norm, we have

$$\max_{1\leq j\leq p} \|\phi_j\|_\infty \leq L.$$

We set for any vector $\beta = (\beta_j)_{1\leq j\leq p} \in \mathbb{R}^p$

$$f_\beta = \sum_{j=1}^p \beta_j \phi_j.$$

The goal is to select a vector $\hat{\beta} \in \mathbb{R}^p$ such that $f_{\hat{\beta}}$ is close to f^*. For this purpose, we set:

$$\hat{\beta} := \arg\min_{\beta\in\mathbb{R}^p} \left\{ \frac{1}{2} C(\beta) + \lambda \sum_{j=1}^p |\beta_j| \right\},$$

where $\lambda := 2L\sqrt{\frac{2}{n}\log\left(\frac{2p}{\delta}\right)}$, with $\delta > 0$ and

$$C(\beta) := \|f_\beta\|^2 - \frac{2}{n}\sum_{i=1}^n f_\beta(X_i).$$

1. Show that

$$\mathbb{E}[C(\beta)] = \|f_\beta - f^*\|^2 - \|f^*\|^2.$$

   Explain why this equality justifies the use of $f_{\hat{\beta}}$ to estimate f^*.

   Correction : *The equality is obvious since $\mathbb{E}[f_\beta(X_i)] = \langle f_\beta, f^*\rangle$. Minimizing $\mathbb{E}[C(\beta)]$ with respect to β is equivalent to minimizing $\beta \mapsto \|f_\beta - f^*\|$. But $\mathbb{E}[C(\beta)]$ is unknown, which is not the case of $C(\beta)$ that should be close to its expectation.*

2. We recall the Hoeffding inequality: *If we consider* n *independent variables* $Y_1, \dots, Y_n$, *such that for any* i,
$$a_i \leq Y_i \leq b_i,$$
where the a_i*'s and the* b_i*'s are non-random, then for any* $t \geq 0$,
$$\mathbb{P}\left(\left|\sum_{i=1}^{n}(Y_i - \mathbb{E}[Y_i])\right| \geq t\right) \leq 2\exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right).$$
We set
$$\Omega := \bigcap_{j=1}^{p}\left\{\frac{2}{n}\left|\sum_{i=1}^{n}(\phi_j(X_i) - \mathbb{E}[\phi_j(X_i)])\right| \leq \lambda\right\}.$$
Show that
$$\mathbb{P}(\Omega) \geq 1 - \delta.$$
Correction : *We take* $Y_i = \phi_j(X_i)$, $a_i = -L$, $b_i = L$ *and* $t = n\lambda/2$. *Then,*
$$\begin{aligned}\mathbb{P}(\Omega) &\geq 1 - 2p\exp\left(-\frac{2t^2}{n(2L)^2}\right)\\ &\geq 1 - 2p\exp\left(-\frac{n\lambda^2}{8L^2}\right) = 1 - \delta.\end{aligned}$$

3. We fix $\beta \in \mathbb{R}^p$.

   (a) Show that
$$\|f_{\hat{\beta}}\|^2 - \frac{2}{n}\sum_{i=1}^{n} f_{\hat{\beta}}(X_i) + 2\lambda\sum_{j=1}^{p}|\hat{\beta}_j| \leq \|f_\beta\|^2 - \frac{2}{n}\sum_{i=1}^{n} f_\beta(X_i) + 2\lambda\sum_{j=1}^{p}|\beta_j|.$$
   Correction : *We have:*
$$\frac{1}{2}C(\hat{\beta}) + \lambda\sum_{j=1}^{p}|\hat{\beta}_j| \leq \frac{1}{2}C(\beta) + \lambda\sum_{j=1}^{p}|\beta_j|,$$
   which is equivalent to
$$\|f_{\hat{\beta}}\|^2 - \frac{2}{n}\sum_{i=1}^{n} f_{\hat{\beta}}(X_i) + 2\lambda\sum_{j=1}^{p}|\hat{\beta}_j| \leq \|f_\beta\|^2 - \frac{2}{n}\sum_{i=1}^{n} f_\beta(X_i) + 2\lambda\sum_{j=1}^{p}|\beta_j|.$$

(b) Deduce then

$$\|f_{\hat\beta}-f^*\|^2 \le \|f_\beta-f^*\|^2+\sum_{j=1}^p(\hat\beta_j-\beta_j)\frac{2}{n}\sum_{i=1}^n(\phi_j(X_i)-\mathbb{E}[\phi_j(X_i)])+2\lambda\sum_{j=1}^p(|\beta_j|-|\hat\beta_j|).$$

<u>Correction :</u> *We have:*

$$\begin{aligned}\|f_{\hat\beta}-f^*\|^2 &= \|f_{\hat\beta}\|+\|f^*\|^2-2\langle f_{\hat\beta},f^*\rangle\\ &\le \|f^*\|^2-2\langle f_{\hat\beta},f^*\rangle+\frac{2}{n}\sum_{i=1}^n f_{\hat\beta}(X_i)-2\lambda\sum_{j=1}^p|\hat\beta_j|+\|f_\beta\|^2-\frac{2}{n}\sum_{i=1}^n f_\beta(X_i)+2\lambda\sum_{j=1}^p|\beta_j|\\ &\le \|f_\beta-f^*\|^2-2\langle f_{\hat\beta}-f_\beta,f^*\rangle+\frac{2}{n}\sum_{i=1}^n f_{\hat\beta}(X_i)-\frac{2}{n}\sum_{i=1}^n f_\beta(X_i)+2\lambda\sum_{j=1}^p(|\beta_j|-|\hat\beta_j|)\\ &\le \|f_\beta-f^*\|^2+\frac{2}{n}\sum_{j=1}^p(\hat\beta_j-\beta_j)\sum_{i=1}^n\phi_j(X_i)-2\sum_{j=1}^p(\hat\beta_j-\beta_j)\langle\phi_j,f^*\rangle+2\lambda\sum_{j=1}^p(|\beta_j|-|\hat\beta_j|)\\ &\le \|f_\beta-f^*\|^2+\frac{2}{n}\sum_{j=1}^p(\hat\beta_j-\beta_j)\sum_{i=1}^n(\phi_j(X_i)-\mathbb{E}[\phi_j(X_i)])+2\lambda\sum_{j=1}^p(|\beta_j|-|\hat\beta_j|).\end{aligned}$$

(c) Finally, conclude that on Ω,

$$\|f_{\hat\beta}-f^*\|^2+\lambda\sum_{j=1}^p|\beta_j-\hat\beta_j| \le \|f_\beta-f^*\|^2+4\lambda\sum_{j\in S(\beta)}|\beta_j-\hat\beta_j|, \tag{A.2}$$

where $S(\beta)$ is the support of β:

$$S(\beta):=\{j\in\{1,\dots,p\}:\beta_j\neq 0\}.$$

<u>Correction :</u> *We have on Ω:*

$$\|f_{\hat\beta}-f^*\|^2 \le \|f_\beta-f^*\|^2+\lambda\sum_{j=1}^p|\hat\beta_j-\beta_j|+2\lambda\sum_{j=1}^p(|\beta_j|-|\hat\beta_j|).$$

Therefore,

$$\begin{aligned}\|f_{\hat\beta}-f^*\|^2+\lambda\sum_{j=1}^p|\hat\beta_j-\beta_j| &\le \|f_\beta-f^*\|^2+2\lambda\sum_{j=1}^p|\hat\beta_j-\beta_j|+2\lambda\sum_{j=1}^p(|\beta_j|-|\hat\beta_j|)\\ &\le \|f_\beta-f^*\|^2+4\lambda\sum_{j\in S(\beta)}|\hat\beta_j-\beta_j|,\end{aligned}$$

since $|\beta_j-\hat\beta_j|+(|\beta_j|-|\hat\beta_j|)=0$ if $j\notin S(\beta)$ and $|\beta_j|-|\hat\beta_j|\le|\beta_j-\hat\beta_j|$.

4. We now wish to bound the last term of (A.2). We introduce the symmetric matrix G whose elements are given by

$$G_{jk} := \int_{\mathbb{R}} \phi_j(x)\phi_k(x)dx$$

and assume that r_p, the smallest eigenvalue of G, satisfies $r_p > 0$.

(a) Prove that for any $\beta \in \mathbb{R}^p$,

$$\|f_{\hat\beta} - f_\beta\|^2 \geq r_p \sum_{j=1}^{p} (\hat\beta_j - \beta_j)^2.$$

Correction : *We have for any $\beta \in \mathbb{R}^p$,*

$$\|f_\beta\|^2 = \| \sum_{j=1}^{p} \beta_j \phi_j \|^2 = \beta^* G \beta \geq r_p \|\beta\|_2^2.$$

Therefore,

$$\|f_{\hat\beta} - f_\beta\|^2 \geq r_p \sum_{j=1}^{p} (\hat\beta_j - \beta_j)^2.$$

(b) Deduce

$$\sum_{j \in S(\beta)} |\beta_j - \hat\beta_j| \leq \sqrt{\frac{\mathrm{card}(S(\beta))}{r_p}} \left(\|f_{\hat\beta} - f^*\| + \|f_\beta - f^*\| \right).$$

Correction : *We have:*

$$\begin{aligned}
\sum_{j \in S(\beta)} |\beta_j - \hat\beta_j| &\leq \sqrt{card(S(\beta))} \sqrt{\sum_{j \in S(\beta)} (\beta_j - \hat\beta_j)^2} \\
&\leq \sqrt{\frac{card(S(\beta))}{r_p}} \|f_{\hat\beta} - f_\beta\| \\
&\leq \sqrt{\frac{card(S(\beta))}{r_p}} \left(\|f_{\hat\beta} - f^*\| + \|f_\beta - f^*\| \right).
\end{aligned}$$

(c) Show that for any constant α,

$$4\lambda \sum_{j \in S(\beta)} |\beta_j - \hat\beta_j| \leq \frac{4\lambda^2 \mathrm{card}(S(\beta))}{\alpha r_p} + 2\alpha \left(\|f_{\hat\beta} - f^*\|^2 + \|f_\beta - f^*\|^2 \right).$$

Correction : *Since* $2ab \leq \alpha^{-1}a^2 + \alpha b^2$*,*

$$4\lambda \sum_{j\in S(\beta)} |\beta_j - \hat{\beta}_j| \leq 2 \times 2\lambda\sqrt{\frac{card(S(\beta))}{r_p}} \times \left(\|f_{\hat{\beta}} - f^*\| + \|f_\beta - f^*\|\right)$$
$$\leq \alpha^{-1}\left(2\lambda\sqrt{\frac{card(S(\beta))}{r_p}}\right)^2 + \alpha\left(\|f_{\hat{\beta}} - f^*\| + \|f_\beta - f^*\|\right)^2$$
$$\leq \frac{4\lambda^2 card(S(\beta))}{\alpha r_p} + 2\alpha\left(\|f_{\hat{\beta}} - f^*\|^2 + \|f_\beta - f^*\|^2\right).$$

(d) Finally, prove that on Ω,

$$\|f_{\hat{\beta}} - f^*\|^2 \leq \inf_{\beta\in\mathbb{R}^p}\left\{3\|f_\beta - f^*\|^2 + \frac{32\lambda^2 \mathrm{card}(S(\beta))}{r_p}\right\}.$$

Give an interpretation of this result.

Correction : *We take* $\alpha = 1/4$*. Then,*

$$4\lambda \sum_{j\in S(\beta)} |\beta_j - \hat{\beta}_j| \leq \frac{16\lambda^2 card(S(\beta))}{r_p} + \frac{1}{2}\|f_{\hat{\beta}} - f^*\|^2 + \frac{1}{2}\|f_\beta - f^*\|^2.$$

Using (A.2)*,*

$$\|f_{\hat{\beta}} - f^*\|^2 \leq 3\|f_\beta - f^*\|^2 + \frac{32\lambda^2 card(S(\beta))}{r_p}.$$

Problem 2: Gauss-Markov property and regularization via elastic-net.

We consider the multivariate linear regression model :

$$Y = X\beta^* + \varepsilon$$

with $Y = (Y_1, \ldots, Y_n)^T$ the vector of observations. The matrix X of size $n \times p$ is assumed to be known. The vector $\beta^* \in \mathbb{R}^p$ is the vector to be estimated. Finally, the error vector is $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$ and satisfies $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$ with $\sigma^2 > 0$ unknown. We denote $(X_1, \ldots, X_p)$ the columns of X. The ℓ_2-norm is denoted $\|\cdot\|_2$, whereas the ℓ_1-norm is denoted $\|\cdot\|_1$. For any matrix A, we denote A^T its transpose matrix and for any estimate $\hat{\beta}$, $\mathrm{var}(\hat{\beta})$ denotes its variance-covariance matrix. For the first two questions, we assume that the rank of X is p (with $p < n$) and

1. We consider $\hat{\beta}^{ols}$ the ordinary least-squares estimate.

(a) Recall the expression of $\hat{\beta}^{ols}$ in function of X and Y.
Correction :

$$\hat{\beta}^{ols} = (X^T X)^{-1} X^T Y$$

(b) Show that

$$\mathbb{E}[\hat{\beta}^{ols}] = \beta^* \quad \text{and} \quad \mathrm{var}(\hat{\beta}^{ols}) = \sigma^2 (X^T X)^{-1}.$$

Correction : *We have:*

$$\mathbb{E}[\hat{\beta}^{ols}] = (X^T X)^{-1} X^T \mathbb{E}[Y] = (X^T X)^{-1} X^T X \beta^* = \beta^*$$
$$var(\hat{\beta}^{ols}) = (X^T X)^{-1} X^T var(Y)((X^T X)^{-1} X^T)^T = \sigma^2 (X^T X)^{-1}.$$

2. We consider $\hat{\beta}$ a linear estimate of β^*: $\hat{\beta} = CY$, where C is a (non-random) $p \times n$-matrix. We assume that $\hat{\beta}$ is non-biased.

(a) Prove that $CX = I_p$.
Correction : *We have for any* β^*,

$$\beta^* = \mathbb{E}[\hat{\beta}] = \mathbb{E}[CY] = CX\beta^*.$$

Since this is true for any vector β^*, *we have* $CX = I_p$.

(b) Show that

$$\mathrm{var}(\hat{\beta}) = \sigma^2 CC^T = \sigma^2 (C - (X^T X)^{-1} X^T)(C - (X^T X)^{-1} X^T)^T + \sigma^2 (X^T X)^{-1}.$$

Correction : *We have:*

$$var(\hat{\beta}) = var(CY) = C var(Y) C^T = \sigma^2 CC^T.$$

Furthermore

$$\begin{aligned} CC^T &= (C - (X^T X)^{-1} X^T + (X^T X)^{-1} X^T)(C - (X^T X)^{-1} X^T + (X^T X)^{-1} X^T)^T \\ &= (C - (X^T X)^{-1} X^T)(C - (X^T X)^{-1} X^T)^T + (X^T X)^{-1} \\ &\quad + 2(C - (X^T X)^{-1} X^T) X (X^T X)^{-1} \\ &= (C - (X^T X)^{-1} X^T)(C - (X^T X)^{-1} X^T)^T + (X^T X)^{-1}. \end{aligned}$$

(c) Show that for any vector $x \in \mathbb{R}^p$,

$$x^T \mathrm{var}(\hat{\beta}) x \geq x^T \mathrm{var}(\hat{\beta}^{ols}) x.$$

The last inequality shows that among unbiased linear estimates, $\hat{\beta}^{ols}$ *is the estimate with minimal variance-covariance matrix (the Gauss-Markov property).*
Correction : *Since* $(C - (X^T X)^{-1} X^T)(C - (X^T X)^{-1} X^T)^T$ *is a non-negative symmetric matrix, the inequality is obvious.*

3. From now on, we do not consider that the rank of X is p. We now consider for $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$,
$$\tilde{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \|Y - X\beta\|_2^2 + \lambda_1 \|\beta\|_1 + \lambda_2 \|\beta\|_2^2 \right\}.$$

   (a) What's the name of this procedure? Correction : *Elastic net* What does it correspond when $\lambda_1 > 0$ and $\lambda_2 = 0$? Correction : *Lasso* What does it correspond when $\lambda_2 > 0$ and $\lambda_1 = 0$? Correction : *Ridge*

   (b) We assume $\lambda_2 > 0$. Show that $\tilde{\beta}$ exists and is the unique minimizer of C, with
$$C(\beta) := \|Y - X\beta\|_2^2 + \lambda_1 \|\beta\|_1 + \lambda_2 \|\beta\|_2^2.$$
   Correction : *C is a continuous function. Since $\lambda_2 > 0$, C is strictly convex.*

   (c) Give a condition under which $\tilde{\beta}$ may have some zero coordinates.
   Correction : $\lambda_1 > 0$

   (d) We now consider j and k such that $\tilde{\beta}_j \times \tilde{\beta}_k > 0$. Prove that
$$-2X_j^T(Y - X\tilde{\beta}) + \lambda_1 \text{sign}(\tilde{\beta}_j) + 2\lambda_2 \tilde{\beta}_j = 0$$
   and
$$-2X_k^T(Y - X\tilde{\beta}) + \lambda_1 \text{sign}(\tilde{\beta}_k) + 2\lambda_2 \tilde{\beta}_k = 0,$$
   where for any $x \in \mathbb{R}^*$, $\text{sign}(x)$ denotes the sign of x.
   Correction : *Since $\hat{\beta}_j$ and $\hat{\beta}_k$ are different from 0, it's just a consequence of computations of the partial derivatives of the criterion C with respect to β_j and β_k.*

   (e) Under assumptions of the previous question, deduce
$$|\tilde{\beta}_j - \tilde{\beta}_k| \leq \frac{\|Y\|_2 \|X_j - X_k\|_2}{\lambda_2}.$$
   Correction : *Since $sign(\tilde{\beta}_k) = sign(\tilde{\beta}_j)$, we have:*
$$\begin{aligned} \lambda_2 |\tilde{\beta}_j - \tilde{\beta}_k| &= |(X_j - X_k)^T (Y - X\tilde{\beta})| \\ &\leq \|X_j - X_k\|_2 \times \|Y - X\tilde{\beta}\|_2 \\ &\leq \|Y\|_2 \|X_j - X_k\|_2, \end{aligned}$$
   since $C(\tilde{\beta}) \leq C(0)$, so
$$\|Y - X\tilde{\beta}\|_2^2 \leq \|Y - X\tilde{\beta}\|_2^2 + \lambda_1 \|\tilde{\beta}\|_1 + \lambda_2 \|\tilde{\beta}\|_2^2 \leq \|Y\|_2^2.$$

   (f) We estimate β^* by $\hat{\beta} = (1 + \lambda_2)\tilde{\beta}$. Show that
$$\hat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \beta^T \left(\frac{X^T X + \lambda_2 I_p}{1 + \lambda_2} \right) \beta - 2Y^T X\beta + \lambda_1 \|\beta\|_1 \right\}.$$

Correction : *We use: for any fonctions f and g, if there exists α such that for any x, $g(x) = f(\alpha x)$, then*

$$\arg\min_x f(x) = \alpha \times \arg\min_x g(x).$$

Then,

$$\begin{aligned}
\tilde{\beta} &= \arg\min_{\beta\in\mathbb{R}^p} \left\{\|Y - X\beta\|_2^2 + \lambda_1\|\beta\|_1 + \lambda_2\|\beta\|_2^2\right\} \\
&= \arg\min_{\beta\in\mathbb{R}^p} \left\{\beta^T(X^TX + \lambda_2 I_p)\beta - 2Y^TX\beta + \lambda_1\|\beta\|_1\right\} \\
&= \arg\min_{\beta\in\mathbb{R}^p} \left\{(1+\lambda_2)\left(\beta^T(X^TX + \lambda_2 I_p)\beta - 2Y^TX\beta + \lambda_1\|\beta\|_1\right)\right\} \\
&= \arg\min_{\beta\in\mathbb{R}^p} \left\{((1+\lambda_2)\beta)^T\left(\frac{X^TX + \lambda_2 I_p}{1+\lambda_2}\right)((1+\lambda_2)\beta) - 2Y^TX(1+\lambda_2)\beta + \lambda_1\|(1+\lambda_2)\beta\|_1\right\}.
\end{aligned}$$

and

$$(1+\lambda_2)\tilde{\beta} = \arg\min_{u\in\mathbb{R}^p} \left\{u^T\left(\frac{X^TX + \lambda_2 I_p}{1+\lambda_2}\right)u - 2Y^TXu + \lambda_1\|u\|_1\right\}.$$

Therefore,

$$\hat{\beta} = \arg\min_{\beta\in\mathbb{R}^p} \left\{\beta^T\left(\frac{X^TX + \lambda_2 I_p}{1+\lambda_2}\right)\beta - 2Y^TX\beta + \lambda_1\|\beta\|_1.\right\}.$$

(g) By expressing $\hat{\beta}$ as the Lasso estimate for a special regression problem, show that $\hat{\beta}$ can be viewed as a stabilized version of the Lasso estimate.

Correction : *We set*

$$\tilde{Y} := \begin{pmatrix} Y \\ 0_p \end{pmatrix} \quad \tilde{X} := \frac{1}{\sqrt{1+\lambda_2}}\begin{pmatrix} X \\ \sqrt{\lambda_2} I_p \end{pmatrix}$$

So,

$$\tilde{X}^T\tilde{X} = \frac{1}{1+\lambda_2}(X^TX + \lambda_2 I_p)$$

and

$$\begin{aligned}
\hat{\beta} &= \arg\min_{\beta\in\mathbb{R}^p} \left\{\beta^T\tilde{X}^T\tilde{X}\beta - 2\tilde{Y}^T\tilde{X}\beta + \lambda_1\|\beta\|_1\right\} \\
&= \arg\min_{\beta\in\mathbb{R}^p} \left\{\|\tilde{Y} - \tilde{X}\beta\|_2^2 + \lambda_1\|\beta\|_1\right\}
\end{aligned}$$

Eigenvalues of $\tilde{X}^T\tilde{X}$ are larger than $\lambda_2/(1+\lambda_2)$

(h) We assume that the matrix X satisfies $X^TX = I_p$. For any $j \in \{1,\dots,p\}$, give the expression of $\hat{\beta}_j$ with respect to $\hat{\beta}_j^{ols}$.

Correction : *We have:*

$$\hat{\beta} = \arg\min_{\beta\in\mathbb{R}^p} \left\{\|\beta\|_2^2 - 2Y^T X\beta + \lambda_1\|\beta\|_1\right\}$$
$$= \arg\min_{\beta\in\mathbb{R}^p} \left\{\sum_{j=1}^{p} \left(\beta_j^2 + \beta_j X_j^T Y + \lambda_1|\beta_j|\right)\right\}$$

Therefore,

$$\hat{\beta}_j = sign(\hat{\beta}_j^{ols})(|\hat{\beta}_j^{ols}| - \lambda_1/2)_+$$

A.3 Exam 2018-2019

High-dimensional statistics

Correction of the EXAM (duration 2h30)

Documents, calculators, phones and smartphones are forbidden

Exercise 1

Given a collection of models $\mathcal{M}$, we study two model selection procedures based on the minimization of two criteria defined for any $m \in \mathcal{M}$ by

$$C_1(m) = C(m) + \alpha_1 \times \mathrm{card}(m), \quad C_2(m) = C(m) + \alpha_2 \times \mathrm{card}(m),$$

where $m \mapsto C(m)$ is a non-negative function defined on $\mathcal{M}$ and $0 < \alpha_1 < \alpha_2 < \infty$. If $\hat{m}_1$ minimizes $m \mapsto C_1(m)$ and $\hat{m}_2$ minimizes $m \mapsto C_2(m)$, prove that $\mathrm{card}(\hat{m}_2) \leq \mathrm{card}(\hat{m}_1)$.

Correction : *Let m a model such that $card(m) > card(\hat{m}_1)$. We show that $C_2(m) > C_2(\hat{m}_1)$. Indeed,*

$$\begin{aligned} C_2(m) &= C(m) + \alpha_2 \times card(m) \\ &= C(m) + \alpha_1 \times card(m) + (\alpha_2 - \alpha_1) \times card(m) \\ &= C_1(m) + (\alpha_2 - \alpha_1) \times card(m) \\ &\geq C_1(\hat{m}_1) + (\alpha_2 - \alpha_1) \times card(m) \\ &> C_1(\hat{m}_1) + (\alpha_2 - \alpha_1) \times card(\hat{m}_1) \\ &= C(\hat{m}_1) + \alpha_1 \times card(\hat{m}_1) + (\alpha_2 - \alpha_1) \times card(\hat{m}_1) \\ &= C_2(\hat{m}_1). \end{aligned}$$

Exercise 2

We consider the multivariate linear regression model :

$$Y = X\beta^* + \varepsilon$$

with $Y = (Y_1, \ldots, Y_n)^T$ the vector of observations. The matrix X is assumed to be known and of size $n \times 2$. The rank of X is 2 (with $2 < n$) and $\beta^* \in \mathbb{R}^2$ is the vector to be estimated. Finally, the error vector is $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$ and satisfies $\mathbb{E}[\varepsilon] = 0$, $\mathrm{var}(\varepsilon) = \sigma^2 I_n$, with $\sigma^2 > 0$ unknown. We denote (X_1, X_2) the columns of X. The ℓ_2-norm is denoted $\|\cdot\|$. For any matrix A, we denote A^T its transpose matrix and for any estimate $\hat{\beta}$, $\mathrm{var}(\hat{\beta})$ denotes its variance-covariance matrix.

1. Prove that for any estimator $\hat{\beta}$ of β^*, we have

$$\mathbb{E}[\|\hat{\beta} - \beta^*\|^2] = \sum_{j=1}^{2} (\text{var}(\hat{\beta}_j) + (\mathbb{E}[\hat{\beta}_j] - \beta_j^*)^2).$$

   Correction : *Obvious*

2. We consider $\hat{\beta}^{ols}$ the ordinary least-squares estimate. Prove that

$$\mathbb{E}[\|\hat{\beta}^{ols} - \beta^*\|^2] = \sigma^2 \left(((X^T X)^{-1})_{11} + ((X^T X)^{-1})_{22} \right).$$

   Correction : *We know that* $\mathbb{E}[\hat{\beta}^{ols}] = \beta^*$ *and* $var(\hat{\beta}^{ols}) = \sigma^2 (X^T X)^{-1}$. *This leads to the result, using Question 1.*

3. We consider the estimate $\tilde{\beta} = (\tilde{\beta}_1, 0)^T$, with $\tilde{\beta}_1$ the ordinary least-squares estimate computed in the wrong model

$$Y = \beta_1^* X_1 + \varepsilon.$$

   Give the expression of $\tilde{\beta}_1$ and show that

$$\mathbb{E}[(\tilde{\beta}_1 - \beta_1^*)^2] = \sigma^2 (X_1^T X_1)^{-1} + [(X_1^T X_1)^{-1} X_1^T X_2 \beta_2^*]^2$$

   and

$$\mathbb{E}[\|\tilde{\beta} - \beta^*\|^2] = \sigma^2 (X_1^T X_1)^{-1} + [(X_1^T X_1)^{-1} X_1^T X_2 \beta_2^*]^2 + (\beta_2^*)^2.$$

   Correction : *The expression of* $\tilde{\beta}_1$ *is*

$$\tilde{\beta}_1 = (X_1^T X_1)^{-1} X_1^T Y,$$

   which leads to

$$\begin{aligned} \mathbb{E}[\tilde{\beta}_1] &= (X_1^T X_1)^{-1} X_1^T \mathbb{E}[Y] \\ &= (X_1^T X_1)^{-1} X_1^T (\beta_1^* X_1 + \beta_2^* X_2) \\ &= \beta_1^* + \beta_2^* (X_1^T X_1)^{-1} X_1^T X_2 \end{aligned}$$

   and

$$(\mathbb{E}[\tilde{\beta}_1] - \beta_1^*)^2 = (\beta_2^*)^2 ((X_1^T X_1)^{-1} X_1^T X_2)^2.$$

   Furthermore,

$$var(\tilde{\beta}_1) = (X_1^T X_1)^{-1} X_1^T \times var(Y) \times X_1 (X_1^T X_1)^{-1} = \sigma^2 (X_1^T X_1)^{-1}.$$

4. Prove that when $|\beta_2^*| \neq 0$ but small enough, then

$$\mathbb{E}[\|\tilde{\beta} - \beta^*\|^2] < \mathbb{E}[\|\hat{\beta}^{ols} - \beta^*\|^2].$$

Indication: You can use the inequality $((X^TX)^{-1})_{11} > (X_1^TX_1)^{-1}$.

Correction : *If* β_2^* *is such that*

$$0 < (\beta_2^*)^2 \leq \frac{\sigma^2(((X^TX)^{-1})_{11} - (X_1^TX_1)^{-1})}{2(1 + ((X_1^TX_1)^{-1}X_1^TX_2)^2)},$$

then

$$\begin{aligned}
\mathbb{E}[\|\tilde{\beta} - \beta^*\|^2] &= \sigma^2(X_1^TX_1)^{-1} + [(X_1^TX_1)^{-1}X_1^TX_2\beta_2^*]^2 + (\beta_2^*)^2 \\
&\leq \sigma^2(X_1^TX_1)^{-1} + \frac{\sigma^2(((X^TX)^{-1})_{11} - (X_1^TX_1)^{-1})}{2} \\
&< \sigma^2((X^TX)^{-1})_{11} \\
&\leq \mathbb{E}[\|\hat{\beta}^{ols} - \beta^*\|^2].
\end{aligned}$$

5. Even if $\beta_2^* \neq 0$ is estimated by 0 and $\tilde{\beta}_1$ is computed in a wrong model, explain why the previous result is not so surprising.

Correction : *A sparse estimate may be better that a non-biased estimate since it has some zero coordinates whose variance is equal to zero.*

Problem

We recall the definition of a multiresolution analysis:

Definition A.1. *A multiresolution analysis is a sequence of nested vector spaces*

$$\{0\} \subset \cdots \subset V_{j+1} \subset V_j \subset V_{j-1} \subset \cdots \subset \mathbb{L}_2(\mathbb{R})$$

such that, for any $j \in \mathbb{Z}$, *if* P_{V_j} *is the orthogonal projection on* V_j, *for any* $f \in \mathbb{L}_2(\mathbb{R})$,

1. $\|P_{V_j}f - f\|_2 \overset{j\to-\infty}{\longrightarrow} 0$
2. $\|P_{V_j}f\|_2 \overset{j\to+\infty}{\longrightarrow} 0$
3. $f \in V_j \iff x \mapsto f(x/2) \in V_{j+1}$ *for any* $j \in \mathbb{Z}$
4. $f \in V_j \iff x \mapsto f(x + 2^jk) \in V_j$ *for any* $k \in \mathbb{Z}$
5. $\exists\, \phi$ *such that* $(\phi_k)_{k\in\mathbb{Z}}$ *is an orthonormal basis of* V_0 *with for any* $x \in \mathbb{R}$, $\phi_k(x) = \phi(x-k)$.

Furthermore, setting

$$\phi_{jk}(t) = \frac{1}{\sqrt{2^j}}\phi\left(\frac{t-k2^j}{2^j}\right), \quad t \in \mathbb{R},$$

for any $j \in \mathbb{Z}$, $(\phi_{jk})_{k\in\mathbb{Z}}$ is an orthonormal basis of V_j.

For any $j \in \mathbb{Z}$, the detail space W_j is defined as the orthogonal complement of V_j in V_{j-1}:

$$V_j \oplus W_j = V_{j-1}.$$

The goal of this problem is to prove the following theorem (providing the definition of a wavelet ψ in function of its associated scaling function ϕ).

Theorem A.1. *Let h a conjugate mirror filter. We define the function g as*

$$\widehat{g}(w) = e^{-iw}\overline{m_0(w+\pi)},$$

where m_0 is the transfer function associated with h. We define the real-valued function ψ such that

$$\widehat{\psi}(w) = \widehat{g}\left(\frac{w}{2}\right)\widehat{\phi}\left(\frac{w}{2}\right).$$

We set for any $j \in \mathbb{Z}$ and any $k \in \mathbb{Z}$

$$\psi_{jk}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-k2^j}{2^j}\right), \quad t \in \mathbb{R}.$$

Then, for any $j \in \mathbb{Z}$, $(\psi_{jk})_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j. Furthermore, $(\psi_{jk})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an orthonormal basis of $\mathbb{L}_2(\mathbb{R})$.

To prove this theorem, you can use the following proposition established in the course.

Proposition A.1. *We have*

1. $\widehat{\phi}(2w) = \widehat{\phi}(w)m_0(w)$, $w \in \mathbb{R}$
2. m_0 *is* 2π*-periodic and* $m_0(0) = 1$
3. $|m_0(w)|^2 + |m_0(w+\pi)|^2 = 1$, $w \in \mathbb{R}$

We denote $\langle \cdot, \cdot \rangle$ the scalar product associated with the $\mathbb{L}_2$-norm and $\star$ the standard convolution product between two functions.

1. The goal of this question is to prove that functions of V_0 are orthogonal to the functions $(\psi_{0k})_{k\in\mathbb{Z}}$. We often use $\psi(\cdot - k) = \psi_{0k}$ and $\phi(\cdot - k) = \phi_{0k}$.

(a) Denoting for any $t \in \mathbb{R}$, $\tilde{\psi}(t) = \psi(-t)$, first prove that

$$\sum_{k\in\mathbb{Z}} (\phi \star \tilde{\psi})(k) e^{-ikw} = \sum_{k\in\mathbb{Z}} \hat{\phi}(w + 2k\pi)\overline{\hat{\psi}(w + 2k\pi)}.$$

Indication: Use that ψ is a real-valued function and the Poisson formula: for any function h,

$$\sum_{k\in\mathbb{Z}} h(k)e^{-ikw} = \sum_{k\in\mathbb{Z}} \widehat{h}(w + 2k\pi).$$

Correction : *We have:*

$$\sum_{k\in\mathbb{Z}} (\phi \star \tilde{\psi})(k) e^{-ikw} = \sum_{k\in\mathbb{Z}} \widehat{(\phi \star \tilde{\psi})}(w + 2k\pi) = \sum_{k\in\mathbb{Z}} \hat{\phi}(w + 2k\pi)\hat{\tilde{\psi}}(w + 2k\pi)$$

and, since ψ is a real-valued function

$$\hat{\tilde{\psi}}(w) = \int e^{-itw}\tilde{\psi}(t)dt = \int e^{itw}\psi(t)dt = \overline{\hat{\psi}(w)}.$$

(b) Deduce that for any $n \in \mathbb{Z}$ and any $p \in \mathbb{Z}$, $\langle \phi(\cdot - n), \psi(\cdot - p)\rangle = 0$ if and only if

$$\sum_{k\in\mathbb{Z}} \hat{\phi}(w + 2k\pi)\overline{\hat{\psi}(w + 2k\pi)} = 0. \tag{A.3}$$

Correction :

$$\begin{aligned}\langle \phi(\cdot - n), \psi(\cdot - p)\rangle &= \int \phi(t - n)\psi(t - p)dt \\ &= \int \phi(u)\psi(u + n - p)du \\ &= (\phi \star \tilde{\psi})(p - n).\end{aligned}$$

Therefore, for any $n \in \mathbb{Z}$ and any $p \in \mathbb{Z}$, $\langle \phi(\cdot - n), \psi(\cdot - p)\rangle = 0$ if and only if for any $k \in \mathbb{Z}$, $(\phi \star \tilde{\psi})(k) = 0$, which leads to the result.

(c) Establish that for any $w \in \mathbb{R}$,

$$m_0(w)\overline{\hat{g}(w)} + m_0(w + \pi)\overline{\hat{g}(w + \pi)} = 0.$$

Correction : *We have:*

$$m_0(w)\overline{\hat{g}(w)} + m_0(w + \pi)\overline{\hat{g}(w + \pi)} = m_0(w)e^{iw}m_0(w + \pi) - m_0(w + \pi)e^{iw}m_0(w) = 0.$$

(d) Prove that the previous equality yields

$$\sum_{k\in\mathbb{Z}} m_0\left(\frac{w}{2}+k\pi\right)\overline{\hat{g}\left(\frac{w}{2}+k\pi\right)}\left|\hat{\phi}\left(\frac{w}{2}+k\pi\right)\right|^2=0.$$

Indication: Use that for any $w\in\mathbb{R}$, $\sum_{k\in\mathbb{Z}}\left|\hat{\phi}\left(w+2k\pi\right)\right|^2=1$.

Correction : *We have*

$$\begin{aligned}A:=&\sum_{k\in\mathbb{Z}} m_0\left(\frac{w}{2}+k\pi\right)\overline{\hat{g}\left(\frac{w}{2}+k\pi\right)}\left|\hat{\phi}\left(\frac{w}{2}+k\pi\right)\right|^2\\
=&\sum_{p\in\mathbb{Z}} m_0\left(\frac{w}{2}+(2p+1)\pi\right)\overline{\hat{g}\left(\frac{w}{2}+(2p+1)\pi\right)}\left|\hat{\phi}\left(\frac{w}{2}+(2p+1)\pi\right)\right|^2\\
&+\sum_{p\in\mathbb{Z}} m_0\left(\frac{w}{2}+2p\pi\right)\overline{\hat{g}\left(\frac{w}{2}+2p\pi\right)}\left|\hat{\phi}\left(\frac{w}{2}+2p\pi\right)\right|^2\\
=&\, m_0\left(\frac{w}{2}+\pi\right)\overline{\hat{g}\left(\frac{w}{2}+\pi\right)}\sum_{p\in\mathbb{Z}}\left|\hat{\phi}\left(\frac{w}{2}+(2p+1)\pi\right)\right|^2\\
&+m_0\left(\frac{w}{2}\right)\overline{\hat{g}\left(\frac{w}{2}\right)}\sum_{p\in\mathbb{Z}}\left|\hat{\phi}\left(\frac{w}{2}+2p\pi\right)\right|^2\\
=&\, m_0\left(\frac{w}{2}+\pi\right)\overline{\hat{g}\left(\frac{w}{2}+\pi\right)}+m_0\left(\frac{w}{2}\right)\overline{\hat{g}\left(\frac{w}{2}\right)}=0\end{aligned}$$

(e) Using Proposition A.1, prove that (A.3) is satisfied.

Correction :

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\hat{\phi}(w+2k\pi)\overline{\hat{\psi}(w+2k\pi)}&=\sum_{k\in\mathbb{Z}}\hat{\phi}\left(\frac{w}{2}+k\pi\right)m_0\left(\frac{w}{2}+k\pi\right)\overline{\hat{\psi}\left(w+2k\pi\right)}\\
&=\sum_{k\in\mathbb{Z}} m_0\left(\frac{w}{2}+k\pi\right)\overline{\hat{g}\left(\frac{w}{2}+k\pi\right)}\left|\hat{\phi}\left(\frac{w}{2}+k\pi\right)\right|^2\\
&=0.\end{aligned}$$

(f) Conclude that functions of V_0 are orthogonal to the functions $(\psi_{0k})_{k\in\mathbb{Z}}$.

Correction : *Using Questions 1)(b) and 1)(e), since functions* $(\phi_k)_{k\in\mathbb{Z}}$ *span* V_0, *functions of* V_0 *are orthogonal to the functions* $(\psi_{0k})_{k\in\mathbb{Z}}$.

2. We now study the functions $(\psi_{0k})_{k\in\mathbb{Z}}$.

(a) Show that

$$\sum_{k\in\mathbb{Z}}|\hat{\psi}(w+2k\pi)|^2=1,\quad w\in\mathbb{R}.$$

Indication: Use that for any $w \in \mathbb{R}$, $\sum_{k\in\mathbb{Z}} \left|\hat{\phi}\left(w+2k\pi\right)\right|^2 = 1$.

Correction :

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\hat{\psi}(w+2k\pi)|^2 &= \sum_{k\in\mathbb{Z}} \left|\hat{g}\left(\frac{w}{2}+k\pi\right)\right|^2 \left|\hat{\phi}\left(\frac{w}{2}+k\pi\right)\right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|m_0\left(\frac{w}{2}+(k+1)\pi\right)\right|^2 \left|\hat{\phi}\left(\frac{w}{2}+k\pi\right)\right|^2 \\
&= \sum_{p\in\mathbb{Z}} \left|m_0\left(\frac{w}{2}+(2p+2)\pi\right)\right|^2 \left|\hat{\phi}\left(\frac{w}{2}+(2p+1)\pi\right)\right|^2 \\
&\quad + \sum_{p\in\mathbb{Z}} \left|m_0\left(\frac{w}{2}+(2p+1)\pi\right)\right|^2 \left|\hat{\phi}\left(\frac{w}{2}+2p\pi\right)\right|^2 \\
&= \left|m_0\left(\frac{w}{2}\right)\right|^2 \sum_{p\in\mathbb{Z}} \left|\hat{\phi}\left(\frac{w}{2}+(2p+1)\pi\right)\right|^2 \\
&\quad + \left|m_0\left(\frac{w}{2}+\pi\right)\right|^2 \sum_{p\in\mathbb{Z}} \left|\hat{\phi}\left(\frac{w}{2}+2p\pi\right)\right|^2 \\
&= \left|m_0\left(\frac{w}{2}\right)\right|^2 + \left|m_0\left(\frac{w}{2}+\pi\right)\right|^2 = 1
\end{aligned}$$

(b) Prove that for any $n \in \mathbb{Z}$ and any $p \in \mathbb{Z}$,

$$\langle \psi(\cdot - n), \psi(\cdot - p)\rangle = 1_{\{n=p\}}.$$

Indication: The arguments are similar to those of the Question 1.

Correction : *For any* $n \in \mathbb{Z}$ *and any* $p \in \mathbb{Z}$

$$\begin{aligned}
\langle \psi(\cdot - n), \psi(\cdot - p)\rangle &= \int \psi(t-n)\psi(t-p)dt \\
&= \int \psi(u)\psi(u+n-p)du \\
&= (\psi \star \tilde{\psi})(p-n).
\end{aligned}$$

The Poisson formula and Question 2)(a) give:

$$\sum_{k\in\mathbb{Z}} (\psi \star \tilde{\psi})(k)e^{-ikw} = \sum_{k\in\mathbb{Z}} |\hat{\psi}(w+2k\pi)|^2 = 1,$$

which is a constant function. By identification, $(\psi \star \tilde{\psi})(k) = 0$ *for* $k \neq 0$ *and* $(\psi \star \tilde{\psi})(0) = 1$. *This implies*

$$\langle \psi(\cdot - n), \psi(\cdot - p)\rangle = 1_{\{n=p\}}.$$

(c) Deduce that functions $(\psi_{0k})_{k\in\mathbb{Z}}$ are orthonormal.

Correction : *Obvious*

3. In this question, we show that $V_{-1} = V_0 \oplus \tilde{W}_0$, where $\tilde{W}_0$ is the space spanned by the functions $(\psi_{0k})_{k\in\mathbb{Z}}$.

(a) Show that the inclusion $V_{-1} \subset V_0 \oplus \tilde{W}_0$ is equivalent to the property:

$$\forall\,(a_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z}),\ \exists\,(b_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z}) \text{ and } (c_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z}) \text{ such that}$$

$$\sum_{k\in\mathbb{Z}} a_k\sqrt{2}\phi(2t-k) = \sum_{k\in\mathbb{Z}} b_k\phi(t-k) + \sum_{k\in\mathbb{Z}} c_k\psi(t-k), \quad t \in \mathbb{R}. \tag{A.4}$$

Correction : *The space V_{-1} is spanned by functions $\phi_{-1k} : t \mapsto \sqrt{2}\phi(2t-k)$. The space V_0 is spanned by functions $\phi_{0k} : t \mapsto \phi(t-k)$. These arguments give the result.*

(b) By setting for any $w \in \mathbb{R}$,

$$\hat{a}(w) = \sum_{k\in\mathbb{Z}} a_k e^{-ikw}, \quad \hat{b}(w) = \sum_{k\in\mathbb{Z}} b_k e^{-ikw}, \quad \hat{c}(w) = \sum_{k\in\mathbb{Z}} c_k e^{-ikw},$$

show that (A.4) is satisfied if

$$\hat{a}\left(\frac{w}{2}\right) = \sqrt{2}\left(\hat{b}(w)m_0\left(\frac{w}{2}\right) + \hat{c}(w)\hat{g}\left(\frac{w}{2}\right)\right), \quad w \in \mathbb{R}. \tag{A.5}$$

Correction : *We have*

$$\text{(A.4)} \iff \int e^{-itw}\sum_{k\in\mathbb{Z}} a_k\sqrt{2}\phi(2t-k)dt = \int e^{-itw}\sum_{k\in\mathbb{Z}} b_k\phi(t-k)dt + \int e^{-itw}\sum_{k\in\mathbb{Z}} c_k\psi(t-k)dt, \forall\, w \in$$

$$\iff \sum_{k\in\mathbb{Z}} a_k e^{-ikw/2}\int \phi(u)e^{-iwu/2}\frac{du}{\sqrt{2}} = \sum_{k\in\mathbb{Z}} b_k e^{-ikw}\int \phi(u)e^{-iwu}du + \sum_{k\in\mathbb{Z}} c_k e^{-ikw}\int \psi(u)e^{-iwu}d$$

$$\iff \hat{a}\left(\frac{w}{2}\right)\hat{\phi}\left(\frac{w}{2}\right) = \sqrt{2}\left(\hat{b}(w)\hat{\phi}(w) + \hat{\psi}(w)\hat{c}(w)\right), \forall\, w \in \mathbb{R}$$

$$\iff \hat{a}\left(\frac{w}{2}\right)\hat{\phi}\left(\frac{w}{2}\right) = \sqrt{2}\left(\hat{b}(w)\hat{\phi}\left(\frac{w}{2}\right)m_0\left(\frac{w}{2}\right) + \hat{c}(w)\hat{g}\left(\frac{w}{2}\right)\hat{\phi}\left(\frac{w}{2}\right)\right), \forall\, w \in \mathbb{R}$$

Therefore, (A.5) *implies* (A.4).

(c) By setting

$$\hat{b}(w) = \frac{1}{\sqrt{2}}\left(\hat{a}\left(\frac{w}{2}\right)\overline{m_0\left(\frac{w}{2}\right)} + \hat{a}\left(\frac{w}{2}+\pi\right)\overline{m_0\left(\frac{w}{2}+\pi\right)}\right),$$

$$\hat{c}(w) = \frac{1}{\sqrt{2}}\left(\hat{a}\left(\frac{w}{2}\right)\overline{\hat{g}\left(\frac{w}{2}\right)} + \hat{a}\left(\frac{w}{2}+\pi\right)\overline{\hat{g}\left(\frac{w}{2}+\pi\right)}\right),$$

show that (A.5) is satisfied.

Correction : *We have for $w \in \mathbb{R}$,*

$$\begin{aligned}\sqrt{2}\left(\hat{b}(w)m_0\left(\frac{w}{2}\right)+\hat{c}(w)\hat{g}\left(\frac{w}{2}\right)\right) &= \hat{a}\left(\frac{w}{2}\right)\left|m_0\left(\frac{w}{2}\right)\right|^2+\hat{a}\left(\frac{w}{2}+\pi\right)\overline{m_0\left(\frac{w}{2}+\pi\right)}m_0\left(\frac{w}{2}\right)\\ &\quad+\hat{a}\left(\frac{w}{2}\right)\left|g\left(\frac{w}{2}\right)\right|^2+\hat{a}\left(\frac{w}{2}+\pi\right)\overline{g\left(\frac{w}{2}+\pi\right)}g\left(\frac{w}{2}\right)\\ &= \hat{a}\left(\frac{w}{2}\right)\left|m_0\left(\frac{w}{2}\right)\right|^2+\hat{a}\left(\frac{w}{2}+\pi\right)\overline{m_0\left(\frac{w}{2}+\pi\right)}m_0\left(\frac{w}{2}\right)\\ &\quad+\hat{a}\left(\frac{w}{2}\right)\left|m_0\left(\frac{w}{2}+\pi\right)\right|^2-\hat{a}\left(\frac{w}{2}+\pi\right)m_0\left(\frac{w}{2}\right)\overline{m_0\left(\frac{w}{2}+\pi\right)}\\ &= \hat{a}\left(\frac{w}{2}\right).\end{aligned}$$

(d) Prove that $V_0 \oplus \tilde{W}_0 \subset V_{-1}$ and deduce that $W_0 = \tilde{W}_0$.

Correction : *We already know that $V_0 \subset V_{-1}$. To show that $\tilde{W}_0 \subset V_{-1}$, we just have to show that*

$$\forall\,(c_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z}),\ \exists\,(a_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z}) \text{ such that}$$

$$\sum_{k\in\mathbb{Z}} c_k\psi(t-k) = \sum_{k\in\mathbb{Z}} a_k\sqrt{2}\phi(2t-k), \quad t\in\mathbb{R},$$

which is equivalent to

$$\hat{a}\left(\frac{w}{2}\right)\hat{\phi}\left(\frac{w}{2}\right) = \sqrt{2}\hat{c}(w)\hat{g}\left(\frac{w}{2}\right)\hat{\phi}\left(\frac{w}{2}\right), \forall\, w\in\mathbb{R}.$$

So, we have to set $\hat{a}$ so that $\forall\, w \in \mathbb{R}$,

$$\hat{a}\left(\frac{w}{2}\right) = \sqrt{2}\hat{c}(w)\hat{g}\left(\frac{w}{2}\right)$$

to obtain the result.
These results imply $V_0 \oplus \tilde{W}_0 = V_{-1} = V_0 \oplus W_0$. This yields $\tilde{W}_0 = W_0$.

4. Show that for any $j \in \mathbb{Z}$, $(\psi_{jk})_{k\in\mathbb{Z}}$ constitutes an orthonormal system of functions. Show also that for any $j \in \mathbb{Z}$ and for any function $f \in V_j$, $\langle f, \psi_{jk}\rangle = 0$ for any $k \in \mathbb{Z}$.

Correction : *We have*

$$\begin{aligned}\langle \psi_{jk}, \psi_{jk'}\rangle &= \frac{1}{2^j}\int \psi(2^{-j}t-k)\psi(2^{-j}t-k')dt\\ &= \int \psi(u-k)\psi(u-k')du = 1_{\{k=k'\}},\end{aligned}$$

which proves the first point. For the second one, we just have to prove that for any k and any k', $\langle \phi_{jk}, \psi_{jk'}\rangle = 0$. By using similar computations and Question 1), the result is satisfied.

5. Prove that for any $j \in \mathbb{Z}$, $(\psi_{jk})_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j. Deduce that the spaces $(W_j)_{j\in\mathbb{Z}}$ are orthogonal and for any $L < J$,

$$V_L = \left[\oplus_{j=L+1}^{J} W_j\right] \oplus V_J.$$

Correction : *$f \in V_{j-1} = V_j \oplus W_j \iff f(2^j\cdot) \in V_{-1} = V_0 \oplus W_0$. Therefore, there exist $(a_k)_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ such that for any $x \in \mathbb{R}$,*

$$f(2^j x) = \sum_{k\in\mathbb{Z}} a_k \phi(x-k) + \sum_{k\in\mathbb{Z}} b_k \psi(x-k).$$

Therefore, for any $x \in \mathbb{R}$,

$$f(x) = \sum_{k\in\mathbb{Z}} a_k \phi(2^{-j}x-k) + \sum_{k\in\mathbb{Z}} b_k \psi(2^{-j}x-k).$$

We obtain that $\{(\phi_{jk})_{k\in\mathbb{Z}}, (\psi_{jk})_{k\in\mathbb{Z}}\}$ is an orthonormal basis of $V_{j-1} = V_j \oplus W_j$ and $(\psi_{jk})_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j.
We have for any $j \in \mathbb{Z}$ that $W_{j+1} \subset V_j \perp W_j$. Therefore, the spaces W_j are orthogonal.
Finally,

$$\begin{aligned} V_L &= V_{L+1} \oplus W_{L+1} \\ &= V_{L+2} \oplus W_{L+2} \oplus W_{L+1} \\ &\cdots \\ &= V_J \oplus W_J \oplus \cdots \oplus W_{L+2} \oplus W_{L+1} \\ &= \left[\oplus_{j=L+1}^{J} W_j\right] \oplus V_J. \end{aligned}$$

6. Finally, prove that $(\psi_{jk})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an orthonormal basis of $\mathbb{L}_2(\mathbb{R})$.

Correction : *We start from*

$$V_L = \left[\oplus_{j=L+1}^{J} W_j\right] \oplus V_J$$

with $L < J$. Now take $L \to -\infty$ and $J \to +\infty$, we obtain

$$\mathbb{L}_2(\mathbb{R}) = \oplus_{j=-\infty}^{+\infty} W_j.$$

Since for any j, $(\psi_{jk})_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j and the spaces W_j are orthogonal, $(\psi_{jk})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an orthonormal basis of $\mathbb{L}_2(\mathbb{R})$.

A.4 Exam 2019-2020

High-dimensional statistics

EXAM: duration 2h30

Documents, calculators, phones and smartphones are forbidden

Exercise 1

Given $\beta^* \in \mathbb{R}^p$ and a matrix X of size $n \times p$, and whose lines are denoted $x_1^T, \dots, x_n^T$, so that

$$X = \begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{pmatrix},$$

we consider the regression model

$$Y_i = x_i^T \beta^* + \varepsilon_i, \quad \varepsilon_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2), \quad i = 1, \dots, n, \tag{A.6}$$

with $\sigma^2 > 0$. We denote

$$Y = (Y_1, \dots, Y_n)^T, \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^T$$

and we define the degree of freedom of a function $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ with coordinates g_i by

$$\mathrm{df}(g) = \frac{1}{\sigma^2} \sum_{i=1}^n \mathrm{cov}(g_i(Y), Y_i).$$

Model (A.15) can be rewritten

$$Y = X\beta^* + \varepsilon$$

and we assume that $\mathrm{rank}(X) = p$. In the sequel, we denote $\|\cdot\|$ the ℓ_2-norm on $\mathbb{R}^n$.

1. We consider $\hat{\beta} \in \mathbb{R}^p$ any estimate of β^* and we set $g(Y) = X\hat{\beta}$, so that $g_i(Y)$ is the ith coordinate of $X\hat{\beta}$:
$$g_i(Y) = (X\hat{\beta})_i.$$
We denote
$$C_p := \|Y - X\hat{\beta}\|^2 - n\sigma^2 + 2\sigma^2 \mathrm{df}(X\hat{\beta}).$$

(a) Establish that $\mathbb{E}[\|\varepsilon\|^2] = n\sigma^2$.

Correction : *Obvious since*

$$\|\varepsilon\|^2 = \sum_{i=1}^{n} \varepsilon_i^2.$$

(b) Prove that for any $i \in \{1, \dots, n\}$,

$$\mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[Y_i] - (X\hat{\beta})_i)] = \mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[(X\hat{\beta})_i] - (X\hat{\beta})_i)].$$

Correction : *We observe that for any* $i \in \{1, \dots, n\}$,

$$\begin{aligned}\mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[Y_i] - (X\hat{\beta})_i)] &= \mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[Y_i] - \mathbb{E}[(X\hat{\beta})_i] + \mathbb{E}[(X\hat{\beta})_i] - (X\hat{\beta})_i)] \\ &= 0 + \mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[(X\hat{\beta})_i] - (X\hat{\beta})_i)].\end{aligned}$$

(c) Prove that

$$\mathbb{E}[C_p] = \mathbb{E}[\|X\hat{\beta} - X\beta^*\|^2].$$

Indication: *We recall that*

$$cov(g_i(Y), Y_i) = \mathbb{E}[(g_i(Y) - \mathbb{E}[g_i(Y)])(Y_i - \mathbb{E}[Y_i])].$$

Correction : *We have:*

$$\begin{aligned}\mathbb{E}[\|X\hat{\beta} - X\beta^*\|^2] &= \mathbb{E}[\|X\hat{\beta} - Y + Y - X\beta^*\|^2] \\ &= \mathbb{E}[\|X\hat{\beta} - Y\|^2] + \mathbb{E}[\|Y - X\beta^*\|^2] - 2\mathbb{E}\left[\sum_{i=1}^{n} (Y - X\beta^*)_i (Y - X\hat{\beta})_i\right] \\ &= \mathbb{E}[\|X\hat{\beta} - Y\|^2] + \mathbb{E}[\|\varepsilon\|^2] \\ &\quad - 2\mathbb{E}\left[\sum_{i=1}^{n} (Y_i - \mathbb{E}[Y_i])(Y_i - \mathbb{E}[Y_i] + \mathbb{E}[Y_i] - (X\hat{\beta})_i)\right] \\ &= \mathbb{E}[\|X\hat{\beta} - Y\|^2] - \mathbb{E}[\|\varepsilon\|^2] - 2\mathbb{E}\left[\sum_{i=1}^{n} (Y_i - \mathbb{E}[Y_i])(\mathbb{E}[Y_i] - (X\hat{\beta})_i)\right].\end{aligned}$$

So, since $\mathbb{E}[\|\varepsilon\|^2] = n\sigma^2$ *and using the result of the previous question,*

$$\begin{aligned}\mathbb{E}[\|X\hat{\beta} - X\beta^*\|^2] &= \mathbb{E}[\|X\hat{\beta} - Y\|^2] - n\sigma^2 - 2\sum_{i=1}^{n} \mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[(X\hat{\beta})_i] - (X\hat{\beta})_i)] \\ &= \mathbb{E}[\|X\hat{\beta} - Y\|^2] - n\sigma^2 + 2\sigma^2 df(X\hat{\beta}) \\ &= \mathbb{E}[C_p].\end{aligned}$$

2. We now estimate $X\beta^*$ with

$$g(Y) = X(X^TX)^{-1}X^TY.$$

   (a) Prove that

$$\mathrm{df}(g) = \sum_{i=1}^{n} x_i^T(X^TX)^{-1}x_i.$$

Correction : *For this case, $g_i(Y) = x_i^T(X^TX)^{-1}X^TY$. We denote 1_i the vector whose components are all equal to 0 except the ith component equal to 1. Since $X^T \times 1_i = x_i$*

$$\begin{aligned} df(g) &= \sigma^{-2}\sum_{i=1}^{n} \mathbb{E}[x_i^T(X^TX)^{-1}X^T\varepsilon \times \varepsilon_i] \\ &= \sum_{i=1}^{n} x_i^T(X^TX)^{-1}X^T\sigma^{-2}\mathbb{E}[\varepsilon_i\varepsilon] \\ &= \sum_{i=1}^{n} x_i^T(X^TX)^{-1}X^T \times 1_i \\ &= \sum_{i=1}^{n} x_i^T(X^TX)^{-1}x_i. \end{aligned}$$

   (b) Deduce that

$$\mathrm{df}(g) = \mathrm{Trace}(X(X^TX)^{-1}X^T).$$

Correction : *Obvious.*

   (c) Finally, prove that

$$\mathrm{df}(g) = p.$$

Correction : *The matrix $X(X^TX)^{-1}X^T$ is the projection matrix on $\mathcal{I}m(X)$, so, since $rank(X) = p$, $df(g) = Trace(X(X^TX)^{-1}X^T) = p$.*

Exercise 2

We consider the model of Exercise 1 written

$$Y = X\beta^* + \varepsilon.$$

We use notations of Exercise 1. We consider the ridge estimate: for $\lambda > 0$,

$$\hat{\beta}_\lambda := (X^TX + \lambda I_p)^{-1}X^TY.$$

We denote $(\mu_1, \ldots, \mu_p)$ the eigenvalues of the matrix $X^T X$ and $(U_1, \ldots, U_p)$ the associated orthonormal basis of eigenvectors. So, we can write

$$X^T X = UDU^T,$$

with U an orthogonal matrix whose columns are given by the U_j's and D the diagonal matrix with the μ_j's on the diagonal:

$$U = [U_1, \ldots, U_P], \quad D = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_p \end{pmatrix}.$$

1. Establish the following bias-variance decomposition:

$$\mathbb{E}[\|\hat{\beta}_\lambda - \beta^*\|^2] = \|\mathbb{E}[\hat{\beta}_\lambda] - \beta^*\|^2 + \mathbb{E}\Big[\|\hat{\beta}_\lambda - \mathbb{E}[\hat{\beta}_\lambda]\|^2\Big].$$

Correction : *Obvious.*

2. Prove that

$$\mathbb{E}[\hat{\beta}_\lambda] = (X^T X + \lambda I_p)^{-1} X^T X \beta^*.$$

Correction : *Obvious.*

3. Deduce that

$$\mathbb{E}[\hat{\beta}_\lambda] - \beta^* = -\lambda (X^T X + \lambda I_p)^{-1} \beta^*.$$

Correction : *We have:*

$$\begin{aligned} \mathbb{E}[\hat{\beta}_\lambda] - \beta^* &= (X^T X + \lambda I_p)^{-1} X^T X \beta^* - \beta^* \\ &= (X^T X + \lambda I_p)^{-1} (X^T X - X^T X - \lambda I_p) \beta^* \\ &= -\lambda (X^T X + \lambda I_p)^{-1} \beta^*. \end{aligned}$$

4. Finally, establish that

$$\|\mathbb{E}[\hat{\beta}_\lambda] - \beta^*\|^2 = \lambda^2 \beta^{*T} (X^T X + \lambda I_p)^{-2} \beta^*.$$

Correction : *Obvious.*

5. Prove that

$$\hat{\beta}_\lambda - \mathbb{E}[\hat{\beta}_\lambda] = (X^T X + \lambda I_p)^{-1} X^T \varepsilon.$$

Correction : *Obvious.*

6. Establish the decomposition

$$(X^TX + \lambda I_p)^{-1}X^TX(X^TX + \lambda I_p)^{-1} = U\tilde{D}U^T,$$

where

$$\tilde{D} = \begin{pmatrix} \frac{\mu_1}{(\mu_1+\lambda)^2} & & & \\ & \frac{\mu_2}{(\mu_2+\lambda)^2} & & \\ & & \ddots & \\ & & & \frac{\mu_p}{(\mu_p+\lambda)^2} \end{pmatrix}.$$

Correction : *The eigenvalues of X^TX are $(\mu_1, \ldots, \mu_p)$ and $(U_1, \ldots, U_p)$ is the associated orthonormal basis of eigenvectors. So, in this basis, the eigenvalues of $X^TX + \lambda I_p$ are $(\mu_1+\lambda, \ldots, \mu_p+\lambda)$ and the eigenvalues of $(X^TX+\lambda I_p)^{-1}X^TX(X^TX+\lambda I_p)^{-1}$ in this basis are the diagonal elements of $\tilde{D}$.*

7. Finally, establish

$$\mathbb{E}\Big[\|\hat{\beta}_\lambda - \mathbb{E}[\hat{\beta}_\lambda]\|^2\Big] = \sigma^2 \sum_{j=1}^{p} \frac{\mu_j}{(\mu_j + \lambda)^2}.$$

Correction : *We simply use the following result: For any deterministic matrix A with n columns,*

$$\mathbb{E}[\|A\varepsilon\|^2] = \sigma^2 Tr(AA^T).$$

Problem

We wish to estimate a function $f \in \mathbb{L}_2(\mathbb{R})$ decomposed on a wavelet basis denoted $(\psi_{jk})_{jk}$:

$$f = \sum_{j=-1}^{+\infty} \sum_{k \in \mathcal{K}_j} \beta_{jk}\psi_{jk},$$

where for any $j \geq -1$, $\mathcal{K}_j$ is the set of integers k such that

$$\forall\, k \notin \mathcal{K}_j, \quad \beta_{jk} = 0.$$

In the previous decomposition, the coefficients $(\beta_{-1k})_k$ correspond to the approximation coefficients. We assume that we observe a noisy version of wavelet coefficients β_{jk}. The noise is assumed to be Gaussian and we consider the following model:

$$X_{jk} = \beta_{jk} + \frac{\sigma}{\sqrt{n}} z_{jk}, \quad j \geq -1, \quad k \in \mathcal{K}_j,$$

where the z_{jk}'s are i.i.d. $\mathcal{N}(0,1)$. The noise level $\frac{\sigma}{\sqrt{n}}$ is assumed to be known. For practical reasons, we only estimate a finite set of wavelet coefficients. This set will have the form

$$\Gamma = \{(j,k) : \quad -1 \leq j \leq J, \ k \in \mathcal{K}_j\}$$

with J an integer. We consider η_{jk} a threshold (defined below) and we set for any j and any k,

$$\hat{\beta}_{jk} = X_{jk} 1_{\{|X_{jk}| > \eta_{jk}\}}.$$

The estimate of f is then

$$\hat{f} = \sum_{j=-1}^{J} \sum_{k \in \mathcal{K}_j} \hat{\beta}_{jk} \psi_{jk}.$$

For any $k \in \mathcal{K}_{-1}$, we set $\eta_{-1k} = 0$ and for $0 \leq j \leq J$ and $k \in \mathcal{K}_j$,

$$\eta_{jk} = \sigma \sqrt{\frac{2\gamma \log n}{n}}.$$

We take γ a constant larger than 1 and such that

$$\operatorname{card}(\Gamma) \leq n^{\frac{\gamma}{8}}.$$

The goal of the problem is to study the $\mathbb{L}_2$-risk of the estimate $\hat{f}$. In the sequel, we denote $\|\cdot\|$ the $\mathbb{L}_2$-norm.

1. Prove that if $Z \sim \mathcal{N}(0,1)$, then for any $x > 0$,

$$\mathbb{P}(|Z| \geq x) \leq \frac{2}{x\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

<u>Correction :</u> *We just use:*

$$\begin{aligned}\mathbb{P}(|Z| \geq x) &= \frac{2}{\sqrt{2\pi}} \int_x^{+\infty} \exp\left(-\frac{t^2}{2}\right) dt \\ &\leq \frac{2}{\sqrt{2\pi}} \int_x^{+\infty} \frac{t}{x} \exp\left(-\frac{t^2}{2}\right) dt = \frac{2}{x\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).\end{aligned}$$

2. Establish the following equality:

$$\mathbb{E}\left[(X_{-1k} - \beta_{-1k})^2\right] = \frac{\sigma^2}{n}.$$

<u>Correction :</u> *It's obvious since the z_{jk}'s are i.i.d. $\mathcal{N}(0,1)$.*

3. Then prove

$$\mathbb{E}\left[\|\hat{f}-f\|^2\right]=\frac{\sigma^2}{n}\mathrm{card}(\mathcal{K}_{-1})+\sum_{j=0}^{J}\sum_{k\in\mathcal{K}_j}\mathbb{E}\left[(\hat{\beta}_{jk}-\beta_{jk})^2\right]+\sum_{j>J}\sum_{k\in\mathcal{K}_j}\beta_{jk}^2. \tag{A.7}$$

Correction : *We have:*

$$\begin{aligned}\mathbb{E}\left[\|\hat{f}-f\|^2\right]&=\mathbb{E}\left[\left\|\sum_{j=-1}^{+\infty}\sum_{k\in\mathcal{K}_j}\beta_{jk}\psi_{jk}-\sum_{j=-1}^{J}\sum_{k\in\mathcal{K}_j}\hat{\beta}_{jk}\psi_{jk}\right\|^2\right]\\&=\mathbb{E}\left[\left\|\sum_{j=-1}^{J}\sum_{k\in\mathcal{K}_j}(\beta_{jk}-\hat{\beta}_{jk})\psi_{jk}+\sum_{j>J}\sum_{k\in\mathcal{K}_j}\beta_{jk}\psi_{jk}\right\|^2\right].\end{aligned}$$

By using Parseval's identity,

$$\begin{aligned}\mathbb{E}\left[\|\hat{f}-f\|^2\right]&=\sum_{k\in\mathcal{K}_{-1}}\mathbb{E}\left[(X_{-1k}-\beta_{-1k})^2\right]+\sum_{j=0}^{J}\sum_{k\in\mathcal{K}_j}\mathbb{E}\left[(\hat{\beta}_{jk}-\beta_{jk})^2\right]+\sum_{j>J}\sum_{k\in\mathcal{K}_j}\beta_{jk}^2\\&=\frac{\sigma^2}{n}card(\mathcal{K}_{-1})+\sum_{j=0}^{J}\sum_{k\in\mathcal{K}_j}\mathbb{E}\left[(\hat{\beta}_{jk}-\beta_{jk})^2\right]+\sum_{j>J}\sum_{k\in\mathcal{K}_j}\beta_{jk}^2.\end{aligned}$$

4. In the sequel, we fix $j\in\{0,\dots,J\}$ and $k\in\mathcal{K}_j$. We wish to provide a control of $\mathbb{E}\left[(\hat{\beta}_{jk}-\beta_{jk})^2\right]$.

   (a) For this purpose, first prove that

   $$\mathbb{E}\left[(\hat{\beta}_{jk}-\beta_{jk})^2\right]\leq A+B+C+D,$$

   with

   $$A:=\mathbb{E}\left[(X_{jk}-\beta_{jk})^2 1_{\left\{|X_{jk}-\beta_{jk}|>\frac{\eta_{jk}}{2}\right\}}\right],\quad B:=\beta_{jk}^2 1_{\left\{|\beta_{jk}|\leq 2\eta_{jk}\right\}},$$

   $$C:=\mathbb{E}\left[(X_{jk}-\beta_{jk})^2 1_{\left\{|\beta_{jk}|>\frac{\eta_{jk}}{2}\right\}}\right],\quad D:=\mathbb{E}\left[\beta_{jk}^2 1_{\left\{|X_{jk}-\beta_{jk}|>\eta_{jk}\right\}}\right].$$

   Indication: *Distinguish cases according to whether $|X_{jk}|$ is larger than η_{jk} or not, and whether $|\beta_{jk}|$ is large or not.*

<u>Correction :</u>

$$\begin{aligned}\mathbb{E}\left[(\hat{\beta}_{jk}-\beta_{jk})^2\right] &= \mathbb{E}\left[(X_{jk}-\beta_{jk})^2 1_{\{|X_{jk}|>\eta_{jk}\}}\right] + \mathbb{E}\left[\beta_{jk}^2 1_{\{|X_{jk}|\le\eta_{jk}\}}\right]\\ &= \mathbb{E}\left[(X_{jk}-\beta_{jk})^2 1_{\{|X_{jk}|>\eta_{jk}\}} 1_{\{|\beta_{jk}|\le\frac{\eta_{jk}}{2}\}}\right] + \mathbb{E}\left[\beta_{jk}^2 1_{\{|X_{jk}|\le\eta_{jk}\}} 1_{\{|\beta_{jk}|\le 2\eta_{jk}\}}\right]\\ &+ \mathbb{E}\left[(X_{jk}-\beta_{jk})^2 1_{\{|X_{jk}|>\eta_{jk}\}} 1_{\{|\beta_{jk}|>\frac{\eta_{jk}}{2}\}}\right] + \mathbb{E}\left[\beta_{jk}^2 1_{\{|X_{jk}|\le\eta_{jk}\}} 1_{\{|\beta_{jk}|> 2\eta_{jk}\}}\right]\\ &\le A+B+C+D.\end{aligned}$$

(b) By using the definition of η_{jk}, establish that

$$D \le \beta_{jk}^2 \times \frac{1}{\sqrt{\pi}\sqrt{\gamma\log n}} n^{-\gamma}.$$

<u>Correction :</u> *We use for any $x>0$,*

$$\mathbb{P}(|z_{jk}|\ge x) \le \frac{2}{x\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right).$$

We have:

$$\begin{aligned}D &= \beta_{jk}^2 \times \mathbb{P}(|X_{jk}-\beta_{jk}|>\eta_{jk})\\ &\le \beta_{jk}^2 \times \mathbb{P}(|z_{jk}|>\sqrt{2\gamma\log n})\\ &\le \beta_{jk}^2 \times \frac{2}{\sqrt{2\pi}\sqrt{2\gamma\log n}} n^{-\gamma}\\ &= \beta_{jk}^2 \times \frac{1}{\sqrt{\pi}\sqrt{\gamma\log n}} n^{-\gamma}.\end{aligned}$$

(c) By using $\mathbb{E}[z_{jk}^4]=3$, prove that

$$A \le \frac{C_A\sigma^2}{n}(\gamma\log n)^{-\frac14} n^{-\frac{\gamma}{8}},$$

with C_A a numerical constant.

<u>Correction :</u> *Since*

$$A = \mathbb{E}\left[(X_{jk}-\beta_{jk})^2 1_{\{|X_{jk}-\beta_{jk}|>\frac{\eta_{jk}}{2}\}}\right],$$

the Cauchy-Schwarz inequality gives

$$\begin{aligned}A^2 &\le \mathbb{E}\left[(X_{jk}-\beta_{jk})^4\right] \times \mathbb{P}\left(|X_{jk}-\beta_{jk}|>\frac{\eta_{jk}}{2}\right)\\ &\le \frac{3\sigma^4}{n^2} \times \mathbb{P}(|z_{jk}|>\sqrt{2^{-1}\gamma\log n})\\ &\le \frac{3\sigma^4}{n^2} \times \frac{2}{\sqrt{2\pi}\sqrt{2^{-1}\gamma\log n}} n^{-\frac{\gamma}{4}}\end{aligned}$$

and

$$A \leq \frac{C_A \sigma^2}{n} (\gamma \log n)^{-\frac{1}{4}} n^{-\frac{\gamma}{8}},$$

with

$$C_A^2 = \frac{6}{\sqrt{\pi}}.$$

(d) Prove that for n large enough

$$B \leq 8\gamma \min\left(\frac{\sigma^2 \log n}{n}, \beta_{jk}^2\right).$$

Correction : *We have that*

$$B = \beta_{jk}^2 1_{\{|\beta_{jk}| \leq 2\eta_{jk}\}}$$

and

$$\eta_{jk} = \sigma \sqrt{\frac{2\gamma \log n}{n}}.$$

If $\frac{\sigma^2 \log n}{n} \geq \beta_{jk}^2$, *the inequality is obvious since* $8\gamma > 1$. *Otherwise,* $\frac{\sigma^2 \log n}{n} < \beta_{jk}^2$, *but*

$$B \leq \beta_{jk}^2 \times \frac{4\eta_{jk}^2}{\beta_{jk}^2} = 8\gamma \times \frac{\sigma^2 \log n}{n}.$$

(e) Prove that

$$C = \frac{\sigma^2}{n} 1_{\{|\beta_{jk}| > \frac{\eta_{jk}}{2}\}}$$

and that for n large enough

$$C \leq \min\left(\frac{\sigma^2 \log n}{n}, \beta_{jk}^2\right).$$

Correction : *The first point is obvious. So,*

$$C = \frac{\sigma^2}{n} 1_{\{|\beta_{jk}| > \frac{\eta_{jk}}{2}\}}$$

and

$$\eta_{jk} = \sigma \sqrt{\frac{2\gamma \log n}{n}}.$$

If $\frac{\sigma^2 \log n}{n} \leq \beta_{jk}^2$, *the inequality is obvious when* $\log n \geq 1$. *Otherwise,* $\frac{\sigma^2 \log n}{n} > \beta_{jk}^2$, *but*

$$C \leq \frac{\sigma^2}{n} \times \frac{4\beta_{jk}^2}{\eta_{jk}^2} = \frac{2\beta_{jk}^2}{\gamma \log n} \leq \beta_{jk}^2,$$

when n is large enough so that $\gamma \log n \geq 2$.

5. Finally, conclude that for n large enough,

$$\mathbb{E}\left[\|\hat{f}-f\|^2\right] \leq \frac{\sigma^2}{n}\mathrm{card}(\mathcal{K}_{-1}) + C_1\sum_{j=0}^{J}\sum_{k\in\mathcal{K}_j}\min\left(\frac{\sigma^2\log n}{n},\beta_{jk}^2\right) + \sum_{j>J}\sum_{k\in\mathcal{K}_j}\beta_{jk}^2 + \frac{C_2}{n},$$

where C_1 and C_2 are two constants depending on $\|f\|$, γ and σ^2.

Correction : *Using previous inequalities, for n large enough,*

$$\mathbb{E}\left[(\hat{\beta}_{jk}-\beta_{jk})^2\right] \leq \beta_{jk}^2 \times \frac{1}{\sqrt{\pi}\sqrt{\gamma\log n}}n^{-\gamma} + \frac{C_A\sigma^2}{n}(\gamma\log n)^{-\frac{1}{4}}n^{-\frac{\gamma}{8}} + (8\gamma+1)\min\left(\frac{\sigma^2\log n}{n},\beta_{jk}^2\right).$$

We have

$$\sum_{j=0}^{J}\sum_{k\in\mathcal{K}_j}\beta_{jk}^2 \times \frac{1}{\sqrt{\pi}\sqrt{\gamma\log n}}n^{-\gamma} \leq \frac{\|f\|^2}{\sqrt{\pi}\sqrt{\gamma\log n}}n^{-\gamma}$$

and

$$\sum_{j=0}^{J}\sum_{k\in\mathcal{K}_j}\frac{C_A\sigma^2}{n}(\gamma\log n)^{-\frac{1}{4}}n^{-\frac{\gamma}{8}} \leq \frac{C_A\sigma^2}{n}(\gamma\log n)^{-\frac{1}{4}}\,card(\Gamma)n^{-\frac{\gamma}{8}} \leq \frac{C_A\sigma^2}{n}(\gamma\log n)^{-\frac{1}{4}}.$$

Since γ is larger than 1, we obtain the result.

A.5 Exam 2020-2021

High-dimensional statistics

EXAM: duration 2h30

Documents, calculators, phones and smartphones are forbidden

Exercise 1

For $n \in \mathbb{N}^*$ and $\sigma > 0$, we consider the following statistical model

$$X = \beta^* + \sigma Z,$$

with $X = (X_1, \dots, X_n)^T \in \mathbb{R}^n$ and $Z = (Z_1, \dots, Z_n)^T \in \mathbb{R}^n$ so that the Z_i's are i.i.d with common distribution $\mathcal{N}(0, 1)$. The goal is to estimate $\beta^* = (\beta_1^*, \dots, \beta_n^*)^T \in \mathbb{R}^n$ by using the observation X. We denote $\|\cdot\|$ the classical ℓ_2-norm.

1. (a) We consider $\hat{\beta}^1$ the penalized estimate defined by

$$\hat{\beta}^1 \in \arg\min_{\beta \in \mathbb{R}^n} \left\{ \|X - \beta\|^2 + 2\lambda \text{pen}(\beta) \right\}$$

with $\lambda > 0$ and where the penalty pen : $\mathbb{R}^n \mapsto \mathbb{R}_+$ is a function depending only on β. Write the penalty pen corresponding to the Lasso estimate.

Correction : *$pen(\beta) = \|\beta\|_1 = \sum_{i=1}^n |\beta_i|$.*

(b) Establish that for any $i \in \{1, \dots, n\}$, $\hat{\beta}_i^1$, the ith coordinate of $\hat{\beta}^1$, is obtained by minimizing the function

$$\Theta : t \in \mathbb{R} \mapsto t^2 - 2tX_i + 2\lambda|t|.$$

Correction : *We have:*

$$\|X - \beta\|^2 + 2\lambda pen(\beta) = \sum_{i=1}^n \left[\beta_i^2 - 2\beta_i X_i + X_i^2 + 2\lambda|\beta_i| \right]$$

and minimization is obtained coordinatewise.

(c) Prove that for any $i \in \{1, \ldots, n\}$,

$$\hat{\beta}_i^1 = \mathrm{sign}(X_i)(|X_i| - \lambda)_+,$$

where $\mathrm{sign}(X_i) \in \{+1, -1\}$ denotes the sign of X_i and $(|X_i| - \lambda)_+ = \max(|X_i| - \lambda; 0)$.

Correction : *On* $\mathbb{R}_+$, $\Theta(t) = t^2 - 2tX_i + 2\lambda t$ *and*

$$\arg\min_{t \in \mathbb{R}_+} \Theta(t) = \max(X_i - \lambda; 0)$$

On $\mathbb{R}_-$, $\Theta(t) = t^2 - 2tX_i - 2\lambda t$ *and*

$$\arg\min_{t \in \mathbb{R}_-} \Theta(t) = \min(X_i + \lambda; 0)$$

- *If* $X_i \geq \lambda$,

$$\Theta(X_i - \lambda) = X_i^2 + \lambda^2 - 2\lambda X_i - 2(X_i - \lambda)X_i + 2\lambda(X_i - \lambda) = -X_i^2 - \lambda^2 + 2\lambda X_i = -(X_i - \lambda)^2 \leq 0.$$

- *If* $X_i \in [-\lambda; \lambda]$, $\Theta(0) = 0$.
- *If* $X_i \leq -\lambda$,

$$\Theta(X_i + \lambda) = X_i^2 + \lambda^2 + 2\lambda X_i - 2(X_i + \lambda)X_i - 2\lambda(X_i + \lambda) = -X_i^2 - \lambda^2 - 2\lambda X_i = -(X_i + \lambda)^2 \leq 0.$$

Finally,

$$\hat{\beta}_i^1 = \left\{ \begin{array}{ccc} X_i - \lambda & \text{if} & X_i \geq \lambda \\ 0 & \text{if} & X_i \in [-\lambda; \lambda] \\ X_i + \lambda & \text{if} & X_i \leq -\lambda \end{array} \right\} = \mathit{sign}(X_i)(|X_i| - \lambda)_+.$$

2. In the sequel, for $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ a measurable function, we denote $\hat{\beta} = F(X)$ an estimate of β^*. We denote $(F_1, \ldots, F_n)$ the $\mathbb{R}$-valued components of F that are assumed to be C^1, so that $F(X) = (F_1(X), \ldots, F_n(X))$. We consider $g : \mathbb{R} \mapsto \mathbb{R}$ assumed to be C^1 such that $\mathbb{E}[|g'(Z_1)|] < \infty$. We denote ϕ the density of Z_1.

   (a) Prove that for any $t \in \mathbb{R}$,

$$\phi'(t) = -t\phi(t).$$

Correction : *The density of* Z_1 *is* $\phi(t) = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)$. *Therefore*

$$\phi'(t) = -t \times \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) = -t\phi(t).$$

(b) Establish that

$$\mathbb{E}[g'(Z_1)] = \int_0^{+\infty} g'(x) \left(\int_x^{+\infty} t\phi(t)dt \right) dx - \int_{-\infty}^0 g'(x) \left(\int_{-\infty}^x t\phi(t)dt \right) dx.$$

Indication : Use $\mathbb{E}[g'(Z_1)] = \mathbb{E}[g'(Z_1)1_{\{Z_1>0\}}] + \mathbb{E}[g'(Z_1)1_{\{Z_1<0\}}]$.

Correction :

$$\begin{aligned}
\mathbb{E}[g'(Z_1)] &= \mathbb{E}[g'(Z_1)1_{\{Z_1>0\}}] + \mathbb{E}[g'(Z_1)1_{\{Z_1<0\}}] \\
&= \int_0^{+\infty} g'(x)\phi(x)dx + \int_{-\infty}^0 g'(x)\phi(x)dx \\
&= \int_0^{+\infty} g'(x) \left(\int_x^{+\infty} (-\phi'(t))dt \right) dx + \int_{-\infty}^0 g'(x) \left(\int_{-\infty}^x \phi'(t)dt \right) dx \\
&= \int_0^{+\infty} g'(x) \left(\int_x^{+\infty} t\phi(t)dt \right) dx - \int_{-\infty}^0 g'(x) \left(\int_{-\infty}^x t\phi(t)dt \right) dx.
\end{aligned}$$

(c) Conclude that

$$\mathbb{E}[g'(Z_1)] = \mathbb{E}[Z_1 g(Z_1)]. \tag{A.8}$$

Correction : *Fubini's theorem holds since*

$$\begin{aligned}
\int_0^{+\infty} |g'(x)| \int_0^{+\infty} t\phi(t)1_{\{t>x\}}dtdx &= -\int_0^{+\infty} |g'(x)| \int_0^{+\infty} \phi'(t)1_{\{t>x\}}dtdx \\
&\le \mathbb{E}[|g'(Z_1)|] < \infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{-\infty}^0 |g'(x)| \left(\int_{-\infty}^0 |t|\phi(t)1_{\{t<x\}}dt \right) dx &= \int_{-\infty}^0 |g'(x)| \int_{-\infty}^0 \phi'(t)1_{\{t<x\}}dtdx \\
&\le \mathbb{E}[|g'(Z_1)|] < \infty.
\end{aligned}$$

Therefore, since

$$\int_{-\infty}^{+\infty} t\phi(t)dt = \mathbb{E}[Z_1] = 0,$$

we have

$$\begin{aligned}
\mathbb{E}[g'(Z_1)] &= \int_0^{+\infty} t\phi(t) \left(\int_0^t g'(x)dx \right) dt - \int_{-\infty}^0 t\phi(t) \left(\int_t^0 g'(x)dx \right) dt \\
&= \int_0^{+\infty} t\phi(t)(g(t) - g(0))dt - \int_{-\infty}^0 t\phi(t)(g(0) - g(t))dt \qquad \text{(A.9)} \\
&= \int_{-\infty}^{+\infty} t\phi(t)g(t)dt = \mathbb{E}[Z_1 g(Z_1)].
\end{aligned}$$

3. In the sequel, we fix $i \in \{1,\dots,n\}$ and introduce $G_i : \mathbb{R}^n \mapsto \mathbb{R}$ such that for any $u \in \mathbb{R}^n$,

$$G_i(u) = F_i(\beta^* + \sigma u)$$

and $X^{(-i)}$ the vector of $\mathbb{R}^{n-1}$ built from X by removing its ith component:

$$X^{(-i)} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n).$$

(a) Prove that the ith partial derivative of F_i satisfies

$$\frac{\partial F_i}{\partial x_i}(X) = \frac{1}{\sigma}\frac{\partial G_i}{\partial x_i}(Z).$$

Correction : *We have, with $u = (u_1, \dots, u_n)^T$,*

$$F_i(u) = G_i\left(\frac{u_1 - \beta_1^*}{\sigma}, \dots, \frac{u_n - \beta_n^*}{\sigma}\right),$$

which implies

$$\frac{\partial F_i}{\partial x_i}(u) = \frac{1}{\sigma}\frac{\partial G_i}{\partial x_i}\left(\frac{u_1 - \beta_1^*}{\sigma}, \dots, \frac{u_n - \beta_n^*}{\sigma}\right).$$

With $u = X$, we obtain:

$$\frac{\partial F_i}{\partial x_i}(X) = \frac{1}{\sigma}\frac{\partial G_i}{\partial x_i}(Z).$$

(b) By using (A.8), deduce that

$$\mathbb{E}\left[\frac{\partial F_i}{\partial x_i}(X) \mid X^{(-i)}\right] = \frac{1}{\sigma}\mathbb{E}\big[Z_i G_i(Z) \mid X^{(-i)}\big].$$

Correction : *We have $X^{(-i)} = \beta^{*(-i)} + \sigma Z^{(-i)}$. So, by using* (A.8),

$$\begin{aligned}\mathbb{E}\left[\frac{\partial F_i}{\partial x_i}(X) \mid X^{(-i)}\right] &= \frac{1}{\sigma}\mathbb{E}\left[\frac{\partial G_i}{\partial x_i}(Z) \mid Z^{(-i)}\right] \\ &= \frac{1}{\sigma}\mathbb{E}\big[Z_i G_i(Z_1, \dots, Z_{i-1}, Z_i, Z_{i+1}, \dots, Z_n) \mid Z^{(-i)}\big] \\ &= \frac{1}{\sigma}\mathbb{E}\big[Z_i G_i(Z) \mid X^{(-i)}\big].\end{aligned}$$

(c) Conclude that

$$\mathbb{E}\left[\frac{\partial F_i}{\partial x_i}(X)\right] = \frac{1}{\sigma^2}\mathbb{E}\big[(X_i - \beta_i^*)F_i(X)\big].$$

Correction : *Finally,*

$$\begin{aligned}\mathbb{E}\left[\frac{\partial F_i}{\partial x_i}(X)\right] &= \mathbb{E}\left[\mathbb{E}\left[\frac{\partial F_i}{\partial x_i}(X) \mid X^{(-i)}\right]\right] \\ &= \mathbb{E}\left[\frac{1}{\sigma}\mathbb{E}\big[Z_i G_i(Z_1, \dots, Z_{i-1}, Z_i, Z_{i+1}, \dots, Z_n) \mid (X^{(-i)}\big]\right] \\ &= \frac{1}{\sigma^2}\mathbb{E}\left[\mathbb{E}\big[(X_i - \beta_i^*)F_i(X) \mid (X^{(-i)}\big]\right] \\ &= \frac{1}{\sigma^2}\mathbb{E}\big[(X_i - \beta_i^*)F_i(X)\big].\end{aligned}$$

4. Let

$$C = \|X - F(X)\|^2 + 2\sigma^2 \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}(X) - n\sigma^2.$$

Prove that

$$\mathbb{E}[C] = \mathbb{E}[\|F(X) - \beta^*\|^2].$$

Correction : *We have*

$$\begin{aligned}\mathbb{E}[C] &= \mathbb{E}[\|X - F(X)\|^2] + 2\sigma^2\mathbb{E}\left[\sum_{i=1}^n \frac{\partial F_i}{\partial x_i}(X)\right] - n\sigma^2 \\ &= \mathbb{E}[\|X - \beta^*\|^2] + \mathbb{E}[\|F(X) - \beta^*\|^2] + 2\sum_{i=1}^n \mathbb{E}[(X_i - \beta_i^*)(\beta_i^* - F_i(X))] + 2\sigma^2\mathbb{E}\left[\sum_{i=1}^n \frac{\partial F_i}{\partial x_i}(X)\right] - \\ &= \mathbb{E}[\|F(X) - \beta^*\|^2] - 2\sum_{i=1}^n \mathbb{E}[(X_i - \beta_i^*)F_i(X)] + 2\sigma^2\mathbb{E}\left[\sum_{i=1}^n \frac{\partial F_i}{\partial x_i}(X)\right],\end{aligned}$$

where we have used $\mathbb{E}[X_i - \beta_i^*] = 0$ *and*

$$\mathbb{E}[\|X - \beta^*\|^2] = \sigma^2\mathbb{E}[\|Z\|^2] = \sigma^2\sum_{i=1}^n \mathbb{E}[Z_i^2] = n\sigma^2.$$

Using the previous question, we have

$$\mathbb{E}[C] = \mathbb{E}[\|F(X) - \beta^*\|^2].$$

5. We now assume that the estimate $F(X)$ depends on a hyperparameter λ; we write $F(X) \equiv F^\lambda(X)$.

   (a) Deduce a method to select the hyperparameter λ.
   Correction : *Since C is an unbiased estimate of $\mathbb{E}[\|F(X)-\beta^*\|^2]$, we naturally minimize the function*

   $$\lambda \mapsto \|X - F^\lambda(X)\|^2 + 2\sigma^2 \sum_{i=1}^n \frac{\partial F_i^\lambda}{\partial x_i}(X)$$

   (b) We now assume that for all $i \in \{1,\dots,n\}$, $F_i^\lambda(X)$ only depends on X_i and there exists H_i^λ such that for any $x \in \mathbb{R}$,

   $$F_i^\lambda(x) = \int_0^x H_i^\lambda(t)dt.$$

   Prove that

   $$\tilde{C}^\lambda = \|X - F^\lambda(X)\|^2 + 2\sigma^2 \sum_{i=1}^n H_i^\lambda(X_i) - n\sigma^2$$

   satisfies

   $$\mathbb{E}[\tilde{C}^\lambda] = \mathbb{E}[\|F^\lambda(X) - \beta^*\|^2].$$

   Correction : *We assume that F_i^λ is absolutely continuous on $\mathbb{R}$: there exists H_i^λ such that for any $x \in \mathbb{R}$,*

   $$F_i^\lambda(x) = \int_0^x H_i^\lambda(t)dt.$$

   We can check that all previous computations hold by replacing $\frac{\partial F_i^\lambda}{\partial x_i}(X)$ by H_i^λ (see (A.9)*).*

   (c) We consider the soft thresholding rule and estimate each coordinate β_i^* by $F_i^\lambda(X) = \mathrm{sign}(X_i)(|X_i| - \lambda)_+$. Determine a good criterion to select λ.
   Correction : *We set*

   $$H_i^\lambda(x) = 1_{\{|x|>\lambda\}}, \quad x \in \mathbb{R}.$$

   By distinguishing the cases $x > 0$ and $x < 0$, we obtain

   $$F_i^\lambda(x) = \int_0^x H_i^\lambda(t)dt.$$

   We obtain the SURE criterion (Stein Unbiased Risk Estimate criterion):

   $$\tilde{C}^\lambda = \|X - F^\lambda(X)\|^2 + 2\sigma^2 card\{i : \ |X_i| > \lambda\} - n\sigma^2.$$

Exercise 2

In the sequel, we denote

$$\mathbb{L}_2(\mathbb{R}) := \left\{ f : \mathbb{R} \mapsto \mathbb{C} : \quad \|f\|_2^2 := \int_{\mathbb{R}} |f(t)|^2 dt < \infty \right\}$$

endowed with the Euclidian scalar product:

$$\langle f, g \rangle := \int_{\mathbb{R}} f(t)\overline{g(t)} dt.$$

For any $f \in \mathbb{L}_2(\mathbb{R})$, we denote $\widehat{f}$ its Fourier transform:

$$\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-it\xi} f(t) dt, \quad \xi \in \mathbb{R}.$$

We recall the inversion formula:

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\xi} \widehat{f}(\xi) d\xi, \quad t \in \mathbb{R}$$

and the Plancherel formula:

$$\int_{\mathbb{R}} f(t)\overline{g(t)} dt = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi, \quad \int_{\mathbb{R}} |f(t)|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi.$$

We recall that a multiresolution analysis $V = (V_j)_{j \in \mathbb{Z}}$ is a sequence of nested vector spaces satisfying

$$\{0\} \subset \cdots \subset V_{j+1} \subset V_j \subset V_{j-1} \subset \cdots \subset \mathbb{L}_2(\mathbb{R})$$

such that, for any $j \in \mathbb{Z}$, if P_{V_j} is the orthogonal projection on V_j, for any $f \in \mathbb{L}_2(\mathbb{R})$,

1. $\|P_{V_j} f - f\|_2 \overset{j \to -\infty}{\longrightarrow} 0$
2. $\|P_{V_j} f\|_2 \overset{j \to +\infty}{\longrightarrow} 0$
3. $f \in V_j \iff x \mapsto f(x/2) \in V_{j+1}$ for any $j \in \mathbb{Z}$
4. $f \in V_j \iff x \mapsto f(x + 2^j k) \in V_j$ for any $(j, k) \in \mathbb{Z}^2$
5. $\exists\, \phi$ such that $(\phi_k)_{k \in \mathbb{Z}}$ is an orthonormal basis of V_0 with for any $x \in \mathbb{R}$, $\phi_k(x) = \phi(x - k)$.

In the sequel, we set

$$\phi(t) = \frac{\sin(\pi t)}{\pi t}, \quad t \in \mathbb{R}.$$

The goal is to prove that the sequence of vector spaces $(V_j)_{j\in\mathbb{Z}}$, defined by

$$V_j = \left\{ f \in \mathbb{L}_2(\mathbb{R}) : \quad \mathrm{supp}(\widehat{f}) \subset [-2^{-j}\pi, 2^{-j}\pi) \right\}, \quad j \in \mathbb{Z},$$

is a multiresolution analysis associated with ϕ.

1. Establish that

$$\widehat{\phi}(\xi) = 1_{[-\pi,\pi)}(\xi), \quad \xi \in \mathbb{R}.$$

Indication : Use the Fourier inversion formula.

Correction : *We have for any $t \in \mathbb{R}$:*

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{it\xi} 1_{[-\pi,\pi)}(\xi) d\xi = \frac{1}{2\pi i t}(e^{it\pi} - e^{-it\pi}) = \frac{\sin(\pi t)}{\pi t}.$$

This provides the result.

2. We consider $f \in V_0$ and we set g such that $\widehat{g}$ is 2π-periodic and

$$\widehat{g}(\xi) := \widehat{f}(\xi), \quad \xi \in [-\pi, \pi).$$

(a) By using that any 2π-periodic function can be decomposed on $(\xi \mapsto e^{ik\xi})_{k\in\mathbb{Z}}$, prove that there exists $(a_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ such that for any $\xi \in \mathbb{R}$,

$$\widehat{g}(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}.$$

Correction : *Since $\widehat{g}$ is 2π-periodic, this is obvious.*

(b) By computing $\widehat{\phi}_k(\xi)$ for any $k \in \mathbb{Z}$ and any $\xi \in \mathbb{R}$, deduce that

$$f(t) = \sum_{k\in\mathbb{Z}} a_k \phi_k(t), \quad t \in \mathbb{R}.$$

Correction : *We have for any $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$,*

$$\widehat{\phi}_k(\xi) = \int_{\mathbb{R}} e^{-it\xi}\phi(t-k)dt = e^{-ik\xi}\int_{\mathbb{R}} e^{-it\xi}\phi(t)dt = e^{-ik\xi} 1_{[-\pi,\pi)}(\xi).$$

Then, we have

$$\begin{aligned}\widehat{f}(\xi) &= \widehat{g}(\xi)1_{[-\pi,\pi)}(\xi)\\ &= \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}1_{[-\pi,\pi)}(\xi)\\ &= \sum_{k\in\mathbb{Z}} a_k \widehat{\phi}_k(\xi).\end{aligned}$$

This gives the result.

3. We study projections on the spaces $(V_j)_{j\in\mathbb{Z}}$.

   (a) Let $j \in \mathbb{Z}$. Prove that for any $f \in \mathbb{L}_2(\mathbb{R})$, the projection of f on V_j is given by the function f_j such that for any $\xi \in \mathbb{R}$,

   $$\widehat{f_j}(\xi) := \widehat{f}(\xi) \times 1_{[-2^{-j}\pi, 2^{-j}\pi)}(\xi).$$

   Correction : *We obviously have that $f_j \in V_j$. Furthermore, for any $g_j \in V_j$ then $\widehat{g_j}$ is supported by $[-2^{-j}\pi, 2^{-j}\pi)$, we have that*

   $$\langle f - f_j, g_j\rangle = \int (f(t) - f_j(t))\overline{g_j(t)}dt = \frac{1}{2\pi}\int_{\mathbb{R}} (\widehat{f}(\xi) - \widehat{f_j}(\xi))\overline{\widehat{g_j}(\xi)}d\xi = 0.$$

   This gives the result.

   (b) Let $f \in \mathbb{L}_2(\mathbb{R})$. Prove that $\|P_{V_j}f - f\|_2 \overset{j\to-\infty}{\longrightarrow} 0$.
   Correction : *We have:*

   $$\|P_{V_j}f - f\|_2^2 = \frac{1}{2\pi}\|\widehat{f_j} - \widehat{f}\|_2^2 = \frac{1}{2\pi}\int_{\xi\notin[-2^{-j}\pi, 2^{-j}\pi)} |\widehat{f}(\xi)|^2 d\xi.$$

   Since $\widehat{f} \in \mathbb{L}_2(\mathbb{R})$, the right hand side goes to 0 when $j \to -\infty$.

   (c) Let $f \in \mathbb{L}_2(\mathbb{R})$. Prove that $\|P_{V_j}f\|_2 \overset{j\to+\infty}{\longrightarrow} 0$.
   Correction : *We have:*

   $$\|P_{V_j}f\|_2^2 = \frac{1}{2\pi}\|\widehat{f_j}\|_2^2 = \frac{1}{2\pi}\int_{\xi\in[-2^{-j}\pi, 2^{-j}\pi)} |\widehat{f}(\xi)|^2 d\xi.$$

   Since $\widehat{f} \in \mathbb{L}_2(\mathbb{R})$, the right hand side goes to 0 when $j \to +\infty$.

4. Establish that $(V_j)_{j\in\mathbb{Z}}$ is a multiresolution analysis.

   Correction :

- *Previous questions give Conditions 1 and 2.*
- *We prove Condition 5. We have that for any* $(k,k')\in\mathbb{Z}^2$,

$$\int_{\mathbb{R}}\phi_k(t)\phi_{k'}(t)dt=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}_k(\xi)\overline{\hat{\phi}_{k'}(\xi)}d\xi=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(k'-k)\xi}d\xi=1_{\{k=k'\}},$$

by using the first part of Question 2b). This result, combined with the second part of Question 2b), shows that Condition 5 is satisfied.

- *We prove Condition 3. Let* $f\in\mathbb{L}_2(\mathbb{R})$. *We denote* $g: x\mapsto f(x/2)$. *We have for* $\xi\in\mathbb{R}$,

$$\widehat{g}(\xi)=\int_{\mathbb{R}}f(x/2)e^{-ix\xi}dx=2\int_{\mathbb{R}}f(t)e^{-2it\xi}dt=2\widehat{f}(2\xi)$$

and for any $j\in\mathbb{Z}$,

$$supp(\widehat{f})\subset[-2^{-j}\pi,2^{-j}\pi)\iff supp(\widehat{g})\subset[-2^{-j-1}\pi,2^{-j-1}\pi)$$

meaning that

$$f\in V_j\iff x\mapsto f(x/2)\in V_{j+1}.$$

- *We prove Condition 4. Let* $f\in\mathbb{L}_2(\mathbb{R})$ *and* $(j,k)\in\mathbb{Z}^2$. *We denote* $g: x\mapsto f(x+2^jk)$. *We have for* $\xi\in\mathbb{R}$,

$$\widehat{g}(\xi)=\int_{\mathbb{R}}f(x+2^jk)e^{-ix\xi}dx=e^{i2^jk\xi}\int_{\mathbb{R}}f(t)e^{-it\xi}dt=e^{i2^jk\xi}\widehat{f}(\xi)$$

and

$$supp(\widehat{f})=supp(\widehat{g})$$

meaning that

$$f\in V_j\iff g\in V_j.$$

A.6 Exam 2021-2022

High-dimensional statistics

EXAM: duration 3h00

Documents, calculators, phones and smartphones are forbidden

Problem 1: Group-Lasso estimation

We consider the multivariate linear regression Gaussian model :

$$Y = X\beta^* + \varepsilon \tag{A.10}$$

with $Y = (Y_1, \dots, Y_n)^T$ the vector of observations. The matrix X of size $n \times p$ is assumed to be known and $\beta^* \in \mathbb{R}^p$ is the vector to be estimated. Finally, the error vector is $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^T$ and satisfies $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$ with $\sigma^2 > 0$ unknown. We denote $(X_1, \dots, X_p)$ the columns of X. The ℓ_2-norm is denoted $\|\cdot\|$. For any matrix A, we denote A^T its transpose matrix and for any estimate $\hat{\beta}$, $\mathrm{var}(\hat{\beta})$ denotes its variance-covariance matrix.

We choose $M \le p$ and let $G_1, \dots, G_M$ some sets so that $\{G_1, \dots, G_M\}$ form a known partition of $\{1, \dots, p\}$ in M sets. That is

$$\bigcup_{j=1}^{M} G_j = \{1, \dots, p\}, \quad G_j \cap G_{j'} = \emptyset, \text{ if } j \neq j'.$$

For any $j \in \{1, \dots, M\}$, we denote $K_j = |G_j|$ the cardinal of G_j and we denote by X_{G_j} the $n \times K_j$ sub-matrix of X formed by the columns indexed by G_j. Finally, for any $\beta \in \mathbb{R}^p$ and any $j \in \{1, \dots, p\}$, we introduce $\beta^{(j)}$ the vector of size K_j defined by

$$\beta^{(j)} = (\beta_k)_{k \in G_j}.$$

We also denote

$$S(\beta) = \big\{ j \in \{1, \dots, M\} : \ \beta^{(j)} \neq 0 \big\}.$$

To estimate $\beta^* = (\beta_1^*, \dots, \beta_p^*)^T \in \mathbb{R}^p$, we consider the Group-Lasso estimator defined by a solution of the following minimization problem

$$\widehat{\beta} \in \arg\min_{\beta \in \mathbb{R}^p} \left\{ \|Y - X\beta\|^2 + \sum_{j=1}^{M} \lambda_j \|\beta^{(j)}\| \right\}, \tag{A.11}$$

where $\lambda_1, \dots, \lambda_M$ are positive parameters specified later.

1. We study the criterion C associated with the minimization problem (A.11) and defined by

$$C : \beta \in \mathbb{R}^p \longmapsto \|Y - X\beta\|^2 + \sum_{j=1}^{M} \lambda_j \|\beta^{(j)}\|.$$

   (a) Show rigorously that the criterion C is convex.

   Correction : *Any (squared) norm is convex. So* $\beta \longmapsto \|Y - X\beta\|^2$ *is convex and for any* j, $\beta \longmapsto \|\beta^{(j)}\|$ *is convex. Since the* λ_j*'s are positive, the penalty is convex and the criterion* C *is convex (the sum of two convex functions is a convex function).*

   (b) Show that if $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are two solutions of the minimization problem (A.11), then

$$X\widehat{\beta}_1 = X\widehat{\beta}_2 \quad \text{and} \quad \sum_{j=1}^{M} \lambda_j \|\widehat{\beta}_1^{(j)}\| = \sum_{j=1}^{M} \lambda_j \|\widehat{\beta}_2^{(j)}\|.$$

   Correction : *We assume that* $X\widehat{\beta}_1 \neq X\widehat{\beta}_2$ *and we consider* $\widehat{\beta} = \frac{1}{2}(\widehat{\beta}_1 + \widehat{\beta}_2)$. *Then, since* $\beta \longmapsto \|\beta\|^2$ *is strictly convex and* $Y - X\widehat{\beta}_1 \neq Y - X\widehat{\beta}_2$,

$$\|Y - X\widehat{\beta}\|^2 = \left\|\frac{1}{2}(Y - X\widehat{\beta}_1) + \frac{1}{2}(Y - X\widehat{\beta}_2)\right\|^2 < \frac{1}{2}\left\|Y - X\widehat{\beta}_1\right\|^2 + \frac{1}{2}\left\|Y - X\widehat{\beta}_2\right\|^2.$$

   The penalty being convex, we obtain

$$C(\widehat{\beta}) < \frac{1}{2}\Big(C(\widehat{\beta}_1) + C(\widehat{\beta}_2)\Big) = C(\widehat{\beta}_1) = C(\widehat{\beta}_2),$$

   which cannot occur. So, $X\widehat{\beta}_1 = X\widehat{\beta}_2$. *The second point is an immediate consequence of this property.*

   (c) Give some conditions on the parameters of the problem so that $\widehat{\beta}$ is the unique solution of the minimization problem (A.11).

   Correction : *The minimization problem has a unique solution if the criterion to be minimized is strictly convex. This is true for instance if the rank of* X *is equal to* p.

   (d) Give the values of the parameters of the problem for which $\widehat{\beta}$ corresponds to the classical Lasso estimator.

   Correction : $M = p$ *and for any* $j \in \{1, \ldots, p\}$, $K_j = 1$, $\lambda_j = \lambda$ *for some* $\lambda > 0$.

   (e) Describe the specificity of the Group-Lasso estimator with respect to the classical Lasso estimator and provide a concrete example for which the Group-Lasso is of interest.

Correction : *For the Group-Lasso estimate, for any $j \in \{1, \dots, M\}$, either $\widehat{\beta}^{(j)} \neq 0$ and all coordinates of $\widehat{\beta}^{(j)}$ are different from 0 or $\widehat{\beta}^{(j)} = 0$ and all coordinates of $\widehat{\beta}^{(j)}$ are equal to 0.*
The Group-Lasso estimator is useful for instance if the vector Y is the response variable corresponding to a disease and each column of X models the influence of a specific gene on Y. All genes can be grouped according to their belonging to chromosones, and a gene has an impact on the disease if and only if the chromosone on which it is located has an impact on the disease.

2. We study the theoretical properties of $\widehat{\beta}$. We consider the event $\mathcal{A}$ so that

$$\mathcal{A} = \bigcap_{j=1}^{M} \mathcal{A}_j, \quad \mathcal{A}_j = \left\{ \|X_{G_j}^T \varepsilon\| \leq \frac{\lambda_j}{4} \right\}.$$

(a) Show rigorously that for any vectors a and b of $\mathbb{R}^p$,

$$a^T b \leq \sum_{j=1}^{M} \|a^{(j)}\| \|b^{(j)}\|.$$

Correction : *We have:*

$$a^T b = \sum_{k=1}^{p} a_k b_k = \sum_{j=1}^{M} \sum_{\ell \in G_j} a_\ell b_\ell$$
$$\leq \sum_{j=1}^{M} \sqrt{\sum_{\ell \in G_j} a_\ell^2} \sqrt{\sum_{\ell \in G_j} b_\ell^2} = \sum_{j=1}^{M} \|a^{(j)}\| \|b^{(j)}\|,$$

where the inequality comes from the Cauchy-Schwarz inequality.

(b) Deduce that for any $\beta \in \mathbb{R}^p$,

$$\|X(\widehat{\beta} - \beta^*)\|^2 \leq \|X(\beta - \beta^*)\|^2 + 2 \sum_{j=1}^{M} \|X_{G_j}^T \varepsilon\| \|\widehat{\beta}^{(j)} - \beta^{(j)}\| + \sum_{j=1}^{M} \lambda_j \|\beta^{(j)}\| - \sum_{j=1}^{M} \lambda_j \|\widehat{\beta}^{(j)}\|.$$

Correction : *By definition of $\widehat{\beta}$, we have:*

$$\|Y - X\widehat{\beta}\|^2 \leq \|Y - X\beta\|^2 + \sum_{j=1}^{M} \lambda_j \|\beta^{(j)}\| - \sum_{j=1}^{M} \lambda_j \|\widehat{\beta}^{(j)}\|,$$

which is equivalent to

$$\begin{aligned}\|X(\widehat{\beta}-\beta^*)\|^2 &\leq \|X(\beta-\beta^*)\|^2 + 2\varepsilon^T X(\widehat{\beta}-\beta) + \sum_{j=1}^M \lambda_j \|\beta^{(j)}\| - \sum_{j=1}^M \lambda_j \|\widehat{\beta}^{(j)}\| \\ &\leq \|X(\beta-\beta^*)\|^2 + 2\sum_{j=1}^M \|(\varepsilon^T X)^{(j)}\| \|\widehat{\beta}^{(j)} - \beta^{(j)}\| + \sum_{j=1}^M \lambda_j \|\beta^{(j)}\| - \sum_{j=1}^M \lambda_j \|\widehat{\beta}^{(j)}\|.\end{aligned}$$

We conclude by observing that for any $\ell \in G_j$,

$$(\varepsilon^T X)_\ell = \varepsilon^T X_\ell = X_\ell^T \varepsilon$$

and $(\varepsilon^T X)^{(j)} = X_{G_j}^T \varepsilon$.

(c) Finally, prove that on the event $\mathcal{A}$,

$$\|X(\widehat{\beta}-\beta^*)\|^2 + \frac{1}{2}\sum_{j=1}^M \lambda_j \|\widehat{\beta}^{(j)} - \beta^{(j)}\| \leq \|X(\beta-\beta^*)\|^2 + 2\sum_{j\in S(\beta)} \lambda_j \min\Big(\|\beta^{(j)}\|; \|\widehat{\beta}^{(j)} - \beta^{(j)}\|\Big). \tag{A.12}$$

Correction : *On the event $\mathcal{A}$,*

$$\|X(\widehat{\beta}-\beta^*)\|^2 + \frac{1}{2}\sum_{j=1}^M \lambda_j \|\widehat{\beta}^{(j)} - \beta^{(j)}\| \leq \|X(\beta-\beta^*)\|^2 + \sum_{j=1}^M \lambda_j \|\widehat{\beta}^{(j)} - \beta^{(j)}\| + \sum_{j=1}^M \lambda_j \Big(\|\beta^{(j)}\| - \|\widehat{\beta}^{(j)}\|\Big).$$

Then, for any $j \notin S(\beta)$,

$$\|\widehat{\beta}^{(j)} - \beta^{(j)}\| + \|\beta^{(j)}\| - \|\widehat{\beta}^{(j)}\| = \|\widehat{\beta}^{(j)}\| - \|\widehat{\beta}^{(j)}\| = 0.$$

For any $j \in S(\beta)$, on the one hand,

$$\|\widehat{\beta}^{(j)} - \beta^{(j)}\| + \|\beta^{(j)}\| - \|\widehat{\beta}^{(j)}\| \leq \|\widehat{\beta}^{(j)}\| + \|\beta^{(j)}\| + \|\beta^{(j)}\| - \|\widehat{\beta}^{(j)}\| = 2\|\beta^{(j)}\|,$$

and on the other hand,

$$\|\widehat{\beta}^{(j)} - \beta^{(j)}\| + \|\beta^{(j)}\| - \|\widehat{\beta}^{(j)}\| \leq 2\|\widehat{\beta}^{(j)} - \beta^{(j)}\|.$$

We obtain the desired result.

(d) Provide a bound on $\mathcal{A}$ of $\|X(\widehat{\beta}-\beta^*)\|^2$ only depending on $\sqrt{\sum_{j\in S(\beta^*)} \lambda_j^2}$ and $\|\beta^*\|$.

Correction : *In the previous expression, we take $\beta = \beta^*$ and we obtain*

$$\begin{aligned}
\|X(\widehat{\beta} - \beta^*)\|^2 &\leq \|X(\widehat{\beta} - \beta^*)\|^2 + \frac{1}{2}\sum_{j=1}^{M} \lambda_j \|\widehat{\beta}^{(j)} - \beta^{*(j)}\| \\
&\leq 2 \sum_{j \in S(\beta^*)} \lambda_j \min\left(\|\beta^{*(j)}\|; \|\widehat{\beta}^{(j)} - \beta^{*(j)}\|\right) \\
&\leq 2 \sum_{j \in S(\beta^*)} \lambda_j \|\beta^{*(j)}\| \\
&\leq 2\sqrt{\sum_{j \in S(\beta^*)} \lambda_j^2} \times \sqrt{\sum_{j \in S(\beta^*)} \|\beta^{*(j)}\|^2} \\
&= 2\|\beta^*\| \sqrt{\sum_{j \in S(\beta^*)} \lambda_j^2}.
\end{aligned}$$

3. We now assume that all eigenvalues of the symmetric matrix $X^T X$ are larger than a constant κ assumed to be positive.

   (a) Prove that

$$\sum_{j \in S(\beta)} \lambda_j \|\widehat{\beta}^{(j)} - \beta^{(j)}\| \leq \sqrt{\sum_{j \in S(\beta)} \lambda_j^2} \times \kappa^{-1/2} \|X(\widehat{\beta} - \beta)\|.$$

   Correction : *We have:*

$$\begin{aligned}
\sum_{j \in S(\beta)} \lambda_j \|\widehat{\beta}^{(j)} - \beta^{(j)}\| &\leq \sqrt{\sum_{j \in S(\beta)} \lambda_j^2} \times \sqrt{\sum_{j \in S(\beta)} \|\widehat{\beta}^{(j)} - \beta^{(j)}\|^2} \\
&\leq \sqrt{\sum_{j \in S(\beta)} \lambda_j^2} \times \|\widehat{\beta} - \beta\| \\
&\leq \sqrt{\sum_{j \in S(\beta)} \lambda_j^2} \times \kappa^{-1/2} \|X(\widehat{\beta} - \beta)\|.
\end{aligned}$$

   (b) Using Inequality (A.12), establish that on the event $\mathcal{A}$,

$$\|X(\widehat{\beta} - \beta^*)\|^2 \leq \inf_{\beta \in \mathbb{R}^p} \left\{ c_1 \|X(\beta - \beta^*)\|^2 + \frac{c_2}{\kappa} \sum_{j \in S(\beta)} \lambda_j^2 \right\},$$

   with c_1 and c_2 two positive absolute constants.

Correction : *We set*

$$A := \|X(\widehat{\beta}-\beta^*)\|^2 + \frac{1}{2}\sum_{j=1}^{M}\lambda_j\|\widehat{\beta}^{(j)}-\beta^{(j)}\|$$

Then, from Inequality (A.12)*, we get for any* $\alpha > 0$*,*

$$\begin{aligned}
A &\le \|X(\beta-\beta^*)\|^2 + 2\sum_{j\in S(\beta)}\lambda_j\|\widehat{\beta}^{(j)}-\beta^{(j)}\| \\
&\le \|X(\beta-\beta^*)\|^2 + 2\sqrt{\sum_{j\in S(\beta)}\lambda_j^2}\times\kappa^{-1/2}\|X(\widehat{\beta}-\beta)\| \\
&\le \|X(\beta-\beta^*)\|^2 + \alpha\|X(\widehat{\beta}-\beta)\|^2 + (\alpha\kappa)^{-1}\sum_{j\in S(\beta)}\lambda_j^2 \\
&\le (1+2\alpha)\|X(\beta-\beta^*)\|^2 + 2\alpha\|X(\widehat{\beta}-\beta^*)\|^2 + (\alpha\kappa)^{-1}\sum_{j\in S(\beta)}\lambda_j^2.
\end{aligned}$$

We take $\alpha = 1/4$ *and we obtain on the event* $\mathcal{A}$*,*

$$\|X(\widehat{\beta}-\beta^*)\|^2 \le \inf_{\beta\in\mathbb{R}^p}\left\{3\|X(\beta-\beta^*)\|^2 + \frac{8}{\kappa}\sum_{j\in S(\beta)}\lambda_j^2\right\}.$$

4. We now assume that all eigenvalues of the symmetric matrix XX^T are smaller than a finite constant Φ. We study $S(\widehat{\beta})$ the support of $\widehat{\beta}$. For this purpose, we recall that $\widehat{\beta}$ is a solution of (A.11) if and only if for any $j \in \{1,\dots,M\}$,

$$\begin{cases} 2X_{G_j}^T(Y-X\widehat{\beta}) = \lambda_j \times \frac{\widehat{\beta}^{(j)}}{\|\widehat{\beta}^{(j)}\|} & \text{if } \widehat{\beta}^{(j)} \neq 0, \\ \left\|2X_{G_j}^T(Y-X\widehat{\beta})\right\| \le \lambda_j & \text{if } \widehat{\beta}^{(j)} = 0. \end{cases}$$

(a) Let $j \in \{1,\dots,M\}$. Prove that if $\widehat{\beta}^{(j)} \neq 0$, on $\mathcal{A}$,

$$\|X_{G_j}^T X(\beta^* - \widehat{\beta})\| \ge \frac{\lambda_j}{4}.$$

Correction : *If* $\widehat{\beta}^{(j)} \neq 0$*, we have*

$$\left\|2X_{G_j}^T(Y-X\widehat{\beta})\right\| = \lambda_j$$

and the triangular inequality gives, on $\mathcal{A}$*,*

$$\begin{aligned}
\|X_{G_j}^T X(\beta^*-\widehat{\beta})\| &\ge \left\|X_{G_j}^T(Y-X\widehat{\beta})\right\| - \left\|X_{G_j}^T(Y-X\beta^*)\right\| \\
&\ge \frac{\lambda_j}{2} - \frac{\lambda_j}{4} = \frac{\lambda_j}{4}.
\end{aligned}$$

(b) Deduce that $|S(\widehat{\beta})|$, the cardinal of $S(\widehat{\beta})$, satisfies on $\mathcal{A}$,

$$|S(\widehat{\beta})| \leq \sum_{j=1}^{M} \frac{16}{\lambda_j^2} \|X_{G_j}^T X(\beta^* - \widehat{\beta})\|^2.$$

Correction : *We have on* $\mathcal{A}$,

$$\begin{aligned} |S(\widehat{\beta})| &= \sum_{j=1}^{M} 1_{\{\widehat{\beta}^{(j)} \neq 0\}} \\ &\leq \sum_{j=1}^{M} 1_{\left\{\|X_{G_j}^T X(\beta^* - \widehat{\beta})\| \geq \frac{\lambda_j}{4}\right\}} \\ &\leq \sum_{j=1}^{M} 1_{\left\{\frac{16}{\lambda_j^2}\|X_{G_j}^T X(\beta^* - \widehat{\beta})\|^2 \geq 1\right\}} \\ &\leq \sum_{j=1}^{M} \frac{16}{\lambda_j^2} \|X_{G_j}^T X(\beta^* - \widehat{\beta})\|^2. \end{aligned}$$

(c) Finally prove that on $\mathcal{A}$,

$$|S(\widehat{\beta})| \leq \frac{16\Phi}{\min_j \lambda_j^2} \|X(\beta^* - \widehat{\beta})\|^2.$$

Correction : *We have on* $\mathcal{A}$,

$$\begin{aligned} |S(\widehat{\beta})| &\leq \sum_{j=1}^{M} \frac{16}{\lambda_j^2} \|X_{G_j}^T X(\beta^* - \widehat{\beta})\|^2 \\ &\leq \frac{16}{\min_j \lambda_j^2} \sum_{j=1}^{M} \|X_{G_j}^T X(\beta^* - \widehat{\beta})\|^2 \\ &\leq \frac{16}{\min_j \lambda_j^2} \|X^T X(\beta^* - \widehat{\beta})\|^2 \\ &\leq \frac{16\Phi}{\min_j \lambda_j^2} \|X(\beta^* - \widehat{\beta})\|^2. \end{aligned}$$

Using the bound of 2d), we obtain

$$|S(\widehat{\beta})| \leq \frac{32\Phi}{\min_j \lambda_j^2} \|\beta^*\| \sqrt{\sum_{j \in S(\beta^*)} \lambda_j^2}.$$

5. The goal of this question is to determine the values of the λ_j's such that the probability of $\mathcal{A}$ is large. In the sequel, we denote Φ_j the largest eigenvalue of the symmetric matrix $X_{G_j} X_{G_j}^T$.

   (a) Prove that
   $$\|X_{G_j}^T \varepsilon\|^2 \leq \sigma^2 \Phi_j \|\sigma^{-1}\varepsilon\|^2.$$

   Correction : *We have*
   $$\|X_{G_j}^T \varepsilon\|^2 = \varepsilon^T X_{G_j} X_{G_j}^T \varepsilon \leq \Phi_j \|\varepsilon\|^2$$
   and we get the result.

   (b) Let $\beta > 1$. By identifying the distribution of $\|\sigma^{-1}\varepsilon\|^2$, determine λ_j such that
   $$\mathbb{P}(\mathcal{A}) \geq 1 - M^{1-\beta}.$$

   Correction : *The random variable $\|\sigma^{-1}\varepsilon\|^2$ is a $\chi^2(n)$-variable. Therefore,*
   $$\begin{aligned}\mathbb{P}(\mathcal{A}^c) &\leq \sum_{j=1}^M \mathbb{P}(\mathcal{A}_j^c)\\ &\leq \sum_{j=1}^M \mathbb{P}\Big(4\|X_{G_j}^T\varepsilon\| > \lambda_j\Big)\\ &\leq \sum_{j=1}^M \mathbb{P}\left(\|\sigma^{-1}\varepsilon\|^2 > \Big(\frac{\lambda_j}{4\sigma\sqrt{\Phi_j}}\Big)^2\right).\end{aligned}$$
   Consequently, taking $q_{n,M,1-\beta}$ the quantile of order $1 - M^{-\beta}$ of the $\chi^2(n)$-distribution, we set
   $$\lambda_j = 4\sigma\sqrt{\Phi_j} \times \sqrt{q_{n,M,1-\beta}}$$
   and
   $$\mathbb{P}(\mathcal{A}^c) \leq \sum_{j=1}^M \mathbb{P}\Big(\|\sigma^{-1}\varepsilon\|^2 > q_{n,M,1-\beta}\Big) \leq \sum_{j=1}^M M^{-\beta} = M^{1-\beta}$$
   and
   $$\mathbb{P}(\mathcal{A}) \geq 1 - M^{1-\beta}.$$

6. We now replace the linear regression Gaussian model (A.10) by the following Poisson model
   $$Y \sim \mathcal{P}oisson\big(\exp(X\beta^*)\big),$$

meaning that we observe $Y = (Y_1, \dots, Y_n)^T \in \mathbb{R}^n$ so that its components are independent and each component Y_i has a Poisson distribution with parameter $\theta_i = \exp\left(\sum_{\ell=1}^p X_{i\ell}\beta^*_\ell\right)$. As previously, the matrix X of size $n \times p$ is assumed to be known and $\beta^* \in \mathbb{R}^p$ is the vector to be estimated.

(a) Give the expression of the log-likelihood associated with the previous Poisson model.

Correction : *We have for any i and any integer k,*

$$\mathbb{P}(Y_i = k) = \exp(-\theta_i)\frac{\theta_i^k}{k!}.$$

Therefore the expression of the log-likelihood at any $\beta \in \mathbb{R}^p$ is

$$\mathcal{L}(\beta) = \sum_{i=1}^n \left(-\exp\left(\sum_{\ell=1}^p X_{i\ell}\beta_\ell\right) + Y_i \sum_{\ell=1}^p X_{i\ell}\beta_\ell - \log(Y_i!) \right).$$

(b) Suggest a Group-Lasso type estimator of β^* based on a convex criterion built from the log-likelihood given in the previous question. Justify your statements.

Correction : *For any $\alpha \in [0;1]$ and any vectors β_1 and β_2 of $\mathbb{R}^p$, we have*

$$\begin{aligned}\exp\left(\sum_{\ell=1}^p X_{i\ell}(\alpha\beta_{1\ell} + (1-\alpha)\beta_{2\ell}\right) &= \exp\left(\alpha\sum_{\ell=1}^p X_{i\ell}\beta_{1\ell} + (1-\alpha)\sum_{\ell=1}^p X_{i\ell}\beta_{2\ell}\right) \\ &\leq \alpha\exp\left(\sum_{\ell=1}^p X_{i\ell}\beta_{1\ell}\right) + (1-\alpha)\exp\left(\sum_{\ell=1}^p X_{i\ell}\beta_{2\ell}\right),\end{aligned}$$

by using the convexity of the function $x \in \mathbb{R} \longmapsto \exp(x)$. Therefore,

$$-\mathcal{L}(\alpha\beta_1 + (1-\alpha)\beta_2) \leq -\alpha\mathcal{L}(\beta_1) - (1-\alpha)\mathcal{L}(\beta_2),$$

meaning that the function $-\mathcal{L}$ is convex. Consequently, a natural Group-Lasso type estimate is

$$\widehat{\beta} \in \arg\min_{\beta\in\mathbb{R}^p}\left\{ -\mathcal{L}(\beta) + \sum_{j=1}^M \lambda_j\|\beta^{(j)}\| \right\},$$

with $\lambda_1, \dots, \lambda_M$ positive parameters.

Exercise 2: Scale-invariant estimators

We consider the multivariate linear regression Gaussian model :

$$Y = X\beta^* + \varepsilon \tag{A.13}$$

with $Y = (Y_1, \dots, Y_n)^T$ the vector of observations and $\beta^* \in \mathbb{R}^p$ the vector to be estimated. The matrix X of size $n \times p$, with $p < n$, is assumed to be known and of rank p. Finally, the error vector, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^T$, is assumed to be centered and its variance-covariance matrix is $\sigma_Y^2 I_n$ where $\sigma_Y^2 > 0$, the variance of the Y_i's, is unknown. The ℓ_2-norm is denoted $\|\cdot\|$, whereas the ℓ_1-norm is denoted $\|\cdot\|_1$.

We consider $\widehat{\beta}$ an estimator of β^* obtained by minimizing some function crit depending on Y and β:

$$\widehat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \mathrm{crit}(Y, \beta).$$

We say that $\widehat{\beta}$ is scale-invariant if for any deterministic constant $s > 0$,

$$\arg\min_{\beta \in \mathbb{R}^p} \mathrm{crit}(sY, s\beta) = \arg\min_{\beta \in \mathbb{R}^p} \mathrm{crit}(Y, \beta).$$

1. Prove that the ordinary least squares estimate is scale-invariant.

   Correction : *For the ordinary least squares estimate*

   $$crit(Y, \beta) = \|Y - X\beta\|^2$$

   and for any $s > 0$

   $$crit(sY, s\beta) = s^2 crit(Y, \beta)$$

   and

   $$\arg\min_{\beta \in \mathbb{R}^p} crit(sY, s\beta) = \arg\min_{\beta \in \mathbb{R}^p} crit(Y, \beta).$$

2. We consider the following estimate

   $$\widehat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \|Y - X\beta\| + \lambda \|\beta\|_1 \right\}$$

   with λ independent of β, Y and σ_Y. Establish that $\widehat{\beta}$ is scale-invariant.

   Correction : *In this case,*

   $$crit(Y, \beta) = \|Y - X\beta\| + \lambda \|\beta\|_1$$

   and we have

   $$crit(sY, s\beta) = s \times crit(Y, \beta)$$

   and

   $$\arg\min_{\beta \in \mathbb{R}^p} crit(sY, s\beta) = \arg\min_{\beta \in \mathbb{R}^p} crit(Y, \beta).$$

3. We consider the following Lasso-estimate

$$\widehat{\beta}_1 = \arg\min_{\beta \in \mathbb{R}^p} \Big\{ \|Y - X\beta\|^2 + \lambda \|\beta\|_1 \Big\}.$$

   (a) We assume that σ_Y is known. By computing the standard deviation of sY_1, for some $s > 0$, provide an expression of λ such that $\widehat{\beta}_1$ is scale-invariant.
   Correction : *We have $\sigma_{sY} = s \times \sigma_Y$. Therefore, by taking λ proportional to σ_Y, say $\lambda = \sigma_Y \mu$ with μ independent of σ_Y, Y and β, we have, with*

$$crit(Y, \beta) = \|Y - X\beta\|^2 + \lambda\|\beta\|_1 = \|Y - X\beta\|^2 + \sigma_Y \mu \|\beta\|_1,$$

$$crit(sY, s\beta) = s^2 crit(Y, \beta)$$

   and

$$\arg\min_{\beta \in \mathbb{R}^p} crit(sY, s\beta) = \arg\min_{\beta \in \mathbb{R}^p} crit(Y, \beta).$$

   (b) We assume that σ_Y is unknown. Suggest an estimate of σ_Y and provide a data-dependent expression of λ such that $\widehat{\beta}_1$ is scale-invariant.
   Correction : *We consider*

$$\widehat{\beta}^{ols}(Y) = \arg\min_{\beta \in \mathbb{R}^p} \|Y - X\beta\| = (X^T X)^{-1} X^T Y$$

   where the last expression is valid since rank$(X) = p$. For any $s > 0$,

$$\widehat{\beta}^{ols}(sY) = s\widehat{\beta}^{ols}(Y).$$

   and we estimate σ_Y by $\widehat{\sigma}_Y$ with

$$\widehat{\sigma}_Y^2 = \frac{\|Y - X\widehat{\beta}^{ols}\|^2}{n - p}$$

   and

$$\widehat{\sigma}_{sY} = s \times \widehat{\sigma}_Y.$$

   Taking $\lambda = \widehat{\sigma}_Y \mu$ with μ independent of σ_Y, Y and β, the estimator $\widehat{\beta}_1$ is scale-invariant.

Exercise 3: Estimation of the mean of functional data

Let Z a random variable taking its values in $\mathbb{H} = \mathbb{L}^2([0,1])$. For all $f, g \in \mathbb{H}$, we denote as usual

$$\langle f, g\rangle_{\mathbb{H}} = \int_0^1 f(t)g(t)dt,$$

and $\|f\|_{\mathbb{H}} = \sqrt{\langle f, f\rangle_{\mathbb{H}}}$. We suppose that there exists a constant $C > 0$ such that $|Z(t)| \leq C$ a.e. in t and a.s.

1. Prove that there exists $m \in \mathbb{H}$ such that, for all $f \in \mathbb{H}$,

$$\mathbb{E}\left[\langle f, Z\rangle_{\mathbb{H}}\right] = \langle f, m\rangle_{\mathbb{H}}$$

and that

$$\|m\|_{\mathbb{H}} \leq C.$$

Correction : *Let $\varphi : f \mapsto \mathbb{E}[\langle f, Z\rangle_{\mathbb{H}}]$. By linearity of the scalar product and the expectation, φ is a linear application from $\mathbb{H}$ to $\mathbb{R}$. Moreover, for all $f \in \mathbb{H}$, by Cauchy-Schwarz inequality,*

$$|\varphi(f)| \leq \mathbb{E}\left[|\langle f, Z\rangle_{\mathbb{H}}|\right] \leq \mathbb{E}\left[\|f\|_{\mathbb{H}}\|Z\|_{\mathbb{H}}\right] \leq C\|f\|_{\mathbb{H}}.$$

Then, the linear form φ is continuous and the result comes directly from the Riesz representation theorem.

2. Let $Z_1, \ldots, Z_n$ be i.i.d. random variables following the same distribution as Z. Prove that

$$\mathbb{E}\left[\left\|\frac{1}{n}\sum_{i=1}^n Z_i - m\right\|_{\mathbb{H}}^2\right] = \frac{\mathbb{E}[\|Z-m\|_{\mathbb{H}}^2]}{n} \leq \frac{4C^2}{n}.$$

Correction :

$$\mathbb{E}\left[\left\|\frac{1}{n}\sum_{i=1}^n Z_i - m\right\|_{\mathbb{H}}^2\right] = \frac{1}{n^2}\mathbb{E}\left[\left\|\sum_{i=1}^n (Z_i - m)\right\|_{\mathbb{H}}^2\right]$$

Let $\tilde{Z}_i = Z_i - m$, we have

$$\mathbb{E}\left[\left\|\sum_{i=1}^n \tilde{Z}_i\right\|_{\mathbb{H}}^2\right] = \sum_{i,j=1}^n \mathbb{E}\left[\langle \tilde{Z}_i, \tilde{Z}_j\rangle_{\mathbb{H}}\right] = \sum_{i,j=1}^n \mathbb{E}\left[\int_0^1 \tilde{Z}_i(t)\tilde{Z}_j(t)dt\right].$$

Using Cauchy-Schwarz inequality and the assumption that $\|Z\|_{\mathbb{H}} \leq C$, we remark that $\tilde{Z}_i\tilde{Z}_j$ is integrable. Then, the Fubini theorem and the fact the the $\tilde{Z}_i$'s are independent and centered gives us

$$\mathbb{E}\left[\int_0^1 \tilde{Z}_i(t)\tilde{Z}_j(t)dt\right] = \int_0^1 \mathbb{E}[\tilde{Z}_i(t)\tilde{Z}_j(t)]dt = \mathbf{1}_{\{i=j\}}\int_0^1 \mathbb{E}[\tilde{Z}_i^2(t)]dt = \mathbb{E}[\|\tilde{Z}\|_{\mathbb{H}}^2]\mathbf{1}_{\{i=j\}}$$

which implies the expected result.

3. We suppose now that Z is continuous a.s. We observe $\{U_{i,h}, i = 1, \dots, n; h = 1, \dots, p\}$ such that

$$U_{i,h} = Z_i(t_h) + \varepsilon_{i,h},$$

where $t_h = h/p$, $h = 1, \dots, p$ and $\{\varepsilon_{i,h}\} \sim_{i.i.d.} \mathcal{N}(0, \sigma^2)$ is independent of $Z_1, \dots, Z_n$. Let $I_h = [(h-1)/p, h/p[$, $h = 1, \dots, p$. We define a simple histogram estimator $\widehat{m}$ as follows

$$\widehat{m}(t) = \frac{1}{n} \sum_{i=1}^{n} \tilde{U}_i$$

where

$$\tilde{U}_i = \sum_{h=1}^{p} U_{i,h} \mathbf{1}_{I_h}(t).$$

(a) Explain why $m^* = \frac{1}{n} \sum_{i=1}^{n} Z_i$ is not an estimator of m.

Correction : *The Z_i's are not observed.*

(b) Verify that

$$\mathbb{E}\left[\left\|\widehat{m} - \mathbb{E}[\tilde{U}_1]\right\|_{\mathbb{H}}^2\right] = \frac{\mathbb{E}[\|\tilde{U}_1 - \mathbb{E}[\tilde{U}_1]\|_{\mathbb{H}}^2]}{n}.$$

Correction : *We apply the result of 2. to the sequence $\tilde{U}_1, \dots, \tilde{U}_n$ which is i.i.d. and verify*

$$\mathbb{E}[\|\tilde{U}_1\|^2] = \mathbb{E}\left[\int_0^1 \left(\sum_{h=1}^{p} U_{1,h} \mathbf{1}_{I_h(t)}\right)^2 dt\right] = \mathbb{E}\left[\sum_{h=1}^{p} \int_{I_h} U_{1,h}^2 dt\right] = \frac{1}{p} \sum_{h=1}^{p} \mathbb{E}[U_{1,h}^2] < +\infty,$$

since $\mathbb{E}[U_{1,h}^2] \leq 2\mathbb{E}[Z_i^2(t_h)] + 2\mathbb{E}[\varepsilon_{i,h}^2] < +\infty$ *for all* h.

(c) Prove that

$$\mathbb{E}[\|\tilde{U}_1 - \mathbb{E}[\tilde{U}_1]\|_{\mathbb{H}}^2 = \frac{1}{p} \sum_{h=1}^{p} \mathrm{Var}(Z_1(t_h)) + \sigma^2.$$

Correction :

$$\begin{aligned} \mathbb{E}[\|\tilde{U}_1 - \mathbb{E}[\tilde{U}_1]\|_{\mathbb{H}}^2 &= \int_0^1 \mathrm{Var}(\tilde{U}_1(t)) dt = \sum_{h,h'=1}^{p} \int_0^1 \mathrm{Cov}(U_{1,h}, U_{1,h'}) \mathbf{1}_{t_h} \mathbf{1}_{t_{h'}} dt \\ &= \sum_{h=1}^{p} \frac{1}{p} \left(\mathrm{Var}(Z_1(t_h)) + \mathrm{Var}(\varepsilon_{1,h})\right) \end{aligned}$$

(d) Calculate $\mathbb{E}[\tilde{U}_1]$.

Correction :

$$\mathbb{E}[\tilde{U}_1] = \sum_{h=1}^{p} \mathbb{E}[U_{i,h}]\mathbf{1}_{I_h}(t) = \sum_{h=1}^{p} m(t_h)\mathbf{1}_{I_h}.$$

(e) We suppose that m is an α-Hölder continuous function, with $\alpha > 0$, i.e., there exists $L > 0$ such that, for all $s, t \in [0,1]$,

$$|m(t) - m(s)| \leq L|t-s|^{\alpha}.$$

Prove the following bound on the risk of the estimator $\widehat{m}$,

$$\mathbb{E}[\|\widehat{m} - m\|_{\mathbb{H}}^2] \leq \frac{L^2}{p^{2\alpha}} + \frac{4C^2 + \sigma^2}{n}.$$

Correction : *We start from the bias-variance decomposition of the risk*

$$\mathbb{E}[\|\widehat{m} - m\|_{\mathbb{H}}^2] = \mathbb{E}\left[\left\|\widehat{m} - \mathbb{E}[\tilde{U}_1]\right\|_{\mathbb{H}}^2\right] + \|\mathbb{E}[\tilde{U}_1] - m\|_{\mathbb{H}}^2.$$

(b) and (c) and the boundedness assumption allow to upper-bound the variance term. For the bias term we get from (d) that

$$\begin{aligned} \|\mathbb{E}[\tilde{U}_1] - m\|_{\mathbb{H}}^2 &= \int_0^1 \left(\sum_{h=1}^{p} m(t_h)\mathbf{1}_{I_h}(t) - m(t)\right)^2 dt \\ &= \sum_{h=1}^{p} \int_{I_h} (m(t_h) - m(t))^2\, dt \leq L^2 \sum_{h=1}^{p} \int_{I_h} |t_h - t|^{2\alpha} dt. \end{aligned}$$

The result comes from the fact that

$$\int_{I_h} |t_h - t|^{2\alpha} dt \leq \int_{I_h} \frac{1}{p^{2\alpha}} dt = \frac{1}{p^{2\alpha+1}}.$$

A.7 Exam 2022-2023

High-dimensional statistics

EXAM: duration 2h30

Documents, calculators, phones and smartphones are forbidden

Exercise 1: Fast wavelet transform

Along this exercice, we use the discrete convolution, denoted $\star_d$, between two sequences of real numbers $a = (a_k)_{k\in\mathbb{Z}}$ and $b = (b_k)_{k\in\mathbb{Z}}$:

$$[a \star_d b](k) := \sum_{\ell\in\mathbb{Z}} a_\ell b_{k-\ell}, \quad k \in \mathbb{Z}.$$

1. We consider a multiresolution analysis:

$$\{0\} \subset \cdots \subset V_{j+1} \subset V_j \subset V_{j-1} \subset \cdots \subset \mathbb{L}_2(\mathbb{R}).$$

We introduce the father wavelet ϕ and for any $(j,k) \in \mathbb{Z}^2$,

$$\phi_{j,k}(t) = \frac{1}{\sqrt{2^j}}\phi\left(\frac{t-k2^j}{2^j}\right) = 2^{-j/2}\phi(2^{-j}t-k), \quad t \in \mathbb{R},$$

so that for any $j \in \mathbb{Z}$, $(\phi_{j,k})_{k\in\mathbb{Z}}$ is an orthonormal basis of V_j.

(a) By using $x \mapsto \phi(x/2) \in V_1$, prove that for any $x \in \mathbb{R}$,

$$\phi(x/2) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k\phi(x-k),$$

with

$$h_k = \frac{1}{\sqrt{2}}\int_{\mathbb{R}}\phi\left(\frac{t}{2}\right)\phi(t-k)dt.$$

<u>Correction :</u> *We have $x \mapsto \phi(x/2) \in V_1 \subset V_0$. So, for any $x \in \mathbb{R}$,*

$$\phi(x/2) = \sum_{k\in\mathbb{Z}}\langle\phi(\cdot/2);\phi_k\rangle\phi_k(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k\phi(x-k),$$

with

$$h_k = \frac{1}{\sqrt{2}}\int_{\mathbb{R}}\phi\left(\frac{t}{2}\right)\phi(t-k)dt.$$

(b) We fix $(j, k) \in \mathbb{Z}^2$. For any $f \in \mathbb{L}^2(\mathbb{R})$, we set

$$\alpha_{j,k} = \langle f; \phi_{j,k} \rangle.$$

i. Show that for any $\ell \in \mathbb{Z}$

$$\langle \phi_{j+1,k}; \phi_{j,\ell} \rangle = h_{\ell-2k}.$$

Correction : *We have:*

$$\begin{aligned}
\langle \phi_{j+1,k}; \phi_{j,\ell} \rangle &:= \int_{\mathbb{R}} \frac{1}{\sqrt{2^{j+1}}} \phi(2^{-(j+1)}t - k) \frac{1}{\sqrt{2^j}} \phi(2^{-j}t - \ell) dt \\
&= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \phi\left(\frac{t - 2k}{2}\right) \phi(t - \ell) dt \\
&= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \phi\left(\frac{t}{2}\right) \phi(t - (\ell - 2k)) dt \\
&= h_{\ell-2k}.
\end{aligned}$$

ii. Justify that we can decompose $\phi_{j+1,k}$ on the $\phi_{j,\ell}$'s and prove:

$$\phi_{j+1,k} = \sum_{\ell \in \mathbb{Z}} h_{\ell-2k} \phi_{j,\ell}.$$

Correction : *Since for any $k \in \mathbb{Z}$, $\phi_{j+1,k} \in V_{j+1} \subset V_j$, we can decompose $\phi_{j+1,k}$ on the $\phi_{j,\ell}$'s. Since $(\phi_{j,\ell})_{\ell \in \mathbb{Z}}$ is an orthonormal basis of V_j, we obtain:*

$$\begin{aligned}
\phi_{j+1,k} &= \sum_{\ell \in \mathbb{Z}} \langle \phi_{j+1,k}; \phi_{j,\ell} \rangle \phi_{j,\ell} \\
&= \sum_{\ell \in \mathbb{Z}} h_{\ell-2k} \phi_{j,\ell}.
\end{aligned}$$

iii. Show that

$$\alpha_{j+1,k} = [\alpha_{j\cdot} \star_d \tilde{h}](2k),$$

where $\alpha_{j\cdot} = (\alpha_{j,\ell})_{\ell \in \mathbb{Z}}$ and for $k \in \mathbb{Z}$, $\tilde{h}_k = h_{-k}$.
Correction : *We have:*

$$\alpha_{j+1,k} = \langle f; \phi_{j+1,k} \rangle = \sum_{\ell \in \mathbb{Z}} h_{\ell-2k} \langle f; \phi_{j,\ell} \rangle = \sum_{\ell \in \mathbb{Z}} h_{\ell-2k} \alpha_{j,\ell} = [\alpha_{j\cdot} \star_d \tilde{h}](2k).$$

2. For any $j \in \mathbb{Z}$, we now introduce W_j as the orthogonal complement of V_j in V_{j-1}:
$$V_j \oplus W_j = V_{j-1}.$$
We introduce the mother wavelet ψ and for any $j \in \mathbb{Z}$ and any $k \in \mathbb{Z}$
$$\psi_{j,k}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-k2^j}{2^j}\right) = 2^{-j/2}\psi(2^{-j}t-k), \quad t \in \mathbb{R}$$
so that for any $j \in \mathbb{Z}$, $(\psi_{j,k})_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j.
   (a) Show that there exists $(\lambda_k)_{k\in\mathbb{Z}}$ such that we can write for any $x \in \mathbb{R}$,
$$\psi(x/2) = \sqrt{2}\sum_{k\in\mathbb{Z}} \lambda_k \phi(x-k). \quad \text{(A.14)}$$
   Correction : *We just use the following fact: the function $x \mapsto \psi(x/2) \in W_1 \subset V_0$.*
   (b) We fix $(j,k) \in \mathbb{Z}^2$. For any $f \in \mathbb{L}^2(\mathbb{R})$, we set
$$\beta_{j,k} = \langle f; \psi_{j,k}\rangle.$$
      i. For any $j \in \mathbb{Z}$, express the coefficients $(\beta_{j+1,k})_{k\in\mathbb{Z}}$ in function of the coefficients $(\alpha_{j,k})_{k\in\mathbb{Z}}$ and $(\lambda_k)_{k\in\mathbb{Z}}$.
      Correction : *As before, we establish that*
$$\psi_{j+1,k} = \sum_{\ell\in\mathbb{Z}} \lambda_{\ell-2k}\phi_{j,\ell}.$$
      We deduce:
$$\beta_{j+1,k} = \langle f; \psi_{j+1,k}\rangle = \sum_{\ell\in\mathbb{Z}} \lambda_{\ell-2k}\langle f; \phi_{j,\ell}\rangle = \sum_{\ell\in\mathbb{Z}} \lambda_{\ell-2k}\alpha_{j,\ell} = [\alpha_{j\cdot} \star_d \tilde{\lambda}](2k),$$
      with for any $k \in \mathbb{Z}$, $\tilde{\lambda}_k = \lambda_{-k}$.
      ii. Show finally that we have for any $j \in \mathbb{Z}$ and any $k \in \mathbb{Z}$,
$$\alpha_{j,k} = \sum_{\ell\in\mathbb{Z}} h_{k-2\ell}\alpha_{j+1,\ell} + \sum_{\ell\in\mathbb{Z}} \lambda_{k-2\ell}\beta_{j+1,\ell}.$$
      Correction : *To prove the result, we decompose $\phi_{j,k} \in V_j = V_{j+1} \oplus W_{j+1}$ and we obtain*
$$\begin{aligned}\phi_{j,k} &= \sum_{\ell\in\mathbb{Z}} \langle \phi_{j,k}; \phi_{j+1,\ell}\rangle \phi_{j+1,\ell} + \sum_{\ell\in\mathbb{Z}} \langle \phi_{j,k}; \psi_{j+1,\ell}\rangle \psi_{j+1,\ell} \\ &= \sum_{\ell\in\mathbb{Z}} h_{k-2\ell}\phi_{j+1,\ell} + \sum_{\ell\in\mathbb{Z}} \lambda_{k-2\ell}\psi_{j+1,\ell}.\end{aligned}$$
      Then, taking the scalar product with f, we obtain the result.

3. We recall the connection between ϕ and ψ: for any $w \in \mathbb{R}$,

$$\hat{\psi}(2w) = e^{-iw}\overline{m_0(w+\pi)}\hat{\phi}(w),$$

where $\hat{\psi}$ and $\hat{\phi}$ denote the Fourier transform of ψ and ϕ respectively and

$$m_0(w) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} h_k e^{-ikw}.$$

(a) Establish that for any $w \in \mathbb{R}$,

$$\hat{\psi}(2w) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} h_{1-k}(-1)^{k+1}e^{-ikw}\hat{\phi}(w).$$

Correction : *For any $w \in \mathbb{R}$,*

$$\begin{aligned}
\hat{\psi}(2w) &= \frac{1}{\sqrt{2}}e^{-iw}\overline{\sum_{k\in\mathbb{Z}} h_k e^{-ik(w+\pi)}}\hat{\phi}(w)\\
&= \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} h_k e^{-iw(1-k)}(-1)^k\hat{\phi}(w)\\
&= \frac{1}{\sqrt{2}}\sum_{\ell\in\mathbb{Z}} h_{1-\ell} e^{-iw\ell}(-1)^{1-\ell}\hat{\phi}(w)\\
&= \frac{1}{\sqrt{2}}\sum_{\ell\in\mathbb{Z}} h_{1-\ell} e^{-iw\ell}(-1)^{1+\ell}\hat{\phi}(w)
\end{aligned}$$

(b) From Equation (A.14), deduce the relationship

$$\lambda_k = (-1)^{k+1}h_{1-k}, \quad k \in \mathbb{Z}.$$

Correction : *Equation* (A.14) *gives: for any $x \in \mathbb{R}$,*

$$\psi(x/2) = \sqrt{2}\sum_{k\in\mathbb{Z}} \lambda_k \phi(x-k).$$

By taking the Fourier transform of both sides, we obtain for any $w \in \mathbb{R}$,

$$\begin{aligned}
\int \psi(x/2)e^{-ixw}dx &= \sqrt{2}\sum_{k\in\mathbb{Z}} \lambda_k \int \phi(x-k)e^{-ixw}dx\\
&= \sqrt{2}\sum_{k\in\mathbb{Z}} \lambda_k e^{-iwk}\hat{\phi}(w)
\end{aligned}$$

and

$$2\hat{\psi}(2w) = \sqrt{2}\sum_{k\in\mathbb{Z}} \lambda_k e^{-iwk}\hat{\phi}(w),$$

which provides the result by using the previous question.

Exercise 2: Degree of freedom

Given $\beta^* \in \mathbb{R}^p$ and a matrix X of size $n \times p$, and whose lines are denoted $x_1^T, \dots, x_n^T$, so that

$$X = \begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{pmatrix},$$

we consider the regression model

$$Y_i = x_i^T \beta^* + \varepsilon_i, \quad \varepsilon_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2), \quad i = 1, \dots, n, \tag{A.15}$$

with $\sigma^2 > 0$. We denote

$$Y = (Y_1, \dots, Y_n)^T, \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^T$$

and we define the degree of freedom of a function $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ by:

$$\mathrm{df}(g) = \frac{1}{\sigma^2} \sum_{i=1}^n \mathrm{cov}(g_i(Y), Y_i),$$

where g_i designs the ith coordinate of g. We recall that

$$\mathrm{cov}(g_i(Y), Y_i) = \mathbb{E}[(g_i(Y) - \mathbb{E}[g_i(Y)])(Y_i - \mathbb{E}[Y_i])].$$

Model (A.15) can be rewritten

$$Y = X\beta^* + \varepsilon$$

and we assume that $\mathrm{rank}(X) = p$. In the sequel, we denote $\|\cdot\|$ the ℓ_2-norm on $\mathbb{R}^n$.

1. We consider $\hat{\beta} \in \mathbb{R}^p$ any estimate of β^* and we set $g(Y) = X\hat{\beta}$, so

   $$g_i(Y) = (X\hat{\beta})_i.$$

   We denote

   $$C_p := \|Y - X\hat{\beta}\|^2 - n\sigma^2 + 2\sigma^2 \mathrm{df}(X\hat{\beta}).$$

   (a) Prove that for any $i \in \{1, \dots, n\}$,

   $$\mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[Y_i] - (X\hat{\beta})_i)] = \mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[(X\hat{\beta})_i] - (X\hat{\beta})_i)].$$

   Correction : *We observe that for any $i \in \{1, \dots, n\}$,*

   $$\begin{aligned}\mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[Y_i] - (X\hat{\beta})_i)] &= \mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[Y_i] - \mathbb{E}[(X\hat{\beta})_i] + \mathbb{E}[(X\hat{\beta})_i] - (X\hat{\beta})_i)] \\ &= 0 + \mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[(X\hat{\beta})_i] - (X\hat{\beta})_i)].\end{aligned}$$

(b) Prove that

$$\mathbb{E}[C_p] = \mathbb{E}[\|X\hat{\beta} - X\beta^*\|^2].$$

Correction : *We have:*

$$\begin{aligned}
\mathbb{E}[\|X\hat{\beta} - X\beta^*\|^2] &= \mathbb{E}[\|X\hat{\beta} - Y + Y - X\beta^*\|^2] \\
&= \mathbb{E}[\|X\hat{\beta} - Y\|^2] + \mathbb{E}[\|Y - X\beta^*\|^2] - 2\mathbb{E}\left[\sum_{i=1}^n (Y - X\beta^*)_i (Y - X\hat{\beta})_i\right] \\
&= \mathbb{E}[\|X\hat{\beta} - Y\|^2] + \mathbb{E}[\|\varepsilon\|^2] \\
&\quad - 2\mathbb{E}\left[\sum_{i=1}^n (Y_i - \mathbb{E}[Y_i])(Y_i - \mathbb{E}[Y_i] + \mathbb{E}[Y_i] - (X\hat{\beta})_i)\right] \\
&= \mathbb{E}[\|X\hat{\beta} - Y\|^2] - \mathbb{E}[\|\varepsilon\|^2] - 2\mathbb{E}\left[\sum_{i=1}^n (Y_i - \mathbb{E}[Y_i])(\mathbb{E}[Y_i] - (X\hat{\beta})_i)\right].
\end{aligned}$$

So, since $\mathbb{E}[\|\varepsilon\|^2] = n\sigma^2$ and using the result of the previous question,

$$\begin{aligned}
\mathbb{E}[\|X\hat{\beta} - X\beta^*\|^2] &= \mathbb{E}[\|X\hat{\beta} - Y\|^2] - n\sigma^2 - 2\sum_{i=1}^n \mathbb{E}[(Y_i - \mathbb{E}[Y_i])(\mathbb{E}[(X\hat{\beta})_i] - (X\hat{\beta})_i)] \\
&= \mathbb{E}[\|X\hat{\beta} - Y\|^2] - n\sigma^2 + 2\sigma^2 df(X\hat{\beta}) \\
&= \mathbb{E}[C_p].
\end{aligned}$$

2. We now estimate $X\beta^*$ with

$$g(Y) = X(X^TX)^{-1}X^TY.$$

(a) Prove that

$$\mathrm{df}(g) = \sum_{i=1}^n x_i^T (X^TX)^{-1} x_i.$$

Correction : *For this case, $g_i(Y) = x_i^T(X^TX)^{-1}X^TY$. We denote 1_i the vector whose components are all equal to 0 except the ith component equal to 1. Since*

$X^T \times 1_i = x_i$

$$\begin{aligned} df(g) &= \sigma^{-2} \sum_{i=1}^{n} \mathbb{E}[x_i^T (X^T X)^{-1} X^T \varepsilon \times \varepsilon_i] \\ &= \sum_{i=1}^{n} x_i^T (X^T X)^{-1} X^T \sigma^{-2} \mathbb{E}[\varepsilon_i \varepsilon] \\ &= \sum_{i=1}^{n} x_i^T (X^T X)^{-1} X^T \times 1_i \\ &= \sum_{i=1}^{n} x_i^T (X^T X)^{-1} x_i. \end{aligned}$$

(b) Deduce that

$$\mathrm{df}(g) = \mathrm{Trace}(X(X^T X)^{-1} X^T).$$

Correction : *Obvious.*

(c) Finally, prove that

$$\mathrm{df}(g) = p.$$

Correction : *The matrix $X(X^T X)^{-1} X^T$ is the projection matrix on $\mathcal{I}m(X)$, so, since rank$(X) = p$, df$(g) =$ Trace$(X(X^T X)^{-1} X^T) = p$.*

Exercise 3: Model selection

We consider the multivariate linear regression model :

$$Y = X\beta^* + \varepsilon$$

with $Y = (Y_1, \ldots, Y_n)^T$ the vector of observations. The matrix X of size $n \times p$ is assumed to be known and is such that its columns, denoted $(X_1, \ldots, X_p)$, are orthogonal and of unit norm (consequently $X^T X = I_p$). The vector $\beta^* \in \mathbb{R}^p$ is unknown. Finally, the error vector is $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$ and satisfies $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$ with $\sigma^2 > 0$ assumed to be known. The classical ℓ_2-norm is denoted $\| \cdot \|_2$. We denote for any model m, a subset of indexes of $\{1, \ldots, p\}$, P_m the projection on span$(X_j : \ j \in m)$. We denote $P_m Y = 0$ if the model m is empty. We estimate $f^* = X\beta^*$ by using $P_{\hat{m}_\lambda} Y$, where for some $\lambda > 0$,

$$\hat{m}_\lambda = \arg \min_{m \subset \{1, \ldots, p\}} \left\{ \|Y - P_m Y\|_2^2 + \lambda \sigma^2 |m| \right\},$$

where $|m|$ is the cardinality of m.

1. Prove that
$$\|Y - P_mY\|_2^2 + \lambda\sigma^2|m| = \|Y\|_2^2 + \sum_{j\in m}\Big(\lambda\sigma^2 - (X_j^TY)^2\Big).$$

   Indication: Use that $\langle X_j; X_k\rangle_{\ell_2} = 1_{\{j=k\}}$.

   Correction : *Since the* X_j*'s are orthonormal, we have:*
$$\begin{aligned}\|Y - P_mY\|_2^2 &= \|Y\|_2^2 + \|P_mY\|_2^2 - 2\langle Y; P_mY\rangle_{\ell_2}\\ &= \|Y\|_2^2 + \|P_mY\|_2^2 - 2\langle P_mY; P_mY\rangle_{\ell_2}\\ &= \|Y\|_2^2 - \|P_mY\|_2^2\\ &= \|Y\|_2^2 - \Big\|\sum_{j\in m}(X_j^TY)X_j\Big\|_2^2\\ &= \|Y\|_2^2 - \sum_{j\in m}(X_j^TY)^2.\end{aligned}$$

   We obtain the result.

2. Deduce that
$$\hat m_\lambda = \left\{j :\ (X_j^TY)^2 > \lambda\sigma^2\right\}.$$

   Correction : *Since* $\|Y\|_2^2$ *does not depend on* m*,*
$$\begin{aligned}\hat m_\lambda &= \arg\min_{m\subset\{1,\dots,p\}}\left\{\|Y - P_mY\|_2^2 + \lambda\sigma^2|m|\right\}\\ &= \arg\min_{m\subset\{1,\dots,p\}}\left\{\sum_{j\in m}\Big(\lambda\sigma^2 - (X_j^TY)^2\Big)\right\}\\ &= \left\{j :\ (X_j^TY)^2 > \lambda\sigma^2\right\}.\end{aligned}$$

3. We assume that $\beta^* = 0$.

   (a) Show that $|\hat m_\lambda|$ has a binomial distribution with parameters (p, q_λ), with $q_\lambda = 1 - F(\lambda)$ and F is the cumulative distribution of a $\chi^2(1)$-variable:
$$|\hat m_\lambda| \sim Bin\big(p, 1 - F(\lambda)\big).$$

   Correction : *If* $\beta^* = 0$*, we have*
$$\begin{aligned}|\hat m_\lambda| &= \sum_{j=1}^p 1_{\{(X_j^T\varepsilon)^2 > \lambda\sigma^2\}}\\ &= \sum_{j=1}^p 1_{\{Z_j^2 > \lambda\}},\end{aligned}$$

with $Z_j = \sigma^{-1} X_j^T \varepsilon$ *and the* Z_j*'s are i.i.d.* $\mathcal{N}(0,1)$*-variables. Therefore,*

$$|\hat{m}_\lambda| \sim Bin\big(p, 1 - F(\lambda)\big).$$

(b) If λ is a constant independent of n and p, evaluate, almost surely, $\lim_{p \to +\infty} \frac{|\hat{m}_\lambda|}{p}$.
Correction : *We apply the strong law of large numbers*

$$\lim_{p \to +\infty} \frac{|\hat{m}_\lambda|}{p} = \lim_{p \to +\infty} \frac{1}{p} \sum_{j=1}^{p} 1_{\{Z_j^2 > \lambda\}} = \mathbb{E}[Z_1^2 > \lambda] \ a.s.$$

Therefore

$$\lim_{p \to +\infty} \frac{|\hat{m}_\lambda|}{p} = 1 - F(\lambda) \ a.s.$$

(c) We take $\lambda = K \log(p)$, where K is a constant independent of n and p. Determine the smallest constant K such that

$$\lim_{p \to +\infty} \mathbb{E}[|\hat{m}_\lambda|] = 0.$$

Indication: *Use*

$$1 - F(\lambda) \sim_{\lambda \to +\infty} \sqrt{\frac{2}{\pi \lambda}} e^{-\lambda/2}.$$

Correction : *We have*

$$\mathbb{E}[|\hat{m}_\lambda|] = p(1 - F(\lambda)).$$

Therefore, when $p \to +\infty$*,*

$$\begin{aligned} \mathbb{E}[|\hat{m}_\lambda|] &= p(1 - F(K \log(p))) \\ &\sim p \sqrt{\frac{2}{\pi K \log(p)}} e^{-K \log(p)/2} \\ &\sim \sqrt{\frac{2}{\pi K \log(p)}} p^{1-K/2}. \end{aligned}$$

and $K = 2$ *is the smallest constant such that*

$$\lim_{p \to +\infty} \mathbb{E}[|\hat{m}_\lambda|] = 0.$$

A.8 Exam 2023-2024

High-dimensional statistics

EXAM: duration 2h30

Documents, calculators, phones and smartphones are forbidden

Exercise 1: Lasso for the logistic model

In the high-dimensional setting, we observe $(Y_1, Y_2, \ldots, Y_n)$, n independent random variables such that for any i, $Y_i \in \{0,1\}$ and its expectation depends on p non-random predictors according to the following transformation

$$\mathbb{E}[Y_i] = \mathbb{P}(Y_i = 1) = 1 - \mathbb{P}(Y_i = 0) = \frac{\exp\left(\sum_{j=1}^p \beta_j^* X_{ij}\right)}{1 + \exp\left(\sum_{j=1}^p \beta_j^* X_{ij}\right)}.$$

In the last expression, X_{ij} denotes the value of the predictor X_j associated with Y_i and β^* is a sparse p-dimensional unknown vector to be estimated. In the sequel, we denote for any $\beta \in \mathbb{R}^p$ and any $i \in \{1, \ldots, n\}$,

$$p_i(\beta) = \frac{\exp\left(\sum_{j=1}^p \beta_j X_{ij}\right)}{1 + \exp\left(\sum_{j=1}^p \beta_j X_{ij}\right)}$$

and $p(\beta) = (p_1(\beta), \ldots, p_n(\beta))^T$.

1. Show that the log-likelihood associated with this model is, for $\beta \in \mathbb{R}^p$,

$$\mathcal{L}(\beta) = \sum_{i=1}^n \left[Y_i \sum_{j=1}^p \beta_j X_{ij} - \log\left(1 + \exp\left(\sum_{j=1}^p \beta_j X_{ij}\right)\right)\right].$$

<u>Correction :</u> *The likelihood is*

$$V(\beta) = \prod_{i=1}^n \left[p_i(\beta)^{Y_i} (1 - p_i(\beta))^{1-Y_i}\right].$$

Therefore,

$$\begin{aligned}\mathcal{L}(\beta) &= \sum_{i=1}^n \left[Y_i \log(p_i(\beta)) + (1 - Y_i)\log(1 - p_i(\beta))\right] \\ &= \sum_{i=1}^n \left[Y_i \sum_{j=1}^p \beta_j X_{ij} - \log\left(1 + \exp\left(\sum_{j=1}^p \beta_j X_{ij}\right)\right)\right].\end{aligned}$$

2. Prove that $\mathcal{L}$ is a concave function on $\mathbb{R}^p$.

<u>Correction :</u> *We first study the function*

$$f(x) = \log\left(1 + e^x\right), \quad x \in \mathbb{R}.$$

We have, for $x \in \mathbb{R}$,

$$f'(x) = \frac{1}{1 + e^{-x}}, \quad f''(x) = \frac{e^{-x}}{(1 + e^{-x})^2},$$

and $f''(x) > 0$. Therefore, f is a convex function. Now, for any $\alpha \in [0, 1]$, and any p-dimensional vectors β and β'

$$\log\left(1 + \exp\left(\sum_{j=1}^p (\alpha\beta_j + (1-\alpha)\beta'_j) X_{ij}\right)\right) = f\left(\sum_{j=1}^p (\alpha\beta_j + (1-\alpha)\beta'_j) X_{ij}\right)$$
$$\leq \alpha f\left(\sum_{j=1}^p \beta_j X_{ij}\right) + (1-\alpha) f\left(\sum_{j=1}^p \beta'_j X_{ij}\right).$$

It yields

$$\sum_{i=1}^n \log\left(1 + \exp\left(\sum_{j=1}^p (\alpha\beta_j + (1-\alpha)\beta'_j) X_{ij}\right)\right) \leq$$
$$\alpha \sum_{i=1}^n \log\left(1 + \exp\left(\sum_{j=1}^p \beta_j X_{ij}\right)\right) + (1-\alpha) \sum_{i=1}^n \log\left(1 + \exp\left(\sum_{j=1}^p \beta'_j X_{ij}\right)\right)$$

Therefore, since $\beta \longmapsto \sum_{i=1}^n Y_i \sum_{j=1}^p \beta_j X_{ij}$ is linear,

$$\mathcal{L}(\alpha\beta + (1-\alpha)\beta') \geq \alpha\mathcal{L}(\beta) + (1-\alpha)\mathcal{L}(\beta')$$

and $\mathcal{L}$ is concave.

3. We estimate β^* by using the estimate

$$\widehat{\beta} \in \arg\min_{\beta \in \mathbb{R}^p} \left\{ -\mathcal{L}(\beta) + \lambda \|\beta\|_{\ell_1} \right\},$$

where $\|\beta\|_{\ell_1}$ is the ℓ_1-norm of the vector β and $\lambda > 0$ is a constant. Justify carefully the introduction of $\widehat{\beta}$ to estimate β^*. In the sequel, we denote for any $\beta \in \mathbb{R}^p$,

$$C_\lambda(\beta) = -\mathcal{L}(\beta) + \lambda \|\beta\|_{\ell_1}.$$

<u>Correction :</u> *Using previous questions, C_λ is convex on $\mathbb{R}^p$. We are in the high-dimensional setting, so we need to use a convex criterion to minimize. The vector β^* is sparse, so it is natural to use a Lasso-type estimate by introducing the ℓ_1-penalty.*

4. Deduce that $\widehat{\beta}$ is a minimizer of C_λ if and only if there exists $w \in \mathbb{R}^p$ such that $\|w\|_\infty \leq 1$ and $w^T\widehat{\beta} = \|\widehat{\beta}\|_1$ and such that

$$X^T\Big(Y - p(\widehat{\beta})\Big) = \lambda w.$$

Correction : *We have*

$$\begin{aligned}\frac{\partial \mathcal{L}(\beta)}{\partial \beta_j} &= \sum_{i=1}^n \left[Y_i X_{ij} - \frac{X_{ij}}{1+\exp\big(-\sum_{\ell=1}^p \beta_\ell X_{i\ell}\big)}\right]\\ &= (X^TY)_j - (X^Tp(\beta))_j.\end{aligned}$$

Since C_λ is a convex function, $\widehat{\beta}$ is a minimizer of C_λ if and only if 0 belongs to the subdifferential of C_λ at $\widehat{\beta}$. We obtain the conclusion.

5. We denote H the Hessian matrix associated with $\mathcal{L}$. It means that for any $(j,k) \in \{1,\dots,p\}$

$$H_{jk}(\beta) = \frac{\partial^2 \mathcal{L}(\beta)}{\partial \beta_j \partial \beta_k}.$$

(a) For any $\beta \in \mathbb{R}^p$, give the expression of $H_{jk}(\beta)$.
Correction : *We have*

$$\frac{\partial \mathcal{L}(\beta)}{\partial \beta_j} = \sum_{i=1}^n \left[Y_i X_{ij} - \frac{X_{ij}}{1+\exp\big(-\sum_{\ell=1}^p \beta_\ell X_{i\ell}\big)}\right].$$

Therefore,

$$\frac{\partial^2 \mathcal{L}(\beta)}{\partial \beta_j \partial \beta_k} = -\sum_{i=1}^n \frac{X_{ij}X_{ik}\exp\big(-\sum_{\ell=1}^p \beta_\ell X_{i\ell}\big)}{\Big(1+\exp\big(-\sum_{\ell=1}^p \beta_\ell X_{i\ell}\big)\Big)^2}.$$

(b) Deduce that for any vector $v \in \mathbb{R}^p$, $v^TH(\beta)v \leq 0$ and

$$v^TH(\beta)v = 0 \iff Xv = 0.$$

Correction :

$$\begin{aligned}v^TH(\beta)v &= \sum_{j=1}^p\sum_{k=1}^p v_jv_kH_{jk}(\beta)\\ &= -\sum_{i=1}^n\sum_{j=1}^p\sum_{k=1}^p v_jv_k\frac{X_{ij}X_{ik}\exp\big(-\sum_{\ell=1}^p \beta_\ell X_{i\ell}\big)}{\Big(1+\exp\big(-\sum_{\ell=1}^p \beta_\ell X_{i\ell}\big)\Big)^2}\\ &= -\sum_{i=1}^n (Xv)_i^2\frac{\exp\big(-\sum_{\ell=1}^p \beta_\ell X_{i\ell}\big)}{\Big(1+\exp\big(-\sum_{\ell=1}^p \beta_\ell X_{i\ell}\big)\Big)^2}.\end{aligned}$$

Therefore, $v^T H(\beta)v \le 0$ *and*

$$v^T H(\beta)v = 0 \iff \text{ for any } i \in \{1, \ldots, n\},\ (Xv)_i = 0 \iff Xv = 0.$$

(c) Prove that if X is full rank, then there exists a unique vector $\widehat{\beta}$ minimizing the function C_λ on $\mathbb{R}^p$.

Correction : *If* X *is full rank, the previous question shows that the log-likelihood* $\mathcal{L}$ *is strictly concave. Then,* C_λ *is strictly convex on* $\mathbb{R}^p$*. We obtain the result.*

6. Describe precisely the tuning of λ by using cross-validation.

Correction : *For any* $i \in \{1, \ldots, n\}$*, we denote* $x_i = (X_{i1}, \ldots, X_{ip})^T$*.*

(a) Choose V *and a discrete set* Λ *of possible values for* λ*.*

(b) Split the training set $\{1, \ldots, n\}$ *into* V *subsets,* $B_1, \ldots, B_V$*, of roughly the same size.*

(c) For each value of $\lambda \in \Lambda$*, for* $k = 1, \ldots, V$*, compute the estimate* $\widehat{\beta}_\lambda^{(-k)}$ *on the training set* $((x_i, Y_i)_{i \in B_\ell})_{\ell \neq k}$ *and record the total error on the validation set* B_k*:*

$$e_k(\lambda) := \frac{1}{card(B_k)} \sum_{i \in B_k} \left(Y_i - \widehat{Y}_{i,\lambda}^{(-k)}\right)^2,$$

where $\widehat{Y}_{i,\lambda}^{(-k)} = 1$ *if* $p_i\left(\widehat{\beta}_\lambda^{(-k)}\right) > 0.5$ *and* $\widehat{Y}_{i,\lambda}^{(-k)} = 0$ *otherwise.*

(d) Compute the average error over all folds,

$$CV(\lambda) := \frac{1}{V} \sum_{k=1}^{V} e_k(\lambda) = \frac{1}{V} \sum_{k=1}^{V} \frac{1}{card(B_k)} \sum_{i \in B_k} \left(Y_i - \widehat{Y}_{i,\lambda}^{(-k)}\right)^2,$$

(e) We choose the value of tuning parameter that minimizes this function CV *on* Λ*:*

$$\hat{\lambda} := \arg\min_{\lambda \in \Lambda} CV(\lambda).$$

Exercise 2: Uniqueness of the Lasso estimate

We consider the linear regression model

$$Y = X\beta^* + \varepsilon,$$

with X the known design matrix whose columns are denoted $X_1, X_2, \dots, X_p$. Here, $\beta^* \in \mathbb{R}^p$ is an unknown vector to estimate and ε is the error vector. We consider the Lasso estimate of β^* defined by

$$\widehat{\beta} \in \arg\min_{\beta \in \mathbb{R}^p} \Big\{ \|Y - X\beta\|_{\ell_2}^2 + \lambda \|\beta\|_{\ell_1} \Big\},$$

where $\lambda > 0$ is a constant. We recall that a vector $\widehat{\beta} \in \mathbb{R}^p$ is a global minimizer of the criterion $C_{\lambda,1}$ defined for $\beta \in \mathbb{R}^p$ by

$$C_{\lambda,1}(\beta) = \|Y - X\beta\|_{\ell_2}^2 + \lambda \|\beta\|_{\ell_1}$$

if and only if there exists w such that

$$\begin{cases} \lambda w & = 2X^T(Y - X\widehat{\beta}), \\ \|w\|_\infty & \leq 1, \\ w_j & = \mathrm{sign}(\widehat{\beta}_j) \qquad \text{if } j \in \{1, \dots, p\} \text{ is such that } \widehat{\beta}_j \neq 0. \end{cases}$$

We set

$$T(\widehat{\beta}) = w.$$

The goal of this exercice is to derive conditions so that $C_{\lambda,1}$ has a unique minimizer. In the sequel, we consider two vectors $\widehat{\beta}^{(1)}$ and $\widehat{\beta}^{(2)}$ that minimize $C_{\lambda,1}$.

1. Show that $X\widehat{\beta}^{(1)} = X\widehat{\beta}^{(2)}$.

   Correction : *Assume that $X\widehat{\beta}^{(1)} \neq X\widehat{\beta}^{(2)}$. We set $u = \big(\widehat{\beta}^{(1)} + \widehat{\beta}^{(2)}\big)/2$. We have*

$$Y - X\beta^{(1)} \neq Y - X\beta^{(2)}.$$

   Since $x \longmapsto \|x\|_{\ell_2}^2$ is strictly convex and $x \longmapsto \|x\|_{\ell_1}$ is convex, we have:

$$\begin{aligned} C_{\lambda,1}(u) &= \|\big(Y - X\beta^{(1)}\big)/2 + \big(Y - X\beta^{(2)}\big)/2\|_{\ell_2}^2 + \lambda \|\beta^{(1)}/2 + \beta^{(2)}/2\|_{\ell_1} \\ &< \frac{1}{2}\Big(\|Y - X\beta^{(1)}\|_{\ell_2}^2 + \lambda \|\beta^{(1)}\|_{\ell_1} + \|Y - X\beta^{(2)}\|_{\ell_2}^2 + \lambda \|\beta^{(2)}\|_{\ell_1} \Big) \end{aligned}$$

   and

$$C_{\lambda,1}(u) < \frac{C_{\lambda,1}(\beta^{(1)}) + C_{\lambda,1}(\beta^{(2)})}{2} = C_{\lambda,1}(\beta^{(1)}) = C_{\lambda,1}(\beta^{(2)}).$$

   We obtain a contradiction. Therefore, $X\widehat{\beta}^{(1)} = X\widehat{\beta}^{(2)}$.

2. Deduce that $T(\widehat{\beta}^{(1)}) = T(\widehat{\beta}^{(2)})$.

   Correction : *We have*

$$\begin{aligned}\lambda T(\widehat{\beta}^{(1)}) &= 2X^T\big(Y - X\widehat{\beta}^{(1)}\big)\\ &= 2X^T\big(Y - X\widehat{\beta}^{(2)}\big)\\ &= \lambda T(\widehat{\beta}^{(2)})\end{aligned}$$

   and, since $\lambda > 0$, $T(\widehat{\beta}^{(1)}) = T(\widehat{\beta}^{(2)})$.

3. We set

$$w = T\big(\widehat{\beta}^{(1)}\big) = T\big(\widehat{\beta}^{(2)}\big)$$

   and

$$J = \Big\{j \in \{1, \dots, p\} : \quad |w_j| = 1\Big\}.$$

   (a) Prove that if $j \in \{1, \dots, p\} \setminus J$, $\widehat{\beta}_j^{(1)} = 0$.

   Correction : *Let* $j \in \{1, \dots, p\} \setminus J$. *If* $\widehat{\beta}_j^{(1)} \neq 0$, *then we can set* $w_j = sign(\widehat{\beta}_j^{(1)})$ *and* $|w_j| = 1$, *which implies* $j \in J$. *contradiction. Therefore,* $\widehat{\beta}_j^{(1)} = 0$.

   (b) We denote $X(J)$ the matrix whose columns are the columns X_j for $j \in J$. The matrix $X(J)$ has n rows and $|J|$ columns. Show that for any $j \in J$,

$$\big(X(J)^T X(J)\widehat{\beta}^{(1)}(J)\big)_j = X_j^T Y - \frac{\lambda}{2} w_j,$$

   where $\widehat{\beta}^{(1)}(J)$ is the vector of size $|J|$ whose components are the $\widehat{\beta}_j^{(1)}$'s for $j \in J$.

   Correction : *We have*

$$2X^T(Y - X\widehat{\beta}^{(1)}) = \lambda w.$$

   Therefore, for $j \in J$,

$$\begin{aligned}X_j^T Y - \frac{\lambda}{2} w_j &= \Big(X^T X\widehat{\beta}^{(1)}\Big)_j\\ &= X_j^T X\widehat{\beta}^{(1)}\\ &= X_j^T \sum_{\ell=1}^{p} \widehat{\beta}_\ell^{(1)} X_\ell\\ &= X_j^T \sum_{\ell \in J} \widehat{\beta}_\ell^{(1)} X_\ell\\ &= \big(X(J)^T X(J)\widehat{\beta}^{(1)}(J)\big)_j.\end{aligned}$$

   We have used that $\widehat{\beta}_\ell^{(1)} = 0$ *if* $\ell \notin J$.

(c) Conclude that if $X(J)^T X(J)$ is invertible, then $\widehat{\beta}^{(1)} = \widehat{\beta}^{(2)}$.

Correction : *We set z the vector of size $|J|$ whose components for $j \in J$ are*

$$z_j = X_j^T Y - \frac{\lambda}{2} w_j.$$

From the previous question, we deduce

$$X(J)^T X(J) \widehat{\beta}^{(1)}(J) = z.$$

Therefore,

$$\widehat{\beta}^{(1)}(J) = \Big(X(J)^T X(J)\Big)^{-1} z.$$

Similarly, we obtain

$$\widehat{\beta}^{(2)}(J) = \Big(X(J)^T X(J)\Big)^{-1} z$$

and $\widehat{\beta}_j^{(2)} = 0$ if $j \notin J$. This implies $\widehat{\beta}^{(1)} = \widehat{\beta}^{(2)}$.

(d) Deduce an algorithm that can be used in practice to check that $C_{\lambda,1}$ has a unique minimizer.

Correction : *We compute $\widehat{\beta}$ a minimizer of $C_{\lambda,1}$. Then, we determine*

$$J = \left\{ j \in \{1, \ldots, p\} : \quad |X_j^T (Y - X\widehat{\beta})| = \frac{\lambda}{2} \right\}.$$

Finally, we check that $X(J)^T X(J)$ is invertible.

Exercise 3: Model selection for functional data

We assume that we observe $Y_1, \dots, Y_n$ such that, for all $i = 1, \dots, n$,

$$Y_i = \langle X_i, \beta^* \rangle + \varepsilon_i$$

- Y_i is the variable of interest (we suppose here that Y_i is a scalar quantity),
- X_i is a fixed functional variable belonging to $\mathbb{L}^2([0,1])$,
- β^* is an unknown element of $\mathcal{F} = \mathbb{L}^2([0,1])$, called *slope function*,
- $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d. centered Gaussian variables. We denote $\sigma^2 = \mathrm{Var}(\varepsilon_1)$.

In the last expression $\langle \cdot, \cdot \rangle$ denotes the $\mathbb{L}^2([0,1])$-scalar product. We suppose that for all $t \in [0,1]$,

$$\widehat{\mu}(t) := \frac{1}{n} \sum_{i=1}^{n} X_i(t) = 0.$$

We consider the empirical covariance operator

$$\widehat{\Gamma} : f \in \mathbb{L}^2([0,1]) \longmapsto \frac{1}{n} \sum_{i=1}^{n} \langle f, X_i \rangle X_i$$

and its eigenelements $(\widehat{\psi}_m, \widehat{\lambda}_m)_{m \geq 1}$ with $\widehat{\lambda}_1 \geq \widehat{\lambda}_2 \geq \cdots \geq \widehat{\lambda}_m \geq \cdots$. Denoting for any integer m such that $m \leq n$,

$$S_m = \mathrm{span}\{\widehat{\psi}_1, \dots, \widehat{\psi}_m\},$$

we consider

$$\gamma_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \langle \beta, X_i \rangle)^2$$

and

$$\widehat{\beta}_m = \arg \min_{\beta \in S_m} \gamma_n(\beta).$$

Finally, we estimate β^* with $\widehat{\beta}_{\widehat{m}}$ with

$$\widehat{m} = \arg \min_{m \in \mathbb{N}^*, m \leq n} \left\{ \gamma_n(\widehat{\beta}_m) + \mathrm{pen}(m) \right\},$$

where $\mathrm{pen}(m)$ is a penalty function only depending on m. In the sequel, we set for any $f \in \mathbb{L}^2([0,1])$,

$$\|f\|_n^2 = \frac{1}{n} \sum_{i=1}^{n} \langle f, X_i \rangle^2, \quad \nu_n(f) = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \langle f, X_i \rangle.$$

1. Show that for any integer m such that $m \leq n$, and any $\beta \in S_m$,

$$\|\widehat{\beta}_{\widehat{m}} - \beta^*\|_n^2 \leq \|\beta - \beta^*\|_n^2 + 2\nu_n(\widehat{\beta}_{\widehat{m}} - \beta) + \text{pen}(m) - \text{pen}(\widehat{m}).$$

Correction : *We have*

$$\begin{aligned}\gamma_n(\widehat{\beta}_{\widehat{m}}) + \text{pen}(\widehat{m}) &\leq \gamma_n(\widehat{\beta}_m) + \text{pen}(m)\\ &\leq \gamma_n(\beta) + \text{pen}(m).\end{aligned}$$

Therefore,

$$\frac{1}{n}\sum_{i=1}^n (Y_i - \langle \widehat{\beta}_{\widehat{m}}, X_i\rangle)^2 + \text{pen}(\widehat{m}) \leq \frac{1}{n}\sum_{i=1}^n (Y_i - \langle \beta, X_i\rangle)^2 + \text{pen}(m),$$

which means

$$\frac{1}{n}\sum_{i=1}^n (\varepsilon_i + \langle \beta^* - \widehat{\beta}_{\widehat{m}}, X_i\rangle)^2 + \text{pen}(\widehat{m}) \leq \frac{1}{n}\sum_{i=1}^n (\varepsilon_i + \langle \beta^* - \beta, X_i\rangle)^2 + \text{pen}(m).$$

We obtain

$$\|\widehat{\beta}_{\widehat{m}} - \beta^*\|_n^2 \leq \|\beta^* - \beta\|_n^2 + 2\nu_n(\widehat{\beta}_{\widehat{m}} - \beta) + \text{pen}(m) - \text{pen}(\widehat{m}),$$

which is the result.

2. Prove that for any $\alpha \in (0, 1)$,

$$\|\widehat{\beta}_{\widehat{m}} - \beta^*\|_n^2 \leq (1 + 2\alpha)\|\beta - \beta^*\|_n^2 + 2\alpha\|\widehat{\beta}_{\widehat{m}} - \beta^*\|_n^2 + \alpha^{-1} \sup_{f \in S_{m\vee\widehat{m}}, \|f\|_n = 1} \nu_n^2(f) + \text{pen}(m) - \text{pen}(\widehat{m}).$$

Correction : *Since $\widehat{\beta}_{\widehat{m}} - \beta \in S_{m\vee\widehat{m}}$, we have:*

$$\begin{aligned}2\nu_n(\widehat{\beta}_{\widehat{m}} - \beta) &\leq 2\|\widehat{\beta}_{\widehat{m}} - \beta\|_n \sup_{f \in S_{m\vee\widehat{m}}, \|f\|_n = 1} \nu_n(f)\\ &\leq \alpha\|\widehat{\beta}_{\widehat{m}} - \beta\|_n^2 + \alpha^{-1} \sup_{f \in S_{m\vee\widehat{m}}, \|f\|_n = 1} \nu_n^2(f)\\ &\leq 2\alpha\|\widehat{\beta}_{\widehat{m}} - \beta^*\|_n^2 + 2\alpha\|\beta - \beta^*\|_n^2 + \alpha^{-1} \sup_{f \in S_{m\vee\widehat{m}}, \|f\|_n = 1} \nu_n^2(f).\end{aligned}$$

This gives the result.

3. Deduce that

$$\|\widehat{\beta}_{\widehat{m}} - \beta^*\|_n^2 \leq 3\|\beta^* - \beta\|_n^2 + 8 \sup_{f \in S_{m\vee\widehat{m}}, \|f\|_n = 1} \nu_n^2(f) + 2\text{pen}(m) - 2\text{pen}(\widehat{m}).$$

Correction : *This is obvious by taking $\alpha = 1/4$.*

4. We study now for any integer m such that $m \leq n$,

$$H_m = \sup_{f \in S_m, \|f\|_n = 1} \nu_n(f).$$

For this purpose, we introduce the matrix Ψ_m such that for $1 \leq i \leq n$ and $1 \leq k \leq m$

$$(\Psi_m)_{ik} = \langle X_i, \widehat{\psi}_k \rangle$$

and P_m the projection matrix on $Im(\Psi_m)$.

(a) Establish that

$$H_m = \frac{1}{\sqrt{n}} \sup_{x \in \mathbb{R}^n, x \in Im(\Psi_m), x^T x = 1} x^T \varepsilon,$$

with $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$.

Correction : *We have:*

$$\begin{aligned} H_m &= \sup_{f \in S_m, \|f\|_n = 1} \nu_n(f) \\ &= \sup_{f \in S_m, \|f\|_n = 1} \frac{1}{n} \sum_{i=1}^n \varepsilon_i \langle f, X_i \rangle. \end{aligned}$$

Given $f = \sum_{k=1}^m u_k \widehat{\psi}_k \in S_m$, *we set*

$$x = \frac{1}{\sqrt{n}} \big(\langle f, X_1 \rangle, \ldots, \langle f, X_n \rangle \big)^T = \frac{1}{\sqrt{n}} \Psi_m \times (u_1, \ldots, u_m)^T \in Im(\Psi_m)$$

and

$$\|f\|_n = 1 \iff x^T x = 1.$$

Therefore,

$$H_m = \frac{1}{\sqrt{n}} \sup_{x \in \mathbb{R}^n, x \in Im(\Psi_m), x^T x = 1} x^T \varepsilon.$$

(b) Deduce that nH_m^2 has a χ^2 distribution.

Correction : *From the previous question, we deduce*

$$nH_m^2 = \sup_{x \in \mathbb{R}^n, x \in Im(\Psi_n), x^T x = 1} \left(x^T P_m \varepsilon \right)^2 = \|P_m \varepsilon\|_{\ell_2}^2$$

and nH_m^2 *has a* χ^2 *distribution with* m *degrees of freedom.*

A.9 Exam 2024-2025

High-dimensional statistics

EXAM: duration 2h30

Documents, calculators, phones and smartphones are forbidden

Exercise 1: Lasso for the Poisson model

In the high-dimensional setting, we observe $(Y_1, Y_2, \dots, Y_n)$, n independent random variables such that for any i, Y_i is a Poisson variable whose mean is

$$\mathbb{E}[Y_i] = \exp\Big(\sum_{j=1}^{p} \beta_j^* X_{ij}\Big).$$

In the last expression, X_{ij} denotes the value of the predictor X_j associated with Y_i and β^* is a sparse p-dimensional unknown vector to be estimated. In the sequel, we denote for any $\beta \in \mathbb{R}^p$ and any $i \in \{1, \dots, n\}$,

$$\lambda_i(\beta) = \exp\Big(\sum_{j=1}^{p} \beta_j X_{ij}\Big)$$

and $\lambda(\beta) = (\lambda_1(\beta), \dots, \lambda_n(\beta))^T$.

1. Show that the log-likelihood associated with this model is, for $\beta \in \mathbb{R}^p$,

$$\mathcal{L}(\beta) = \sum_{i=1}^{n} \left[Y_i \sum_{j=1}^{p} \beta_j X_{ij} - \exp\left(\sum_{j=1}^{p} \beta_j X_{ij} \right) - \log\big(Y_i!\big) \right].$$

Indication : We recall that if Y is a Poisson variable with mean $\mu > 0$, we have for any $k \in \{0, 1, 2, \dots\}$, $\mathbb{P}(Y = k) = \exp(-\mu)\frac{\mu^k}{k!}$.

Correction : *The likelihood is*

$$V(\beta) = \prod_{i=1}^{n} \Big[\exp\big(-\lambda_i(\beta)\big) \frac{\lambda_i(\beta)^{Y_i}}{Y_i!} \Big].$$

Therefore,

$$\begin{aligned}\mathcal{L}(\beta) &= \log\big(V(\beta)\big)\\ &= \sum_{i=1}^n \Big[Y_i\log(\lambda_i(\beta)) - \lambda_i(\beta) - \log\big(Y_i!\big)\Big]\\ &= \sum_{i=1}^n \left[Y_i\sum_{j=1}^p \beta_j X_{ij} - \exp\left(\sum_{j=1}^p \beta_j X_{ij}\right) - \log\big(Y_i!\big)\right].\end{aligned}$$

2. Prove that $\mathcal{L}$ is a concave function on $\mathbb{R}^p$.

   Correction : *Since $\beta\in\mathbb{R}^p \longmapsto \sum_{j=1}^p \beta_j X_{ij}$ is linear and $\beta\in\mathbb{R}^p \longmapsto \exp\Big(\sum_{j=1}^p \beta_j X_{ij}\Big)$ is convex, $\mathcal{L}$ is a concave function on $\mathbb{R}^p$.*

3. We estimate β^* by using the estimate

$$\widehat{\beta} \in \arg\min_{\beta\in\mathbb{R}^p}\Big\{-\mathcal{L}(\beta) + \gamma\|\beta\|_{\ell_1}\Big\},$$

   where $\|\beta\|_{\ell_1}$ is the ℓ_1-norm of the vector β and $\gamma > 0$ is a constant. Justify carefully the introduction of $\widehat{\beta}$ to estimate β^*. In the sequel, we denote for any $\beta\in\mathbb{R}^p$,

$$C_\gamma(\beta) = -\mathcal{L}(\beta) + \gamma\|\beta\|_{\ell_1}.$$

   Correction : *Using previous questions, C_γ is convex on $\mathbb{R}^p$. We are in the high-dimensional setting, so we need to use a convex criterion to minimize. The vector β^* is sparse, so it is natural to use a Lasso-type estimate by introducing the ℓ_1-penalty.*

4. Deduce that $\widehat{\beta}$ is a minimizer of C_γ if and only if there exists $w\in\mathbb{R}^p$ such that $\|w\|_\infty \le 1$ and $w^T\widehat{\beta} = \|\widehat{\beta}\|_1$ and such that

$$X^T\Big(Y - \lambda\big(\widehat{\beta}\big)\Big) = \gamma w.$$

   Correction : *We have*

$$\begin{aligned}\frac{\partial\mathcal{L}(\beta)}{\partial\beta_j} &= \sum_{i=1}^n \left[Y_iX_{ij} - X_{ij}\exp\left(\sum_{\ell=1}^p \beta_\ell X_{i\ell}\right)\right]\\ &= (X^TY)_j - (X^T\lambda(\beta))_j.\end{aligned}$$

   Since C_γ is a convex function, $\widehat{\beta}$ is a minimizer of C_γ if and only if 0 belongs to the subdifferential of C_γ at $\widehat{\beta}$. We obtain the conclusion.

5. We denote H the Hessian matrix associated with $\mathcal{L}$. It means that for any $(j,k) \in \{1,\dots,p\}$
$$H_{jk}(\beta) = \frac{\partial^2 \mathcal{L}(\beta)}{\partial \beta_j \partial \beta_k}.$$
   (a) For any $\beta \in \mathbb{R}^p$, give the expression of $H_{jk}(\beta)$.
   Correction : *We have*
$$\frac{\partial \mathcal{L}(\beta)}{\partial \beta_j} = \sum_{i=1}^n \left[Y_i X_{ij} - X_{ij} \exp\left(\sum_{\ell=1}^p \beta_\ell X_{i\ell} \right) \right].$$
   Therefore,
$$\frac{\partial^2 \mathcal{L}(\beta)}{\partial \beta_j \partial \beta_k} = -\sum_{i=1}^n X_{ij} X_{ik} \exp\left(\sum_{\ell=1}^p \beta_\ell X_{i\ell} \right).$$
   (b) Deduce that for any vector $v \in \mathbb{R}^p$, $v^T H(\beta) v \le 0$ and
$$v^T H(\beta) v = 0 \iff Xv = 0.$$
   Correction :
$$\begin{aligned} v^T H(\beta) v &= \sum_{j=1}^p \sum_{k=1}^p v_j v_k H_{jk}(\beta) \\ &= -\sum_{i=1}^n \sum_{j=1}^p \sum_{k=1}^p v_j v_k X_{ij} X_{ik} \exp\left(\sum_{\ell=1}^p \beta_\ell X_{i\ell} \right) \\ &= -\sum_{i=1}^n (Xv)_i^2 \exp\left(\sum_{\ell=1}^p \beta_\ell X_{i\ell} \right). \end{aligned}$$
   Therefore, $v^T H(\beta) v \le 0$ *and*
$$v^T H(\beta) v = 0 \iff \text{for any } i \in \{1,\dots,n\},\ (Xv)_i = 0 \iff Xv = 0.$$
   (c) Prove that if X is full rank, then there exists a unique vector $\widehat{\beta}$ minimizing the function C_γ on $\mathbb{R}^p$.
   Correction : *If X is full rank, the previous question shows that the log-likelihood $\mathcal{L}$ is strictly concave. Then, C_λ is strictly convex on $\mathbb{R}^p$. We obtain the result.*

Exercise 2: Estimation of the mean of functional data

Let $X_1, \dots, X_n \sim_{i.i.d.} X$ where X is a random variable taking values in the space $\mathbb{L}^2([0,1])$ of square-integrable functions on $[0,1]$ equipped with its usual scalar product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt, \qquad f, g \in \mathbb{L}^2([0,1])$$

and associated norm $\|f\| = \sqrt{\langle f, f \rangle}$, $f \in \mathbb{L}^2([0,1])$. The aim of the exercise is to study several estimators of the mean function $\mu = \mathbb{E}[X]$ from a set of noisy and discretized observations of the X_i's. More precisely, we assume we observe a data matrix

$$\mathbf{Y} = (Y_{i,j})_{1 \le i \le n; j=0,\dots,p},$$

such that

$$Y_{i,j} = X_i(t_j) + \varepsilon_{i,j}, \qquad i = 1, \dots, n; j = 0, \dots, p,$$

with

- $t_0, \dots, t_p$ a regular grid of $[0,1]$ i.e. $t_j = j/p$, $j = 0, \dots, p$;
- $\{\varepsilon_{i,j}\}_{1 \le i \le n; 0 \le j \le p} \sim_{i.i.d.} \mathcal{N}(0, \sigma^2)$ are noise variables assumed to be independent of $X_1, \dots, X_n$.

We study in the exercise two estimators of μ:

- a first estimator defined as a step function as defined in the course

$$\tilde{\mu}(t) = \frac{1}{n} \sum_{i=1}^{n} Y_{i,j}, \qquad \text{for } t \in [t_{j-1}, t_j), j = 1, \dots, p.$$

- a second least-squares estimator

$$\check{\mu}_D(t) = \sum_{k=1}^{D} \check{m}_k \varphi_k(t), \qquad t \in [0,1],$$

  with $\{\varphi_1, \dots, \varphi_D\}$ a sequence of orthonormal functions of $(\mathbb{L}^2([0,1]), \langle \cdot, \cdot \rangle)$ and $\check{\mathbf{m}}_D = (\check{m}_1, \dots, \check{m}_D)^T \in \mathbb{R}^D$ is solution of the minimisation problem

$$\min_{\mathbf{m}=(m_1,\dots,m_D)^T \in \mathbb{R}^D} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{p} \left(Y_{i,j} - \sum_{k=1}^{D} m_k \varphi_k(t_j) \right)^2.$$

The aim is to study the quadratic risk of both estimators.

1. **Study of the estimator $\tilde{\mu}$:** we split the quantity $\tilde{\mu} - \mu$ in three terms

$$\tilde{\mu} - \mu = \widehat{\mu} - \mu + R_{disc} + R_{noise},$$

where

$$\widehat{\mu}(t) = \frac{1}{n}\sum_{i=1}^{n} X_i(t), \qquad t \in [0,1];$$

$$R_{disc}(t) = \frac{1}{n}\sum_{i=1}^{n}(X_i(t_j) - X_i(t)) \qquad \text{for } t \in [t_{j-1}, t_j), j = 1, \ldots, p;$$

$$R_{noise}(t) = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_{i,j} \qquad \text{for } t \in [t_{j-1}, t_j), j = 1, \ldots, p.$$

(a) Assume that $\mathbb{E}[\|X\|^2] < +\infty$, prove that

$$\mathbb{E}[\|\widehat{\mu} - \mu\|^2] = \frac{\mathbb{E}[\|X - \mu\|^2]}{n}.$$

Correction :

$$\mathbb{E}[\|\widehat{\mu} - \mu\|^2] = \mathbb{E}\left[\left\|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right\|^2\right] = \frac{1}{n^2}\sum_{i,i'=1}^{n}\mathbb{E}[\langle X_i - \mu, X_{i'} - \mu\rangle].$$

For $i \neq i'$, since $X_i - \mu$ and $X_{i'} - \mu$ are independent,

$$\mathbb{E}[\langle X_i - \mu, X_{i'} - \mu\rangle] = \langle \mathbb{E}[X_i] - \mu, \mathbb{E}[X_{i'} - \mu]\rangle = 0.$$

The result follows.

(b) Assume that there exist $\alpha \in (0,1]$ and $L > 0$ such that

$$\mathbb{E}[(X(t) - X(s))^2] \leq L|t-s|^{2\alpha}, \qquad \text{for all } t, s \in [0,1].$$

Prove that

$$\mathbb{E}[\|R_{disc}\|^2] \leq \frac{L}{(2\alpha+1)p^{2\alpha+1}}.$$

Correction : *By using e.g. the Cauchy-Schwartz inequality*

$$\begin{aligned}
\mathbb{E}[\|R_{disc}\|^2] &= \mathbb{E}\left[\int_0^1 R_{disc}^2(t)dt\right] = \sum_{j=1}^p \int_{t_{j-1}}^{t_j} \mathbb{E}[R_{disc}^2(t)]dt \\
&= \sum_{j=1}^p \int_{t_{j-1}}^{t_j} \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n (X_i(t_j) - X_i(t))\right)^2\right] dt \\
&= \sum_{j=1}^p \int_{t_{j-1}}^{t_j} \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n (X_i(t_j) - X_i(t))\right)^2\right] dt \\
&\leq \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^p \int_{t_{j-1}}^{t_j} \mathbb{E}\left[(X_i(t_j) - X_i(t))^2\right] dt \\
&\leq \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^p \int_{t_{j-1}}^{t_j} L|t_j - t|^{2\alpha} dt = \frac{L}{(2\alpha+1)p^{2\alpha}}.
\end{aligned}$$

(c) Prove that

$$\mathbb{E}[\|R_{noise}\|^2] = \frac{\sigma^2}{n}.$$

Correction :

$$\begin{aligned}
\mathbb{E}[\|R_{noise}\|^2] &= \mathbb{E}\left[\int_0^1 R_{noise}^2(t)dt\right] = \sum_{j=1}^p \int_{t_{j-1}}^{t_j} \mathbb{E}[R_{noise}^2(t)]dt \\
&= \sum_{j=1}^p \int_{t_{j-1}}^{t_j} \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n \varepsilon_{i,j}\right)^2\right] dt \\
&= \frac{1}{n^2}\sum_{j=1}^p \int_{t_{j-1}}^{t_j} \sum_{i,i'=1}^n \mathbb{E}\left[\varepsilon_{i,j}\varepsilon_{i',j}\right] dt = \frac{\sigma^2}{n}.
\end{aligned}$$

(d) Deduce an upper-bound on the risk of the estimator

$$\mathbb{E}[\|\tilde{\mu} - \mu\|^2].$$

Discuss how this upper-bound varies with the number of individuals n, the number p, the noise variance, and the regularity of the process (quantified by α).

Correction : *We have:*

$$\begin{aligned}\mathbb{E}[\|\tilde{\mu}-\mu\|^2] &\leq 3\mathbb{E}[\|\widehat{\mu}-\mu\|^2]+3\mathbb{E}[\|R_{disc}\|^2]+3\mathbb{E}[\|R_{noise}\|^2]\\ &\leq \frac{3\mathbb{E}[\|X-\mu\|^2]}{n}+\frac{3L}{(2\alpha+1)p^{2\alpha}}+\frac{3\sigma^2}{n}.\end{aligned}$$

The larger n (or the larger p or the larger α), the smaller the risk. These results are expected. In particulars the smoother the signal, the easier the function to be estimated and the smaller the risk. Of course, when σ^2 increases, the risk increases.

2. **Study of the estimator $\breve{\mu}$:**

   (a) Let

$$\gamma_{n,p}(\mathbf{m})=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\left(Y_{i,j}-\sum_{k=1}^{D}m_k\varphi_k(t_j)\right)^2.$$

Verify that

$$\begin{aligned}\gamma_{n,p}(\mathbf{m}) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}(X_i(t_j)-\widehat{\mu}(t_j))^2+\sum_{j=1}^{p}\left(\widehat{\mu}(t_j)-\sum_{k=1}^{D}m_k\varphi_k(t_j)\right)^2+\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}^2\\ &+\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}(X_i(t_j)-\widehat{\mu}(t_j))\varepsilon_{i,j}+\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}\left(\widehat{\mu}(t_j)-\sum_{k=1}^{D}m_k\varphi_k(t_j)\right).\end{aligned}$$

Correction :

$$\begin{aligned}\gamma_{n,p}(\mathbf{m}) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\left(X_i(t_j)-\widehat{\mu}(t_j)+\widehat{\mu}(t_j)-\sum_{k=1}^{D}m_k\varphi_k(t_j)+\varepsilon_{i,j}\right)^2\\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}(X_i(t_j)-\widehat{\mu}(t_j))^2+\sum_{j=1}^{p}\left(\widehat{\mu}(t_j)-\sum_{k=1}^{D}m_k\varphi_k(t_j)\right)^2\\ &+\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}^2+\underbrace{\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}(X_i(t_j)-\widehat{\mu}(t_j))\left(\widehat{\mu}(t_j)-\sum_{k=1}^{D}m_k\varphi_k(t_j)\right)}_{=0}\\ &+\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}(X_i(t_j)-\widehat{\mu}(t_j))\varepsilon_{i,j}+\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}\left(\widehat{\mu}(t_j)-\sum_{k=1}^{D}m_k\varphi_k(t_j)\right)\end{aligned}$$

(b) Let, for two functions $f, g : [0,1] \to \mathbb{R}$, and a vector $u = (u_1, \ldots, u_p)^T \in \mathbb{R}^p$

$$\langle f, u\rangle_p = \sum_{j=1}^{p} f(t_j)u_j, \quad \langle f, g\rangle_p = \sum_{j=1}^{p} f(t_j)g(t_j), \quad \|f\|_p^2 = \sum_{j=1}^{p} f^2(t_j),$$

$$\overline{\boldsymbol{\varepsilon}}_{\bullet} = \left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_{i,j}\right)_{j=1,\ldots,p}$$

and

$$\Pi_D\mu = \sum_{j=1}^{p}\langle \mu, \varphi_j\rangle\varphi_j.$$

Deduce from (a) that

$$\|\widehat{\mu} - \check{\mu}\|_p^2 \le 2\langle \check{\mu} - \Pi_D\mu, \overline{\boldsymbol{\varepsilon}}_{\bullet}\rangle_p + \|\widehat{\mu} - \Pi_D\mu\|_p^2.$$

Indication: Set $\mathbf{m}_D = (\langle\mu, \varphi_1\rangle, \ldots, \langle\mu, \varphi_D\rangle)^T$ *and use*

$$\gamma_{n,p}(\check{\mathbf{m}}_D) \le \gamma_{n,p}(\mathbf{m}_D).$$

Correction : *Let* $\mathbf{m}_D = (\langle\mu, \varphi_1\rangle, \ldots, \langle\mu, \varphi_D\rangle)^T$, *we have, by definition of* $\check{\mathbf{m}}_D$,

$$\gamma_{n,p}(\check{\mathbf{m}}_D) \le \gamma_{n,p}(\mathbf{m}_D).$$

Then, by (a),

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}(X_i(t_j) - \widehat{\mu}(t_j))^2 \quad + \quad & \sum_{j=1}^{p}\left(\widehat{\mu}(t_j) - \sum_{k=1}^{D}\check{m}_k\varphi_k(t_j)\right)^2 + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}^2 \\
& + \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}(X_i(t_j) - \widehat{\mu}(t_j))\varepsilon_{i,j} + \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}\left(\widehat{\mu}(t_j) - \sum_{k=1}^{D}\check{m}_k\varphi_k(t_j)\right) \\
\le \quad & \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}(X_i(t_j) - \widehat{\mu}(t_j))^2 + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}^2 \\
& + \sum_{j=1}^{p}\left(\widehat{\mu}(t_j) - \sum_{k=1}^{D}\langle\mu, \varphi_k\rangle\varphi_k(t_j)\right)^2 + \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}(X_i(t_j) - \widehat{\mu}(t_j))\varepsilon_{i,j} \\
& + \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}\left(\widehat{\mu}(t_j) - \sum_{k=1}^{D}\langle\mu, \varphi_k\rangle\varphi_k(t_j)\right).
\end{aligned}$$

It means that

$$\begin{aligned}\sum_{j=1}^{p}\left(\widehat{\mu}(t_j)-\sum_{k=1}^{D}\check{m}_k\varphi_k(t_j)\right)^2 &\leq -\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}\left(\widehat{\mu}(t_j)-\sum_{k=1}^{D}\check{m}_k\varphi_k(t_j)\right)\\ &\quad+\sum_{j=1}^{p}\left(\widehat{\mu}(t_j)-\sum_{k=1}^{D}\langle\mu,\varphi_k\rangle\varphi_k(t_j)\right)^2\\ &\quad+\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}\left(\widehat{\mu}(t_j)-\sum_{k=1}^{D}\langle\mu,\varphi_k\rangle\varphi_k(t_j)\right)\\ &\leq -\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{p}\varepsilon_{i,j}\sum_{k=1}^{D}\left(\check{m}_k-\langle\mu,\varphi_k\rangle\right)\varphi_k(t_j)\\ &\quad+\sum_{j=1}^{p}\left(\widehat{\mu}(t_j)-\sum_{k=1}^{D}\langle\mu,\varphi_k\rangle\varphi_k(t_j)\right)^2\end{aligned}$$

(c) Prove that, for all $\eta > 0$,

$$\mathbb{E}[2\langle\check{\mu}-\Pi_D\mu,\overline{\boldsymbol{\varepsilon}}_\bullet\rangle_p] \leq \eta\mathbb{E}[\|\Pi_D\mu-\check{\mu}\|_p^2]+\eta^{-1}\sigma^2\frac{p}{n}.$$

Indication: remember that for all $x, y \in \mathbb{R}$ *and* $\eta > 0$, $2xy \leq \eta x^2 + \eta^{-1}y^2$.
Correction : *Using successively Cauchy-Schwarz inequality and the suggested inequality*

$$2\langle\Pi_D\mu-\check{\mu},\overline{\boldsymbol{\varepsilon}}_\bullet\rangle_p \leq 2\|\Pi_D\mu-\check{\mu}\|_p\|\overline{\boldsymbol{\varepsilon}}_\bullet\|_p \leq \eta\|\Pi_D\mu-\check{\mu}\|_p^2+\eta^{-1}\|\overline{\boldsymbol{\varepsilon}}_\bullet\|_p^2.$$

Now

$$\mathbb{E}[\|\overline{\boldsymbol{\varepsilon}}_\bullet\|_p^2] = \mathbb{E}\left[\sum_{j=1}^{p}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_{i,j}\right)^2\right] = \frac{\sigma^2 p}{n}.$$

(d) Deduce that there exists a constant $C > 0$ such that

$$\mathbb{E}[\|\mu-\check{\mu}\|_p^2] \leq C\left(\mathbb{E}[\|\mu-\hat{\mu}\|_p^2]+\|\mu-\Pi_D\mu\|_p^2+\sigma^2\frac{p}{n}\right).$$

Indication: for all $f, g : [0,1] \to \mathbb{R}$, $\|f+g\|_p^2 \leq 2\|f\|_p^2 + 2\|g\|_p^2$.
Correction : *We have*

$$\|\mu-\check{\mu}\|_p^2 \leq 2\|\mu-\widehat{\mu}\|_p^2+2\|\widehat{\mu}-\check{\mu}\|_p^2.$$

From (b) we obtain,

$$\|\mu - \check{\mu}\|_p^2 \leq 2\|\mu - \widehat{\mu}\|_p^2 + 4\langle \check{\mu} - \Pi_D\mu, \overline{\varepsilon}_\bullet\rangle_p + 2\|\widehat{\mu} - \Pi_D\mu\|_p^2.$$

Taking expectation and applying (c) leads to

$$\begin{aligned}
\mathbb{E}[\|\mu - \check{\mu}\|_p^2] &\leq 2\mathbb{E}\left[\|\mu - \widehat{\mu}\|_p^2\right] + 2\eta\mathbb{E}[\|\Pi_D\mu - \check{\mu}\|_p^2] + 2\eta^{-1}\sigma^2\frac{p}{n} + 2\mathbb{E}\left[\|\widehat{\mu} - \Pi_D\mu\|_p^2\right] \\
&\leq 2\mathbb{E}\left[\|\mu - \widehat{\mu}\|_p^2\right] + 4\eta\mathbb{E}[\|\Pi_D\mu - \mu\|_p^2] + 4\eta\mathbb{E}[\|\mu - \check{\mu}\|_p^2] + 2\eta^{-1}\sigma^2\frac{p}{n} \\
&\quad +4\mathbb{E}\left[\|\widehat{\mu} - \mu\|_p^2\right] + 4\mathbb{E}\left[\|\Pi_D\mu - \mu\|_p^2\right].
\end{aligned}$$

Then choosing η such that $1 - 4\eta > 0$ (i.e. $\eta < 1/4$) and $C > 0$ sufficiently large leads to the expected result.

(e) Let $K(s,t) = \mathrm{Cov}(X_1(s), X_1(t))$ be the covariance kernel of X, prove that

$$\mathbb{E}\left[\|\widehat{\mu} - \mu\|_p^2\right] \leq \frac{p}{n}\sup_{t\in[0,1]} K(t,t).$$

Correction : *Remark that*

$$\mathbb{E}[\widehat{\mu}(t)] = \mu(t), \qquad t \in [0,1].$$

Then

$$\begin{aligned}
\mathbb{E}\left[\|\widehat{\mu} - \mu\|_p^2\right] &= \mathbb{E}\left[\sum_{j=1}^p (\widehat{\mu}(t_j) - \mathbb{E}[\widehat{\mu}(t_j)])^2\right] = \sum_{j=1}^p \mathrm{Var}(\widehat{\mu}(t_j)) = \frac{1}{n}\sum_{j=1}^p \mathrm{Var}(X_i(t_j)) \\
&= \frac{1}{n}\sum_{j=1}^p K(t_j, t_j).
\end{aligned}$$

(f) Verify that for any function $f : [0,1] \to \mathbb{R}$ such that there exists $L > 0$ and $\alpha \in (0,1]$ such that, for all $s, t \in [0,1]$

$$|f^2(t) - f^2(s)| \leq L|t - s|^{2\alpha}, \tag{A.16}$$

we have

$$\left|\|f\|^2 - \frac{1}{p}\|f\|_p^2\right| \leq Lp^{-2\alpha}.$$

Correction : *Since $t_j - t_{j-1} = \frac{1}{p}$,*

$$\begin{aligned}
\left|\|f\|^2 - \frac{1}{p}\|f\|_p^2\right| &= \left|\int_0^1 f^2(t)dt - (t_j - t_{j-1})\sum_{j=1}^p f^2(t_j)\right| \\
&= \left|\sum_{j=1}^p \int_{t_{j-1}}^{t_j} \Big(f^2(t) - f^2(t_j)\Big)dt\right| \leq Lp^{-2\alpha}.
\end{aligned}$$

(g) Assume that μ, $\Pi_D\mu$, $\check{\mu}$ and $\hat{\mu}$ verify assumption (A.16), deduce an upper-bound on the risk $\mathbb{E}[\|\check{\mu}-\mu\|^2]$ as a function of σ^2, n, p, α and $\|\Pi_D\mu-\mu\|^2$ and compare with the upper-bound obtained in question 1.(d).

Correction : *We have:*

$$\begin{aligned}
\mathbb{E}[\|\mu-\check{\mu}\|^2] &\leq C\left(\mathbb{E}[\|\hat{\mu}-\mu\|_p^2]+\|\Pi_D\mu-\mu\|_p^2+\sigma^2\frac{p}{n}\right)\\
&\leq C\left(\frac{p}{n}\sup_{t\in[0,1]}K(t,t)+\|\Pi_D\mu-\mu\|_p^2+\sigma^2\frac{p}{n}\right).
\end{aligned}$$

A.10 Exam 2025-2026

High-dimensional statistics

EXAM: duration 2h30

Documents, calculators, phones and smartphones are forbidden

Exercise 1: Multiple testing via e-values

We consider $\mathcal{P}_0$ a set of probability measures. We say that **a non-negative random variable E is an e-value for $\mathcal{P}_0$** if and only if

$$\mathbb{E}_{\mathbb{P}}[E] \le 1, \quad \forall\, \mathbb{P} \in \mathcal{P}_0,$$

where, in the last expression, $\mathbb{E}_{\mathbb{P}}$ denotes the expectation under the probability measure $\mathbb{P}$.

1. We denote $\mathcal{N}(\mu, 1)$ the Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance 1. We then consider $\mathcal{P}_0 = \{\mathcal{N}(0,1)\}$ and $\mathcal{P}_1 = \{\mathcal{N}(\mu,1),\ \mu > 0\}$. We observe X a variable with distribution $\mathbb{P} \in \mathcal{P}_0 \cup \mathcal{P}_1$. For any $\mu > 0$, let

$$E_\mu = \exp\big(\mu X - \mu^2/2\big).$$

Prove that E_μ is an e-value for $\mathcal{P}_0$.

Correction : *Let $\mathbb{P} = \mathcal{N}(0,1)$ and $\mu > 0$.*

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[E_\mu] &= \mathbb{E}_{\mathbb{P}}[\exp\big(\mu X - \mu^2/2\big)] \\ &= \int_{-\infty}^{+\infty} \exp\big(\mu x - \mu^2/2\big) \frac{1}{\sqrt{2\pi}} \exp\big(-x^2/2\big) dx \\ &= \int_{-\infty}^{+\infty} \exp\big(-(x-\mu)^2/2\big) dx = 1.\end{aligned}$$

2. We denote $\mathcal{B}(p)$ the Bernoulli distribution with parameter $p \in (0,1)$. Let $p_0 \in (0,1)$. Then, we consider $\mathcal{P}_0 = \{\mathcal{B}(p_0)\}$ and $\mathcal{P}_1 = \big\{\mathcal{B}(q),\ q \neq p_0\big\}$. We observe $X_1, \dots, X_n$ n i.i.d. random variables with common distribution $\mathbb{P} \in \mathcal{P}_0 \cup \mathcal{P}_1$. For any $q \in (0,1)$ different from p_0, let $S_n = \sum_{i=1}^n X_i$ and

$$E_q = \left(\frac{q}{p_0}\right)^{S_n} \left(\frac{1-q}{1-p_0}\right)^{n-S_n}.$$

Prove that E_q is an e-value for $\mathcal{P}_0$.

Correction : *Let $\mathbb{P} = \mathcal{B}(p_0)$ and $q \neq p_0$. Under $\mathbb{P}$, S_n has a binomial distribution with parameters n and p_0.*

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[E_q] &= \mathbb{E}_{\mathbb{P}}\left[\left(\frac{q}{p_0}\right)^{S_n}\left(\frac{1-q}{1-p_0}\right)^{n-S_n}\right] \\
&= \sum_{k=0}^{n}\binom{n}{k} p_0^k(1-p_0)^{n-k}\left(\frac{q}{p_0}\right)^{k}\left(\frac{1-q}{1-p_0}\right)^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k} q^k(1-q)^{n-k} = 1.
\end{aligned}$$

3. We denote $\mathcal{E}(\lambda)$ the exponential distribution with parameter $\lambda \in \mathbb{R}_+^*$, whose density with respect to the Lebesgue measure on $\mathbb{R}_+^*$ is

$$x \in \mathbb{R}_+^* \longmapsto \lambda \exp(-\lambda x).$$

Let $0 < \lambda_0 < \lambda_1 < \infty$. We then consider $\mathcal{P}_0 = \{\mathcal{E}(\lambda),\ 0 < \lambda \leq \lambda_0\}$ and $\mathcal{P}_1 = \{\mathcal{E}(\lambda),\ \lambda \geq \lambda_1\}$. We observe X a variable with distribution $\mathbb{P} \in \mathcal{P}_0 \cup \mathcal{P}_1$. For any $\lambda' > \lambda_0$, let

$$E_{\lambda'} = \frac{\lambda'}{\lambda_0}\exp((\lambda_0 - \lambda')X).$$

Prove that $E_{\lambda'}$ is an e-value for $\mathcal{P}_0$.

Correction : *Let $0 < \lambda \leq \lambda_0$ and $\mathbb{P} = \mathcal{E}(\lambda)$. Since $\lambda_0 - \lambda' - \lambda < 0$, we have:*

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[E_{\lambda'}] &= \frac{\lambda\lambda'}{\lambda_0}\int_0^{+\infty}\exp((\lambda_0 - \lambda')x - \lambda x)dx \\
&= \frac{\lambda\lambda'}{\lambda_0(\lambda + \lambda' - \lambda_0)} \\
&= \frac{rr'}{r + r' - 1},
\end{aligned}$$

with $r = \lambda/\lambda_0$ and $r' = \lambda'/\lambda_0$. Then $0 < r \leq 1 < r'$ and $\mathbb{E}_{\mathbb{P}}[E_{\lambda'}] \leq 1$ since

$$\frac{rr'}{r + r' - 1} \leq 1 \iff rr' \leq r + r' - 1 \iff (r' - 1)(1 - r) \geq 0,$$

which is true.

4. We now consider the multiple testing setting. For this purpose, we consider K sets of probability measures denoted $\mathcal{P}_1, \ldots, \mathcal{P}_K$. We set

$$I = \{1, \ldots, K\}.$$

We denote $\mathbb{P}$ the true distribution of the observations and assume that $\mathbb{P} \in \bigcup_{k \in I} \mathcal{P}_k$. We denote I_0 the following unknown set:

$$I_0 = \Big\{k \in I : \quad \mathbb{P} \in \mathcal{P}_k\Big\}.$$

To infer I_0, we assume that for all $k \in I$, we are given E_k an e-value for $\mathcal{P}_k$ and we build a multiple testing procedure, namely a procedure that takes as imput the vector of e-values $(E_1, \ldots, E_K)$ and returns a set of indices

$$\hat{R} \subset I,$$

which provides the set of rejected probability mesures.
We call:
- $\hat{R}$: indices of positives,
- $\hat{R} \cap I_0$: indices of false positives,
- $\hat{R} \setminus I_0$: indices of true positives.
We denote:

$$FP = \text{card}(\hat{R} \cap I_0), \quad TP = \text{card}(\hat{R} \setminus I_0).$$

The False Discovery Proportion (FDP) corresponds to

$$FDP = \frac{FP}{FP + TP},$$

with the convention $0/0 = 0$. The False Discovery Rate (FDR) is defined as the mean of the False Discovery Proportion:

$$FDR = \mathbb{E}_{\mathbb{P}}\left[\frac{FP}{FP + TP} 1_{\{FP+TP \geq 1\}}\right].$$

Let $\alpha \in (0, 1)$. We consider the e-BH procedure at level α, meaning that

$$\hat{R} = \left\{k \in I : \quad \hat{k} E_k \geq \frac{K}{\alpha}\right\},$$

with

$$\hat{k} = \max\left\{k \in I : \quad k E_{(k)} \geq \frac{K}{\alpha}\right\},$$

where $(E_{(k)})_{k \in I}$ is the set of ordered e-values in the decreasing order:

$$E_{(1)} \geq E_{(2)} \geq \cdots \geq E_{(K)}$$

and $\max(\emptyset) = 0$.

(a) We assume that $\hat{R}$ is not empty. Show that card$(\hat{R}) = \hat{k}$.
Indication: Study the sign of $\hat{k}E_{(k)} - \frac{K}{\alpha}$ *by distinguishing the cases* $k > \hat{k}$ *and* $k \leq \hat{k}$.

Correction : *Since* $\hat{R}$ *is not empty,* $\hat{k} \neq 0$. *We distinguish two cases:*
- If $k > \hat{k}$, *we have* $kE_{(k)} < \frac{K}{\alpha}$ *and* $E_{(k)} < \frac{K}{\alpha k} < \frac{K}{\alpha \hat{k}}$.
- If $k \leq \hat{k}$, *we have* $E_{(k)} \geq E_{(\hat{k})} \geq \frac{K}{\alpha \hat{k}}$.
We deduce that we have exactly $\hat{k}$ *integers* k *such that* $\hat{k}E_k \geq \frac{K}{\alpha}$. *Therefore, card*$(\hat{R}) = \hat{k}$.

(b) Prove that

$$FDP \leq \alpha \sum_{k \in I_0} \frac{E_k}{K}.$$

Correction : *We have:*

$$\begin{aligned}
\frac{FDP}{\alpha} &= \sum_{k \in I_0} \frac{1_{\{k \in \hat{R}\}}}{\alpha\, card(\hat{R})} \\
&= \sum_{k \in I_0} \frac{1_{\{k \in \hat{R}\}}}{\alpha \hat{k}} \\
&\leq \sum_{k \in I_0} \frac{E_k}{K}.
\end{aligned}$$

We use in the previous computations $0/0 = 0$.

(c) Deduce that

$$FDR \leq \alpha.$$

Correction : *We have:*

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\frac{FP}{FP+TP} 1_{\{FP+TP \geq 1\}}\right] &= \mathbb{E}_{\mathbb{P}}\left[FDP 1_{\{card(\hat{R}) \geq 1\}}\right] \\
&\leq \mathbb{E}_{\mathbb{P}}\left[\alpha \sum_{k \in I_0} \frac{E_k}{K}\right] \\
&\leq \frac{\alpha}{K} \sum_{k \in I_0} \mathbb{E}_{\mathbb{P}}[E_k] \\
&\leq \alpha \frac{card(I_0)}{K} \leq \alpha,
\end{aligned}$$

since for any $k \in I_0$, $\mathbb{P} \in \mathcal{P}_k$ *and then* $\mathbb{E}_{\mathbb{P}}[E_k] \leq 1$. *This gives* $FDR \leq \alpha$.

5. We say that **a non-negative random variable P is a p-value for $\mathcal{P}_0$** if and only if
$$\mathbb{P}(P \le t) \le t, \quad \forall\, t \in (0,1),\ \forall\, \mathbb{P} \in \mathcal{P}_0.$$
Prove that if E is a positive e-value for $\mathcal{P}_0$, then the random variable $P = 1/E$ is a p-value for $\mathcal{P}_0$.

Correction : *We have for any $t \in (0,1)$ and any $\mathbb{P} \in \mathcal{P}_0$,*
$$\mathbb{P}(1/E \le t) = \mathbb{P}(E \ge 1/t) \le t\mathbb{E}_{\mathbb{P}}[E] \le t,$$
by using the Markov inequality.

Exercise 2: Model selection for the ℓ_p-loss

We consider the following classical sequence model where we observe $N \in \mathbb{N}^*$ random variables $Y_1, \ldots, Y_N$ such that
$$Y_n = \theta_n + \sigma\xi_n, \qquad n \in \{1, \ldots, N\}, \tag{A.17}$$
the noise level $\sigma > 0$ is assumed to be known and the ξ_n's are i.i.d. Gaussian centered variables with variance 1. Denoting $[\![1\,;N]\!] = \{1, \ldots, N\}$, we aim at estimating the sequence $\theta = (\theta_n)_{n\in[\![1\,;N]\!]}$ by using observations $Y = (Y_n)_{n\in[\![1\,;N]\!]}$. For $1 \le p < \infty$, we denote $\|\cdot\|_p$ the classical ℓ_p-norm on $\mathbb{R}^N$: for $x \in \mathbb{R}^N$,
$$\|x\|_p^p = \sum_{n=1}^N |x_n|^p.$$
In Model (A.17), we consider the integrated risk associated with the ℓ_p-norm: For any estimator $\hat\theta \in \mathbb{R}^N$, we set
$$R_p\big(\hat\theta, \theta\big) = \mathbb{E}\Big[\|\hat\theta - \theta\|_p^p\Big] = \sum_{n=1}^N \mathbb{E}\big[|\hat\theta_n - \theta_n|^p\big].$$
In the sequel, we consider model selection estimators. For each $m \subset [\![1\,;N]\!]$, called model in the sequel, we set
$$\hat\theta^{(m)} = \big(Y_n 1_{\{n\in m\}}\big)_{n\in[\![1\,;N]\!]},$$
meaning that for $n \in [\![1\,;N]\!]$,
$$\hat\theta_n^{(m)} = \begin{cases} Y_n & \text{if } n \in m,\\ 0 & \text{otherwise.}\end{cases}$$
We denote $|m|$ the cardinal of m. We consider in the sequel a collection $\mathcal{M}$ of models.

1. For $m \in \mathcal{M}$, show that the risk of $\hat{\theta}^{(m)}$ is

$$R_p\big(\hat{\theta}^{(m)}, \theta\big) = \sum_{n \notin m} |\theta_n|^p + \sigma^p \tau_p |m|,$$

where τ_p denotes a constant only depending on p that has to be specified.

Correction : *We have:*

$$\begin{aligned} R_p\big(\hat{\theta}^{(m)}, \theta\big) &= \sum_{n=1}^{N} \mathbb{E}\big[|\hat{\theta}_n^{(m)} - \theta_n|^p\big] \\ &= \sum_{n \in m} \mathbb{E}\big[|Y_n - \theta_n|^p\big] + \sum_{n \notin m} \mathbb{E}\big[|0 - \theta_n|^p\big] \\ &= \sum_{n \notin m} |\theta_n|^p + \sigma^p \tau^p |m|, \end{aligned}$$

with $\tau_p = \mathbb{E}[|\xi_1|^p]$.

2. We take $p = 2$ and assume that the sequence θ belongs to the Sobolev space of smoothness $\alpha > 0$ and radius $L > 0$:

$$W^\alpha(L) = \Big\{ x = (x_n)_{n \in [\![1\,;N]\!]} \in \mathbb{R}^N : \quad \sum_{n=1}^{N} n^{2\alpha} x_n^2 \leq L^2 \Big\}.$$

In this question, we consider the following collection of models:

$$\mathcal{M} = \Big\{ m = \{1, 2, \dots, k\} : \quad 1 \leq k \leq N \Big\}.$$

(a) Let $m \in \mathcal{M}$. By using $\left(\frac{n}{|m|}\right)^{2\alpha} > 1$ for $n \notin m$, prove that

$$\sum_{n \notin m} \theta_n^2 \leq L^2 |m|^{-2\alpha}.$$

Correction : *Let* $m \in \mathcal{M}$. *We take* $k \in [\![1\,;N]\!]$ *such that* $m = \{1, 2, \dots, k\}$. *Then,*

$$\sum_{n \notin m} \theta_n^2 = \sum_{n=k+1}^{N} \theta_n^2 \leq k^{-2\alpha} \sum_{n=k+1}^{N} n^{2\alpha} \theta_n^2 \leq L^2 |m|^{-2\alpha}.$$

(b) In this case, show that, by choosing $m \in \mathcal{M}$ conveniently, if N is large enough,

$$\inf_{m \in \mathcal{M}} R_2\big(\hat{\theta}^{(m)}, \theta\big) \leq C L^{\frac{2}{1+2\alpha}} \sigma^{\frac{4\alpha}{1+2\alpha}} + \sigma^2,$$

where C is an absolute constant.

Correction : *For $p=2$, $\tau_2 = 1$. Therefore, for any $m \in \mathcal{M}$,*

$$\begin{aligned} R_2\big(\hat{\theta}^{(m)}, \theta\big) &= \sum_{n \notin m} \theta_n^2 + \sigma^2 |m| \\ &\leq L^2 |m|^{-2\alpha} + \sigma^2 |m|. \end{aligned}$$

Now, we take $m \in \mathcal{M}$ such that

$$\left(\frac{L}{\sigma}\right)^{\frac{2}{1+2\alpha}} < |m| \leq \left(\frac{L}{\sigma}\right)^{\frac{2}{1+2\alpha}} + 1.$$

We need $N > \left(\frac{L}{\sigma}\right)^{\frac{2}{1+2\alpha}}$. We obtain:

$$R_2\big(\hat{\theta}^{(m)}, \theta\big) \leq 2 L^{\frac{2}{1+2\alpha}} \sigma^{\frac{4\alpha}{1+2\alpha}} + \sigma^2.$$

In the sequel, we do no longer consider the case $p=2$ and we take $p \geq 1$. Given a collection of models $\mathcal{M}$, we wish to select $\hat{m} \in \mathcal{M}$ in the best possible way. For this purpose, we consider a penalty function defined on $\mathcal{M}$, denoted pen. We denote

$$\tilde{\theta} = \hat{\theta}^{(\hat{m})}$$

with

$$\hat{m} = \arg\min_{m \in \mathcal{M}} \left\{ -\sum_{n \in m} |Y_n|^p + \mathrm{pen}(m) \right\}.$$

In the sequel, we establish an oracle inequality for $\tilde{\theta}$.

3. We fix $m \in \mathcal{M}$. We set

$$V_p(m) = \sigma^p \sum_{n \in m} |\xi_n|^p, \quad B_p(m) = \sum_{n \notin m} |\theta_n|^p$$

and

$$J(m) = B_p(m) + \sum_{n \notin m} \big(\sigma^p |\xi_n|^p - |Y_n|^p\big).$$

(a) Show that

$$\|\hat{\theta}^{(m)} - \theta\|_p^p = B_p(m) + V_p(m).$$

Correction : *Obvious.*

(b) We set

$$T_1 = \sigma^p \sum_{n=1}^{N} |\xi_n|^p \quad \text{and} \quad T_2 = \sum_{n=1}^{N} |Y_n|^p.$$

Show that we can write

$$J(m) = \|\hat{\theta}^{(m)} - \theta\|_p^p - 2V_p(m) + \sum_{n\in m} |Y_n|^p + T_1 - T_2.$$

Correction : *We can write*

$$J(m) = B_p(m) + (T_1 - V_p(m)) - \Big(T_2 - \sum_{n\in m} |Y_n|^p\Big)$$
$$= \|\hat{\theta}^{(m)} - \theta\|_p^p - 2V_p(m) + \sum_{n\in m} |Y_n|^p + T_1 - T_2.$$

(c) Deduce that

$$\|\hat{\theta}^{(\hat{m})} - \theta\|_p^p \le \|\hat{\theta}^{(m)} - \theta\|_p^p + \Big[2V_p(\hat{m}) - \text{pen}(\hat{m})\Big] - \Big[2V_p(m) - \text{pen}(m)\Big] + J(\hat{m}) - J(m).$$

Correction : *The definition of* $\hat{m}$ *gives*

$$-\sum_{n\in \hat{m}} |Y_n|^p + pen(\hat{m}) \le -\sum_{n\in m} |Y_n|^p + pen(m).$$

Then, since T_1 *and* T_2 *do not depend on* m,

$$\|\hat{\theta}^{(\hat{m})} - \theta\|_p^p - 2V_p(\hat{m}) - J(\hat{m}) + pen(\hat{m}) \le \|\hat{\theta}^{(m)} - \theta\|_p^p - 2V_p(m) - J(m) + pen(m).$$

Thus

$$\|\hat{\theta}^{(\hat{m})} - \theta\|_p^p \le \|\hat{\theta}^{(m)} - \theta\|_p^p + \Big[2V_p(\hat{m}) - pen(\hat{m})\Big] - \Big[2V_p(m) - pen(m)\Big] + J(\hat{m}) - J(m).$$

(d) Let us denote

$$S_n = |\theta_n|^p + \sigma^p|\xi_n|^p - |Y_n|^p,$$

$m^c = [\![1\,;N]\!] \setminus m$ and $\hat{m}^c = [\![1\,;N]\!] \setminus \hat{m}$. Prove that

$$J(\hat{m}) - J(m) = \sum_{n\in \hat{m}^c \cap m} S_n - \sum_{n\in m^c\cap \hat{m}} S_n. \tag{A.18}$$

Correction : *We have:*

$$\begin{aligned} J(\hat{m}) - J(m) &= \sum_{n \in \hat{m}^c} S_n - \sum_{n \in m^c} S_n \\ &= \left(\sum_{n \in \hat{m}^c \cap m} S_n + \sum_{n \in \hat{m}^c \cap m^c} S_n \right) - \left(\sum_{n \in m^c \cap \hat{m}^c} S_n + \sum_{n \in m^c \cap \hat{m}} S_n \right) \\ &= \sum_{n \in \hat{m}^c \cap m} S_n - \sum_{n \in m^c \cap \hat{m}} S_n. \end{aligned}$$

4. We assume that $p = 1$.

   (a) Prove that for any n, $S_n \geq 0$ and deduce that

$$J(\hat{m}) - J(m) \leq \sum_{n \in m} S_n.$$

Correction : *We have:*

$$S_n = |\theta_n| + \sigma|\xi_n| - |Y_n| \geq |\theta_n + \sigma\xi_n| - |Y_n| \geq 0.$$

Then,

$$J(\hat{m}) - J(m) = \sum_{n \in \hat{m}^c} S_n - \sum_{n \in m^c} S_n \leq \sum_{n \in [\![1\,;N]\!]} S_n - \sum_{n \in m^c} S_n = \sum_{n \in m} S_n.$$

   (b) Prove that for any $n \in m$, $S_n \leq 2\sigma|\xi_n|$ and deduce that

$$\|\tilde{\theta} - \theta\|_1 \leq 3\|\hat{\theta}^{(m)} - \theta\|_1 + \Big[2V_p(\hat{m}) - \text{pen}(\hat{m})\Big] - \Big[2V_p(m) - \text{pen}(m)\Big].$$

Correction : *Since*

$$S_n = |\theta_n| + \sigma|\xi_n| - |\theta_n + \sigma\xi_n| \leq 2\sigma|\xi_n|,$$

we obtain:

$$J(\hat{m}) - J(m) \leq \sum_{n \in m} S_n \leq \sum_{n \in m} 2\sigma|\xi_n| \leq 2\|\hat{\theta}^{(m)} - \theta\|_1$$

and

$$\|\tilde{\theta} - \theta\|_1 \leq 3\|\hat{\theta}^{(m)} - \theta\|_1 + \Big[2V_p(\hat{m}) - pen(\hat{m})\Big] - \Big[2V_p(m) - pen(m)\Big].$$

5. We assume that $1 < p < \infty$. We still wish to bound $J(\hat{m}) - J(m)$ by using (A.18). In the sequel, we shall use the following lemma.

Lemma A.1. *Let $p > 1$. We have the following results.*

1. *Let $K > 0$ and a and b two reals such that $|a| \geq K|b|$. Then,*
$$||a+b|^p - |a|^p - |b|^p| \leq C_{1p}(K)|a|^p,$$
where $C_{1p}(K)$ is a positive constant only depending on p and K such that $\lim_{K\to\infty} C_{1p}(K) = 0$.

2. *There exists a constant C_{2p} such that for any $x > 0$ and $y > 0$,*
$$(x+y)^p \leq \frac{3}{2}x^p + C_{2p}y^p.$$

(a) We assume that $|\theta_n| \geq K\sigma|\xi_n|$. Show that if K is large enough,
$$|S_n| \leq \frac{1}{2}|\theta_n|^p.$$

Correction : *Since $Y_n = \theta_n + \sigma\xi_n$, applying Lemma A.1, we have*
$$\begin{aligned}|S_n| &= \Big||\theta_n|^p + \sigma^p|\xi_n|^p - |Y_n|^p\Big| \\ &\leq C_{1p}(K)|\theta_n|^p \\ &\leq \frac{1}{2}|\theta_n|^p,\end{aligned}$$
choosing K large enough.

(b) We assume that $|\theta_n| < K\sigma|\xi_n|$. Show that for any $K > 0$
$$|S_n| \leq C_{1p}(K^{-1})|\sigma\xi_n|^p.$$

Correction : *Since $Y_n = \theta_n + \sigma\xi_n$, we have:*
$$\begin{aligned}|S_n| &= \Big||\theta_n|^p + \sigma^p|\xi_n|^p - |Y_n|^p\Big| \\ &\leq C_{1p}(K^{-1})|\sigma\xi_n|^p.\end{aligned}$$
using again Lemma A.1 with $|\sigma\xi_n| \geq K^{-1}|\theta_n|$.

(c) Finally, deduce that there exists $K > 0$ such that
$$\sum_{n\in\hat{m}^c\cap m} S_n \leq \frac{1}{2}B_p(\hat{m}) + C_{1p}(K^{-1})V_p(m).$$

Correction : *From previous questions, we have:*

$$\sum_{n\in\hat{m}^c\cap m} S_n \le \frac{1}{2}\sum_{n\in\hat{m}^c}|\theta_n|^p + C_{1p}(K^{-1})\sum_{n\in m}|\sigma\xi_n|^p$$
$$\le \frac{1}{2}B_p(\hat{m}) + C_{1p}(K^{-1})V_p(m).$$

(d) Using that there exists a constant $C_{2p} > 1$ such that for any $x > 0$ and $y > 0$

$$(x+y)^p \le \frac{3}{2}x^p + C_{2p}y^p, \tag{A.19}$$

show that

$$-\sum_{n\in m^c\cap\hat{m}} S_n \le \frac{1}{2}V_p(\hat{m}) + \Big(C_{2p}-1\Big)B_p(m).$$

Correction : *We have:*

$$-\sum_{n\in m^c\cap\hat{m}} S_n = \sum_{n\in m^c\cap\hat{m}}\Big[|Y_n|^p - |\theta_n|^p - \sigma^p|\xi_n|^p\Big]$$
$$= \sum_{n\in m^c\cap\hat{m}}\Big[|\theta_n+\sigma\xi_n|^p - |\theta_n|^p - \sigma^p|\xi_n|^p\Big]$$
$$\le \sum_{n\in m^c\cap\hat{m}}\Big[\frac{3}{2}|\sigma\xi_n|^p + C_{2p}|\theta_n|^p - |\theta_n|^p - \sigma^p|\xi_n|^p\Big]$$
$$\le \frac{1}{2}\sum_{n\in\hat{m}}|\sigma\xi_n|^p + \Big(C_{2p}-1\Big)\sum_{n\in m^c}|\theta_n|^p$$
$$\le \frac{1}{2}V_p(\hat{m}) + \Big(C_{2p}-1\Big)B_p(m).$$

(e) Finally, prove that

$$J(\hat{m}) - J(m) \le \frac{1}{2}\|\hat{\theta}^{(\hat{m})}-\theta\|_p^p + C'_p\|\hat{\theta}^{(m)}-\theta\|_p^p,$$

where C'_p is a constant to be determined.

Correction : *From Equation* (A.18), *we have:*

$$J(\hat{m}) - J(m) = \sum_{n\in\hat{m}^c\cap m} S_n - \sum_{n\in m^c\cap\hat{m}} S_n$$
$$\le \frac{1}{2}B_p(\hat{m}) + C_{1p}(K^{-1})V_p(m) + \frac{1}{2}V_p(\hat{m}) + \Big(C_{2p}-1\Big)B_p(m)$$
$$\le \frac{1}{2}\|\hat{\theta}^{(\hat{m})}-\theta\|_p^p + C'_p\|\hat{\theta}^{(m)}-\theta\|_p^p,$$

with $C'_p = \max\left(C_{2p}-1, C_{1p}(K^{-1})\right)$.

(f) Deduce that

$$\|\hat{\theta}^{(\hat{m})}-\theta\|_p^p \leq 2(1+C'_p)\|\hat{\theta}^{(m)}-\theta\|_p^p+2\left[2V_p(\hat{m})-\text{pen}(\hat{m})\right]-2\left[2V_p(m)-\text{pen}(m)\right].$$

<u>Correction :</u> *Using Question 3b), we have*

$$\|\hat{\theta}^{(\hat{m})}-\theta\|_p^p \leq \|\hat{\theta}^{(m)}-\theta\|_p^p+\Big[2V_p(\hat{m})-pen(\hat{m})\Big]-\Big[2V_p(m)-pen(m)\Big]+J(\hat{m})-J(m).$$

Therefore,

$$\begin{aligned}\|\hat{\theta}^{(\hat{m})}-\theta\|_p^p &\leq \|\hat{\theta}^{(m)}-\theta\|_p^p+\Big[2V_p(\hat{m})-pen(\hat{m})\Big]-\Big[2V_p(m)-pen(m)\Big]+\frac{1}{2}\|\hat{\theta}^{(\hat{m})}-\theta\|_p^p+C'_p\|\hat{\theta}\\ &\leq 2(1+C'_p)\|\hat{\theta}^{(m)}-\theta\|_p^p+2\left[2V_p(\hat{m})-pen(\hat{m})\right]-2\left[2V_p(m)-pen(m)\right].\end{aligned}$$

6. We assume that we can choose the penalty pen such that for any $m\in\mathcal{M}$,

$$\mathbb{E}\big|\text{pen}(m)-2V_p(m)\big| \leq M_p\sigma^p,$$

for M_p a constant only depending on p. Prove that for any $p\geq 1$,

$$R_p\big(\tilde{\theta},\theta\big) \leq 2(1+C'_p)\inf_{m\in\mathcal{M}} R_p\big(\hat{\theta}^{(m)},\theta\big)+2(\text{card}(\mathcal{M})+1)M_p\sigma^p.$$

<u>Correction :</u> *For any $p\geq 1$,*

$$\begin{aligned}\|\hat{\theta}^{(\hat{m})}-\theta\|_p^p &\leq 2(1+C'_p)\|\hat{\theta}^{(m)}-\theta\|_p^p+2\big|2V_p(\hat{m})-pen(\hat{m})\big|+2\big|2V_p(m)-pen(m)\big|\\ &\leq 2(1+C'_p)\|\hat{\theta}^{(m)}-\theta\|_p^p+\sum_{m'\in\mathcal{M}}2\big|2V_p(\hat{m})-pen(\hat{m})\big|1_{\{\hat{m}=m'\}}+2\big|2V_p(m)-pen(m)\big|\\ &\leq 2(1+C'_p)\|\hat{\theta}^{(m)}-\theta\|_p^p+\sum_{m'\in\mathcal{M}}2\big|2V_p(m')-pen(m')\big|+2\big|2V_p(m)-pen(m)\big|\end{aligned}$$

We deduce:

$$\mathbb{E}\|\hat{\theta}^{(\hat{m})}-\theta\|_p^p \leq 2(1+C'_p)\mathbb{E}\Big[\|\hat{\theta}^{(m)}-\theta\|_p^p\Big]+2(card(\mathcal{M})+1)M_p\sigma^p$$

and then the result.

7. We assume that $\mathcal{M}$ is the collection of all possible models and that for any $m\in\mathcal{M}$, $\text{pen}(m)=\sigma^p\log^{\frac{p}{2}}(N)|m|$. Prove that

$$\tilde{\theta}_n = Y_n\times 1_{\left\{|Y_n|>\sigma\sqrt{\log(N)}\right\}}.$$

Correction : *We have:*

$$\tilde{\theta} = \hat{\theta}^{(\hat{m})}$$

with

$$\hat{m} = \arg\min_{m \in \mathcal{M}} \left\{ -\sum_{n \in m} |Y_n|^p + pen(m) \right\}.$$

But

$$\begin{aligned} -\sum_{n \in m} |Y_n|^p + pen(m) &= -\sum_{n \in m} |Y_n|^p + \sigma^p \log^{\frac{p}{2}}(N)|m| \\ &= -\sum_{n \in m} \Big[|Y_n|^p - \sigma^p \log^{\frac{p}{2}}(N) \Big]. \end{aligned}$$

The minimum of the previous sum with respect to $m \subset [\![1\,;N]\!]$ *is achieved by taking* $m = \hat{m}$*, with*

$$\hat{m} = \Big\{ n : \quad |Y_n| > \sigma \sqrt{\log(N)} \Big\}.$$

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