Supplemental material to "Statistical deconvolution of the free Fokker-Planck equation at fixed time".

This is the supplement to the article *Statistical deconvolution of the free Fokker-Planck equation at fixed time*. It gathers the appendices referenced in the main paper namely: Appendix A (Proof of Equation (1.7)), Appendix B (Proof of Theorem 2.6 and Theorem-Definition 2.8), Appendix C (Proof of Lemma 3.4), Appendix D (Proof of Lemma 3.5) and Appendix E (Proof of Corollary 4.3).

Appendix A: Proof of (1.7)

As mentioned in the introduction, a full proof of (1.7) can be found in [1, Theorem 4.3.2]. The proof therein is involved and proceeds backward, showing that the solutions of (1.7) are the eigenvalues of an Hermitian Brownian motion. In this appendix, we use a more direct approach (following for example [19, 34]) that leads to a non rigorous but more intuitive sketch of proof. Recall that $X^n(t) = X^n(0) + H^n(t)$ where $H^n(t)$ is the Hermitian Brownian motion of Definition 2.1. For $k \leq \ell$ and $t > 0$, we denote by $\mathbf{x}_{k\ell}(t) := \mathrm{Re} X^n_{k,\ell}(t)$ and $\mathbf{y}_{k\ell}(t) := \mathrm{Im} X^n_{k,\ell}(t)$ respectively the real and imaginary parts of the entries of the matrix $X^n(t)$. The processes $\mathbf{x}_{k\ell}$ and $\mathbf{y}_{k\ell}$ are semi-martingales and we will assume that for any $m \in \{1, \dots, n\}$, the m-th smallest eigenvalue $\lambda^n_m(t)$ of $X^n(t)$ is a smooth function of $(\mathbf{x}_{k\ell}, \mathbf{y}_{k\ell})_{k \leq \ell}$ so that we can apply Itô's formula[1]:

$$\begin{aligned}\mathrm{d}\lambda_m := &\sum_{k<\ell} \frac{\partial \lambda_m}{\partial x_{k\ell}} \mathrm{d}\mathbf{x}_{k\ell} + \sum_{k<\ell} \frac{\partial \lambda_m}{\partial y_{k\ell}} \mathrm{d}\mathbf{y}_{k\ell} + \sum_{k=1}^{n} \frac{\partial \lambda_m}{\partial x_{kk}} \mathrm{d}\mathbf{x}_{kk} + \frac{1}{4n} \sum_{k<\ell} \left(\frac{\partial^2 \lambda_m}{\partial x_{k\ell}^2} + \frac{\partial^2 \lambda_m}{\partial y_{k\ell}^2} \right) \mathrm{d}t \\ &+ \frac{1}{2n} \sum_{k=1}^{n} \frac{\partial^2 \lambda_m}{\partial x_{kk}^2} \mathrm{d}t, \end{aligned} \tag{A.1}$$

where we have used that, in the range of indices we are interested in, $\langle \mathbf{x}_{ij}, \mathbf{y}_{k\ell} \rangle = 0$; if $i \neq j$, $\mathrm{d}\langle \mathbf{x}_{ij}, \mathbf{x}_{kl} \rangle = \mathrm{d}\langle \mathbf{y}_{ij}, \mathbf{y}_{k\ell} \rangle = \frac{\mathrm{d}t}{2n} \delta_{ik} \delta_{j\ell}$, and $\mathrm{d}\langle \mathbf{x}_{ii}, \mathbf{x}_{ii} \rangle = \frac{\mathrm{d}t}{n}$. We now have to compute the derivatives. It relies on the so-called Hadamard variation formulae, well-known in perturbation theory.

Lemma A.1. *Let H be an Hermitian matrix, with entries $(h_{k\ell} = x_{k\ell} + \mathrm{i} y_{k\ell})_{1 \leq k < \ell \leq n}$. We assume that H has distinct (real) eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $u_1, \dots, u_n$. Then, denoting by u_{km} the k-th component of the vector u_m, we have for all $m \in \{1, \cdots, n\}$:*

$$\begin{aligned} \frac{\partial \lambda_m}{\partial x_{k\ell}} &= \bar{u}_{km} u_{\ell m} + \bar{u}_{\ell m} u_{km}, \text{ for } k < \ell, \\ \frac{\partial \lambda_m}{\partial y_{k\ell}} &= \mathrm{i}(\bar{u}_{km} u_{\ell m} - \bar{u}_{\ell m} u_{km}), \text{ for } k < \ell, \\ \frac{\partial \lambda_m}{\partial x_{kk}} &= |u_{km}|^2, \\ \frac{\partial^2 \lambda_m}{\partial x_{k\ell}^2} &= 2 \sum_{m' \neq m} \frac{1}{\lambda_m - \lambda_{m'}} |\bar{u}_{km'} u_{\ell m} + \bar{u}_{\ell m'} u_{km}|^2, \text{ for } k < \ell \end{aligned}$$

[1] This is far from obvious and the actual rigorous proof does not proceed like that.

$$\frac{\partial^2 \lambda_m}{\partial y_{k\ell}^2} = 2 \sum_{m' \neq m} \frac{1}{\lambda_m - \lambda_{m'}} |\bar{u}_{km'} u_{\ell m} - \bar{u}_{\ell m'} u_{km}|^2, \textit{ for } k < \ell$$

$$\frac{\partial^2 \lambda_m}{\partial x_{kk}^2} = 2 \sum_{m' \neq m} \frac{1}{\lambda_m - \lambda_{m'}} |u_{km}|^2 |u_{km'}|^2.$$

Proof. Again, we assume here that all the functions that we use hereafter are smooth functions of the real and imaginary parts of the entries of the matrix. For $k \leq \ell$, let us denote by ∂ the derivative $\frac{\partial}{\partial x_{k\ell}}$ or $\frac{\partial}{\partial y_{k\ell}}$. The matrix ∂H corresponds to the matrix whose entries are $\partial h_{k\ell}$.

For any $m, m' \in \{1, \ldots, n\}$, we have $H.u_m = \lambda_m u_m$, and $u_m^* u_{m'} = \delta_{mm'}$, where in this proof $\delta_{mm'}$ is the Kronecker symbol equal to 1 if and only if $m = m'$ and 0 otherwise, and where u_m^* is the adjoint vector of u_m defined as the row vector with k-th component $u_{km}^* = \mathrm{Re}(u_{km}) - \mathrm{i}\mathrm{Im}(u_{km})$. Thus,

$$\partial H.u_m + H.\partial u_m = \partial \lambda_m \times u_m + \lambda_m \partial u_m, \tag{A.2}$$

and for all m and m' (possibly equal):

$$\partial u_m^*.u_{m'} + u_m^* \partial u_{m'} = 0. \tag{A.3}$$

Multiplying (A.2) by u_m^* on the left, we get the first Hadamard formula:

$$\partial \lambda_m = u_m^*.\partial H.u_m. \tag{A.4}$$

Now multiplying (A.2) by $u_{m'}^*$ on the left, we get, for $m \neq m'$,

$$u_{m'}^*.\partial H.u_m = (\lambda_m - \lambda_{m'}) u_{m'}^* \partial u_m,$$

so that

$$\partial u_m = \sum_{m'=1}^{n} (u_{m'}^* \partial u_m) u_{m'} = \sum_{m' \neq m} \frac{u_{m'}^*.\partial H.u_m}{\lambda_m - \lambda_{m'}} u_{m'} + (u_m^* \partial u_m) u_m.$$

From there, taking the derivative of the first Hadamard formula (A.4) and using the above equality with (A.3) leads to the second Hadamard equality:

$$\partial^2 \lambda_m = u_m^*.\partial^2 H.u_m + 2 \sum_{m' \neq m} \frac{|u_{m'}^*.\partial H.u_m|^2}{\lambda_m - \lambda_{m'}}. \tag{A.5}$$

Now, for $\partial = \frac{\partial}{\partial x_{k\ell}}$ or $\partial = \frac{\partial}{\partial y_{k\ell}}$, we have that $\partial^2 H = 0$. Moreover, $\frac{\partial H}{\partial x_{k\ell}}$ is the matrix full of zeros except for the terms (k, ℓ) and (ℓ, k) that are equal to 1 and $\frac{\partial H}{\partial y_{k\ell}}$ $(k < \ell)$ is the matrix full of zeros except for the terms (k, ℓ) equal to i and (ℓ, k) that are equal to $-$i. Injecting this information into (A.4) and (A.5) provides the announced derivatives. □

Plugging the formulae of Lemma A.1 into the Itô formula (A.1) above, we get

$$\partial \lambda_m = \frac{1}{\sqrt{n}} \mathrm{d}\beta_m + \frac{1}{n} \sum_{m' \neq m} \frac{1}{\lambda_m - \lambda_{m'}} \mathrm{d}t,$$

with

$$\mathrm{d}\beta_m := \frac{1}{\sqrt{2}}\sum_{k<\ell}((\bar{u}_{km}u_{\ell m} + \bar{u}_{\ell m}u_{km})\mathrm{d}B_{k,\ell} + \mathrm{i}\,(\bar{u}_{km}u_{\ell m} - \bar{u}_{\ell m}u_{km})\mathrm{d}\tilde{B}_{k,\ell}) + \sum_{k=1}^{n}|u_{km}|^2\mathrm{d}B_{kk}.$$

$\beta_1, \dots, \beta_n$ are centered semimartingales. Furthermore,

$$\mathrm{d}\langle \beta_m, \beta_{m'}\rangle_t = \sum_{k,\ell=1}^{n} \bar{u}_{km}u_{\ell m}\bar{u}_{\ell m'}u_{km'}\mathrm{d}t = \delta_{mm'}\mathrm{d}t,$$

so that they are independent standard Brownian motions.

Appendix B: Proof of Theorem 2.6 and Theorem-Definition 2.8

B.1. Proof of Theorem 2.6

The constants of Theorem 2.6 are better than the ones of Arizmendi et al. [2] who work in full generality. We develop here the main steps of the proof in our context, using the explicit formula for the semi-circular distribution. In the whole proof, we consider $z \in \mathbb{C}_{2\sqrt{t}}$.

Step 1: We first prove that the function $L_z(w) = h_{\sigma_t}\big(\widetilde{h}_{\mu_t}(w) - z\big) + z$ is well-defined and analytic on $\mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}$. Since h_{σ_t} is defined on $\mathbb{C}^+$, we need to check that $\widetilde{h}_{\mu_t}(w) - z \in \mathbb{C}^+$ for $w \in \mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}$. This is satisfied since for such w,

$$\mathrm{Im}\big(\widetilde{h}_{\mu_t}(w) - z\big) = \mathrm{Im}\big(w + F_{\mu_t}(w) - z\big) \geq 2\mathrm{Im}(w) - \mathrm{Im}(z) > 0, \tag{B.1}$$

where we have used $\mathrm{Im}F_{\mu_t}(w) \geq \mathrm{Im}(w)$ for the first inequality. Indeed, if $w = w_1 + iw_2$, we have

$$(F_{\mu_t}(w))^{-1} = G_{\mu_t}(w) = \int \frac{d\mu_t(x)}{w_1 + iw_2 - x} = \int \frac{(w_1 - x)d\mu_t(x)}{(w_1 - x)^2 + w_2^2} - iw_2 \int \frac{d\mu_t(x)}{(w_1 - x)^2 + w_2^2}$$

and

$$\begin{aligned}\mathrm{Im}(F_{\mu_t}(w)) &= w_2 \times \frac{\int \frac{d\mu_t(x)}{(w_1-x)^2+w_2^2}}{\left(\int \frac{(w_1-x)d\mu_t(x)}{(w_1-x)^2+w_2^2}\right)^2 + w_2^2\left(\int \frac{d\mu_t(x)}{(w_1-x)^2+w_2^2}\right)^2} \\ &\geq w_2 \times \frac{\int \frac{d\mu_t(x)}{(w_1-x)^2+w_2^2}}{\int \frac{(w_1-x)^2 d\mu_t(x)}{\left((w_1-x)^2+w_2^2\right)^2} + w_2^2 \int \frac{d\mu_t(x)}{\left((w_1-x)^2+w_2^2\right)^2}} = w_2\end{aligned} \tag{B.2}$$

Step 2: We show that $L_z(\mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}) \subset \overline{\mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}}$ and that L_z is not a conformal automorphism.

First, let us show that $L_z\left(\mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}\right) \subset \overline{\mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}}$. Let $w \in \mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}$, we have:

$$\mathrm{Im}\left(L_z(w)\right) = \mathrm{Im}\left[t.G_{\sigma_t}\left(\widetilde{h}_{\mu_t}(w) - z\right) + z\right] = \mathrm{Im}\left(\frac{\widetilde{h}_{\mu_t}(w) - z - \sqrt{\left(\widetilde{h}_{\mu_t}(w) - z\right)^2 - 4t}}{2} + z\right). \tag{B.3}$$

To lower bound the right hand side, note that for all $v \in \mathbb{C}^+$, one can check that:

$$\mathrm{Im}\big(\sqrt{v^2 - 4t}\big) \leq \sqrt{\mathrm{Im}^2(v) + 4t}.$$

Therefore, we have:

$$\mathrm{Im}\left(\sqrt{\left(\widetilde{h}_{\mu_t}(w) - z\right)^2 - 4t}\right) \leq \sqrt{\left[\mathrm{Im}\big(\widetilde{h}_{\mu_t}(w) - z\big)\right]^2 + 4t}.$$

Hence, (B.3) yields:

$$\mathrm{Im}\big(L_z(w)\big) \geq \frac{1}{2}\left[\mathrm{Im}\big(\widetilde{h}_{\mu_t}(w) - z\big) - \sqrt{\left[\mathrm{Im}\big(\widetilde{h}_{\mu_t}(w) - z\big)\right]^2 + 4t}\right] + \mathrm{Im}(z).$$

The function $g(s) = s - \sqrt{s^2 + 4t}$ is non-decreasing on $\mathbb{R}_+$ and for all $s > 0$, $g(s) \geq -2\sqrt{t}$. Thus:

$$\mathrm{Im}\big(L_z(w)\big) \geq \mathrm{Im}(z) - \sqrt{t} > \frac{1}{2}\mathrm{Im}(z), \tag{B.4}$$

since $z \in \mathbb{C}_{2\sqrt{t}}$. This guarantees that $L_z(w) \in \mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}$.

Let us now prove that L_z is not an automorphism of $\mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}$. Consider

$$|L_z(w) - z| = \left|F_{\sigma_t}\left(\widetilde{h}_{\mu_t}(w) - z\right) - \left(\widetilde{h}_{\mu_t}(w) - z\right)\right| = \left|tG_{\sigma_t}\left(\widetilde{h}_{\mu_t}(w) - z\right)\right|.$$

For $v \in \mathbb{C}^+$, if $|v| > 3\sqrt{t}$, since the support of σ_t is $[-2\sqrt{t}, 2\sqrt{t}]$,

$$|tG_{\sigma_t}(v)| = \left|\int_{-2\sqrt{t}}^{2\sqrt{t}} \frac{t}{v - x}\mathrm{d}\sigma_t(x)\right| \leq \sqrt{t}.$$

If $|v| \leq 3\sqrt{t}$,

$$|tG_{\sigma_t}(v)| = \left|\frac{v - \sqrt{v^2 - 4t}}{2}\right| \leq \frac{2|v| + 2\sqrt{t}}{2} \leq 4\sqrt{t}.$$

Hence, for all $w \in \mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}$,

$$|L_z(w) - z| \leq 4\sqrt{t}. \tag{B.5}$$

This implies that $L_z\left(\mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}\right)$ is included in the ball centered at z with radius $4\sqrt{t}$. As a result, L_z is not surjective and hence is not an automorphism of $\mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}$.

Step 3: Existence and uniqueness of w_{fp}, which is a fixed point of L_z.

By Steps 1 and 2, L_z satisfies the assumptions of Denjoy-Wolff's fixed-point theorem (see e.g. [4, 2]). The theorem says that for all $w \in \mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}$ the iterated sequence $L_z^{\circ m}(w) = L_z \circ L_z^{\circ(m-1)}(w)$ converges to the unique Denjoy-Wolff point of L_z which we define as $w_{fp}(z)$. The Denjoy-Wolff point is either a fixed-point of L_z or a point of the boundary of the domain. Let us check that w_{fp} is a fixed point of L_z. For any $z \in \mathbb{C}_{2\sqrt{t}}$, there exists $\gamma > 2$ such that $z \in \mathbb{C}_{\gamma\sqrt{t}}$ and from (B.4), $L_z(\mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}) \subset \mathbb{C}_{(1-\frac{1}{\gamma})\mathrm{Im}(z)}$. Moreover, from (B.5), $L_z(\mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}) \subset B(z, 4\sqrt{t})$. Therefore, $w_{fp}(z) \in \overline{\mathbb{C}_{(1-\frac{1}{\gamma})\mathrm{Im}(z)} \cap B(z, 4\sqrt{t})} \subset \mathbb{C}_{\frac{1}{2}\mathrm{Im}(z)}$, so that it is necessarily a fixed point.

We now define

$$w_1(z) := F_{\mu_t}(w_{fp}(z)) + w_{fp}(z) - z.$$

One can check that

$$\begin{aligned}
F_{\sigma_t}(w_1(z)) &= w_1(z) - h_{\sigma_t}(w_1(z)) && \text{(B.6)}\\
&= F_{\mu_t}(w_{fp}(z)) + w_{fp}(z) - z - h_{\sigma_t}(F_{\mu_t}(w_{fp}(z)) + w_{fp}(z) - z) && \text{(B.7)}\\
&= \tilde{h}_{\mu_t}(w_{fp}(z)) - z - h_{\sigma_t}(\tilde{h}_{\mu_t}(w_{fp}(z)) - z) && \text{(B.8)}\\
&= \tilde{h}_{\mu_t}(w_{fp}(z)) - w_{fp}(z) = F_{\mu_t}(w_{fp}(z)).
\end{aligned}$$

One can therefore rewrite

$$w_1(z) = F_{\sigma_t}(w_1(z)) + w_{fp}(z) - z.$$

From (B.5) and the fact that $w_{fp}(z)$ is a fixed point of L_z, one easily gets that $\lim_{y\to+\infty} w_{fp}(iy)/(iy) = 1$, which implies that $\lim_{y\to+\infty} F_{\mu_t}(w_{fp}(iy))/(iy) = 1$, and $\lim_{y\to+\infty} w_1(iy)/(iy) = 1$.

Now we connect F_{μ_0} to the previous quantities. For z large enough, all the functions we consider are invertible and we have

$$F_{\mu_t}(w_{fp}(z)) + w_{fp}(z) = z + w_1(z) = z + F_{\sigma_t}^{<-1>}(F_{\mu_t}(w_{fp}(z))).$$

On the other hand, for z large enough, using Theorem-definition 2.5 for $\mu_1 = \sigma_t$ and $\mu_2 = \mu_0$, we get

$$F_{\mu_t}(w_{fp}(z)) + w_{fp}(z) = \alpha_1(w_{fp}(z)) + \alpha_2(w_{fp}(z)) = F_{\sigma_t}^{<-1>}(F_{\mu_t}(w_{fp}(z))) + F_{\mu_0}^{<-1>}(F_{\mu_t}(w_{fp}(z))).$$

Comparing the two equalities gives

$$F_{\mu_0}^{<-1>}(F_{\mu_t}(w_{fp}(z))) = z,$$

so that, for z large enough,

$$F_{\mu_t}(w_{fp}(z)) = F_{\mu_0}(z).$$

The two functions being analytic on $\mathbb{C}_{2\sqrt{t}}$, the equality can be extended to any $z \in \mathbb{C}_{2\sqrt{t}}$.

Finally, since

$$w_1(z) = F_{\mu_t}(w_{fp}(z)) + w_{fp}(z) - z = F_{\mu_0}(z) + w_{fp}(z) - z,$$

we have, using (B.2) with μ_0 instead of μ_t,

$$\text{Im}(w_1(z)) = \text{Im}(F_{\mu_0}(z)) + \text{Im}(w_{fp}(z)) - \text{Im}(z) \geq \text{Im}(w_{fp}(z)) \geq \frac{1}{2}\text{Im}(z).$$

This ends the proof of Theorem 2.6.

B.2. Proof of Theorem-Definition 2.8

The proof of this theorem follows the steps of the proof of Theorem 2.6. First, $\widehat{L}_z(w) := t\widehat{G}_{\mu_t^n}(w) + z$ is a well-defined and analytic function on $\mathbb{C}^+$. Let us check that $\widehat{L}_z\big(\mathbb{C}_{\frac{1}{2}\text{Im}(z)}\big) \subset \mathbb{C}_{\frac{1}{2}\text{Im}(z)}$ for $z \in \mathbb{C}_{2\sqrt{t}}$. For $w = u + iv \in \mathbb{C}_{\frac{1}{2}\text{Im}(z)}$,

$$\text{Im}\big(\widehat{G}_{\mu_t^n}(w)\big) = \frac{1}{n}\sum_{j=1}^{n} \text{Im}\Big(\frac{u - \lambda_j^n(t) - iv}{(u - \lambda_j^n(t))^2 + v^2}\Big) > -\frac{1}{v} = -\frac{1}{\text{Im}(w)}. \tag{B.9}$$

Thus,

$$\text{Im}\big(\widehat{L}_z(w)\big) = t\,\text{Im}\big(\widehat{G}_{\mu_t^n}(w)\big) + \text{Im}(z) > -\frac{t}{\text{Im}(w)} + \text{Im}(z) > -\frac{2t}{\text{Im}(z)} + \text{Im}(z) > \frac{1}{2}\text{Im}(z).$$

The second and last inequalities comes from the choice of $w \in \mathbb{C}_{\frac{1}{2}\text{Im}(z)}$, and from $\text{Im}(z) > 2\sqrt{t}$. Moreover, $\widehat{L}_z$ is not an automorphism since:

$$\Big|\widehat{L}_z(w) - z\Big| = \Big|t\widehat{G}_{\mu_t^n}(w)\Big| = \left|\frac{1}{n}\sum_{j=1}^{n}\frac{t}{w - \lambda_j^n(t)}\right| \leq \frac{t}{\text{Im}(w)} \leq \sqrt{t} \tag{B.10}$$

since $\text{Im}(w) > \frac{1}{2}\text{Im}(z) > \sqrt{t}$. We use again the Denjoy-Wolff fixed-point theorem. Because the inclusion of $\widehat{L}_z\big(\mathbb{C}_{\frac{1}{2}\text{Im}(z)}\big)$ into $\mathbb{C}_{\frac{1}{2}\text{Im}(z)}$ is strict, the unique Denjoy-Wolff point of $\widehat{L}_z$ is necessarily a fixed point that we denote $\widehat{w}_{fp}(z)$. From the construction, $\text{Im}(\widehat{w}_{fp}(z)) > \text{Im}(z)/2$. Finally, the last announced estimate is a straightforward consequence of (B.10).

Appendix C: Proof of Lemma 3.4

Recall that $R_{n,t}(z)$ and $\widetilde{R}_{n,t}(z)$ are defined in (3.4) and (3.9), and that

$$n\widetilde{A}_2^n(z) = \sum_{k=1}^{n} \mathbb{E}\left[\big(R_{n,t}(z)\big)_{kk} \mid X^n(0)\right] - \big(\widetilde{R}_{n,t}(z)\big)_{kk}. \tag{C.1}$$

Proceeding as in Dallaporta and Février [21], we introduce some notations. Let $R_{n,t}^{(k)}(z)$ be the resolvent of the $(n-1)\times(n-1)$ obtained from $X^n(t)$ by removing the k-th row and column and $C_{k,t}^{(k)}$ be the $(n-1)$-dimensional vector obtained from the k-th column of $H^n(t)$ by removing its k-th component. Using Schur's complement (see e.g. [3, Appendix A.1]):

$$\Big(\big(R_{n,t}(z)\big)_{kk}\Big)^{-1} = z - (H^n(t))_{kk} - (X^n(0))_{kk} - C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)}.$$

Because $\widetilde{R}_{n,t}(z)$ is a diagonal matrix, we have easily:

$$\begin{aligned}\big(R_{n,t}(z)\big)_{kk} = \quad & \big(\widetilde{R}_{n,t}(z)\big)_{kk} + \big(\widetilde{R}_{n,t}(z)\big)_{kk}.\big(R_{n,t}(z)\big)_{kk}.\Big((H^n(t))_{kk} + C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)} \\ & -\frac{t}{n}\mathbb{E}\big[\mathrm{Tr}\left(R_{n,t}(z) \mid X^n(0)\right)\big]\Big).\end{aligned}$$

Replacing $\big(R_{n,t}(z)\big)_{kk}$ in the right-hand side of the previous formula, we obtain:

$$\begin{aligned}&\big(R_{n,t}(z)\big)_{kk} - \big(\widetilde{R}_{n,t}(z)\big)_{kk} \\ =&\big(\widetilde{R}_{n,t}(z)\big)_{kk}^2.\Big((H^n(t))_{kk} + C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)} - \frac{t}{n}\mathbb{E}\Big[\mathrm{Tr}\Big(R_{n,t}(z)\Big) \mid X^n(0)\Big]\Big) \\ +&\big(\widetilde{R}_{n,t}(z)\big)_{kk}^2.\big(R_{n,t}(z)\big)_{kk}.\Big((H^n(t))_{kk} + C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)} - \frac{t}{n}\mathbb{E}\left[\mathrm{Tr}\left(R_{n,t}(z)\right) \mid X^n(0)\right]\Big)^2.\end{aligned} \tag{C.2}$$

Since $H^n(t)$ and $C_{k,t}^{(k)}$ are independent of $X_n(0)$,

$$\begin{aligned}&\mathbb{E}\Big[\Big| (H^n(t))_{kk} + C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)} - \frac{t}{n}\mathbb{E}\left[\mathrm{Tr}\left(R_{n,t}(z)\right) \mid X^n(0)\right]\Big|^2 \mid X^n(0)\Big] \\ =&\mathbb{E}\Big[\Big| (H^n(t))_{kk} + C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)} - \frac{t}{n}\mathrm{Tr}\big(R_{n,t}^{(k)}(z)\big) + \frac{t}{n}\mathrm{Tr}\big(R_{n,t}^{(k)}(z)\big) - \frac{t}{n}\mathbb{E}\big[\mathrm{Tr}\big(R_{n,t}^{(k)}(z)\big) \mid X^n(0)\big] \\ &+ \frac{t}{n}\mathbb{E}\big[\mathrm{Tr}\big(R_{n,t}^{(k)}(z)\big) \mid X^n(0)\big] - \frac{t}{n}\mathbb{E}\left[\mathrm{Tr}\left(R_{n,t}(z)\right) \mid X^n(0)\right]\Big|^2 \Big| X^n(0)\Big] \\ =&\mathbb{E}\left[(H^n(t))_{kk}^2\right] + \mathbb{E}\Big[\Big|C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)} - \frac{t}{n}\mathrm{Tr}\big(R_{n,t}^{(k)}(z)\big)\Big|^2 \mid X^n(0)\Big] \\ &+ \frac{t^2}{n^2}\Big(\mathrm{Var}\big[\mathrm{Tr}\big(R_{n,t}^{(k)}(z)\big)\big|X^n(0)\big] + \Big|\mathbb{E}\big[\mathrm{Tr}\big(R_{n,t}^{(k)}(z)\big) - \mathrm{Tr}\left(R_{n,t}(z)\right) \mid X^n(0)\big]\Big|^2\Big).\end{aligned} \tag{C.3}$$

We now upper bound each of the term in the right-hand side of (C.3). The first term equals to t/n.

Step 1: We upper bound the second term in (C.3). By Lemma 5 of [21],

$$\mathbb{E}\Big[C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)} \mid X^n(0)\Big] = \frac{t}{n}\mathbb{E}\Big[\mathrm{Tr}\big(R_{n,t}^{(k)}(z)\big) \mid X^n(0)\Big]. \tag{C.4}$$

Thus, the second term in (C.3) equals to $\mathrm{Var}\big(C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)} \mid X^n(0)\big)$ and we have:

$$\begin{aligned}\mathrm{Var}\left[C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)} \mid X^n(0)\right] =&\frac{t^2}{n^2}\mathbb{E}\left[\mathrm{Tr}\big(R_{n,t}^{(k),*}(z).R_{n,t}^{(k)}(z)\big) \mid X^n(0)\right] \\ \leq&\frac{t^2}{n^2}\mathbb{E}\left[\sum_{j=1}^{n}\frac{1}{|z-\lambda_j^{(k)}|^2} \mid X^n(0)\right]\end{aligned}$$

where the $\lambda_j^{(k)}$'s are the eigenvalues of the matrix with resolvent $R_{n,t}^{(k)}(z)$. Hence,

$$\text{Var}\left[C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)} \mid X^n(0)\right] \leq \frac{t^2}{n\text{Im}^2(z)}. \tag{C.5}$$

Step 2: We now upper bound the third and fourth terms of (C.3). Let us denote in the sequel by $\mathbb{E}_k$ the expectation with respect to $\left\{(H^n(t))_{jk} : 1 \leq j \leq n\right\}$, and by $\mathbb{E}_{\leq k}$ the conditional expectation on the sigma-field $\sigma\left(((X^n(0))_{ij}, 1 \leq i \leq j \leq n), ((H^n(t))_{ij}, 1 \leq i \leq j \leq k)\right)$.

We have:

$$\text{Var}\big[\text{Tr}\big(R_{n,t}^{(k)}(z)\big)\big|X^n(0)\big] \leq 2\text{Var}\big[\text{Tr}\big(R_{n,t}(z)\big)\big|X^n(0)\big] + 2\text{Var}\big[\text{Tr}\big(R_{n,t}(z)\big) - \text{Tr}(R_{n,t}^{(k)}(z)\big|X^n(0)\big]. \tag{C.6}$$

For the first term,

$$\begin{aligned}\text{Var}\big[\text{Tr}\big(R_{n,t}(z)\big)\big|X^n(0)\big] &= \sum_{k=1}^n \mathbb{E}\left[\big|\big(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1}\big)\text{Tr}\left(R_{n,t}(z)\right)\big|^2 \mid X^n(0)\right] \\ &= \sum_{k=1}^n \mathbb{E}\left[\left|\big(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1}\big)\big(\text{Tr}(R_{n,t}(z)) - \text{Tr}(R_{n,t}^{(k)}(z))\big)\right|^2 \mid X^n(0)\right],\end{aligned} \tag{C.7}$$

as $\big(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1}\big)\text{Tr}\left(R_{n,t}^{(k)}(z)\right) = 0$. The Schur complement formula (see e.g. [3, Appendix A.1]) gives that:

$$\text{Tr}\big(R_{n,t}(z)\big) - \text{Tr}\big(R_{n,t}^{(k)}(z)\big) = \frac{1 + C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z)^2.C_{k,t}^{(k)}}{z - (H^n(t))_{kk} - (X^n(0))_{kk} - C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)}}. \tag{C.8}$$

Then,

$$\begin{aligned}\left|\text{Tr}\big(R_{n,t}(z)\big) - \text{Tr}\big(R_{n,t}^{(k)}(z)\big)\right| &\leq \frac{\left|1 + C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z)^2.C_{k,t}^{(k)}\right|}{\left|\text{Im}\left(z - (H^n(t))_{kk} - (X^n(0))_{kk} - C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)}\right)\right|} \\ &\leq \frac{1 + \left|C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z)^2.C_{k,t}^{(k)}\right|}{\left|\text{Im}(z) - \text{Im}\left(C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z).C_{k,t}^{(k)}\right)\right|} \\ &\leq \frac{1 + C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z)^*.R^{(k)}(z).C_{k,t}^{(k)}}{\left|\text{Im}(z) + \text{Im}\,(z)\,.C_{k,t}^{(k)*}.R_{n,t}^{(k)}(z)^*.R_{n,t}^{(k)}(z).C_{k,t}^{(k)}\right|} \\ &= \frac{1}{\text{Im}(z)}.\end{aligned} \tag{C.9}$$

The second inequality it due to the fact that $(H^n(t))_{kk}, (X^n(0))_{kk} \in \mathbb{R}$ and the third inequality comes from the following equality: With $\Psi\ :\ M \in \mathcal{H}_n(\mathbb{C}) \mapsto C^*MC$ with $C \in \mathbb{C}^n$, then, for any $z \in \mathbb{C}$ and any resolvent matrix $R(z)$, we have (see [21, Lemma 1])

$$\text{Im}\big(\Psi(R(z))\big) = -\text{Im}(z)\Psi\big(R(z)^* R(z)\big).$$

The bound (C.9) does not depend on $X^n(0)$. Plugging this bound into (C.7), we obtain:

$$\text{Var}\big[\text{Tr}\big(R_{n,t}(z)\big)\big|X^n(0)\big] \leq \frac{4n}{\text{Im}^2(z)}.$$

From there, using (C.6),

$$\text{Var}\big[\text{Tr}\big(R^{(k)}_{n,t}(z)\big)\big|X^n(0)\big] \leq \frac{8n+2}{\text{Im}^2(z)}. \tag{C.10}$$

Similarly, (C.9) also provides an upper bound for the fourth term of (C.3):

$$\Big|\mathbb{E}\big[\text{Tr}\big(R^{(k)}_{n,t}(z)\big) - \text{Tr}\left(R_{n,t}(z)\right) \mid X^n(0)\big]\Big|^2 \leq \frac{1}{\text{Im}^2(z)}. \tag{C.11}$$

Step 3: In conclusion, using (C.3), (C.5), (C.10) and (C.11), we obtain that:

$$\mathbb{E}\left[\left|(H_n(t))_{kk} + C^{(k)*}_{k,t}.R^{(k)}_{n,t}(z).C^{(k)}_{k,t} - \frac{t}{n}\mathbb{E}\left[\text{Tr}\left(R_{n,t}(z)\right) \mid X^n(0)\right]\right|^2 \big|X^n(0)\right]$$
$$\leq \frac{t}{n} + \frac{t^2}{n\text{Im}^2(z)} + (8n+3)\,\frac{t^2}{n^2\text{Im}^2(z)}.$$

Going back to (C.2) and using (C.4) to upper-bound the first term in the right-hand side:

$$\begin{aligned}
&\left|\mathbb{E}\left[\big(R_{n,t}(z)\big)_{kk} - \big(\widetilde{R}_{n,t}(z)\big)_{kk} \mid X^n(0)\right]\right| \\
&\leq \frac{t}{n}\big|\big(\widetilde{R}_{n,t}(z)\big)_{kk}\big|^2.\mathbb{E}\left[\big|\text{Tr}\big(R^{(k)}_{n,t}(z)\big) - \text{Tr}\big(R_{n,t}(z)\big)\big| \;\Big|\; X^n(0)\right] \\
&+ \big|\big(\widetilde{R}_{n,t}(z)\big)_{kk}\big|^2.\mathbb{E}\Big[\big|\big(R_{n,t}(z)\big)_{kk}\big|.\big|(H^n(t))_{kk} + C^{(k)*}_{k,t}.R^{(k)}_{n,t}(z).C^{(k)}_{k,t} \\
&- \frac{t}{n}\mathbb{E}\left[\text{Tr}\left(R_{n,t}(z)\right) \mid X^n(0)\right]\big|^2 \;\Big|\; X^n(0)\Big] \\
&\leq \big|\big(\widetilde{R}_{n,t}(z)\big)_{kk}\big|^2.\left(\frac{t}{n\text{Im}(z)} + \frac{t}{n\text{Im}(z)} + \frac{t^2}{n\text{Im}^3(z)} + \frac{(8n+3)t^2}{n^2\text{Im}^3(z)}\right) \\
&\leq \big|\big(\widetilde{R}_{n,t}(z)\big)_{kk}\big|^2.\frac{1}{n}\left(\frac{2t}{\text{Im}(z)} + \frac{12t^2}{\text{Im}^3(z)}\right).
\end{aligned}$$

Using this upper bound in (C.1), we obtain by summation the result and using that for any k,

$$\big|\widetilde{R}_{n,t}(z)\big)_{kk}\big|^2 \leq \frac{1}{\text{Im}^2(z)}.$$

Appendix D: Proof of Lemma 3.5

From (3.7) and introducing $\overline{w}_1(z)$ such that:

$$G_{\mu_0^n \boxplus \sigma_t}(z) = G_{\mu_0^n}\big(\overline{w}_{fp}(z)\big) = G_{\sigma_t}(\overline{w}_1(z)).$$

We can derive from Theorem-Definition 2.5 that $\overline{w}_{fp}(z)$ solves the equation (i) of Lemma 3.5 and that:

$$z = \overline{w}_{fp}(z) + tG_{\mu_0^n}(\overline{w}_{fp}(z)),$$

for all $z \in \mathbb{C}^+$. The latter equation justifies (ii) of Lemma 3.5.

Appendix E: Proof of Corollary 4.3

Recall that from Proposition 4.2 and Theorem 4.1, the mean integrated square error is

$$MISE = \mathbb{E}\Big[\left\|\widehat{p}_{0,h} - p_0\right\|^2\Big] \leq Le^{-2ah^{-r}} + \frac{\gamma^8}{(\gamma^2-4t)^4}\frac{C_{var}.e^{\frac{2\gamma}{h}}}{n}.$$

Minimizing in h amounts to solving the following equation obtained by taking the derivative in the right hand side of (4.6):

$$\psi(h) := \exp\big(\frac{2\gamma}{h} + \frac{2a}{h^r}\big)h^{r-1} = O(n). \tag{E.1}$$

Consequently for the minimizer h_* of (E.1) we get that

$$\frac{e^{\frac{2\gamma}{h_*}}}{n} = Ch_*^{1-r}e^{-2ah_*^{-r}},$$

for some constant $C > 0$. Hence, in view of (4.6), when $r < 1$ the bias dominates the variance and the contrary occurs when $r > 1$. Thus, there are three cases to consider to derive rates of convergence: $r = 1$, $r < 1$ and $r > 1$. To solve the equation (E.1), we follow the steps of Lacour [25].

Case $r = 1$.

The case where $r = 1$ provides a window $h_* = 2(a+\gamma)/\log n$ and we get

$$MISE = O\left(n^{-\frac{a}{a+\gamma}}\right).$$

Case $r < 1$.

In this case, and in the case $r > 1$, following the ideas in [25], we will look for the bandwidth h expressed as an expansion in $\log(n)$. In this expansion and when $r < 1$, the integer k such that $\frac{k}{k+1} < r \leq \frac{k+1}{k+2}$ will play a role. The optimal bandwidth is of the form:

$$h_* = 2\gamma\Big(\log(n) + (r-1)\log\log(n) + \sum_{i=0}^{k} b_i(\log n)^{r+i(r-1)}\Big)^{-1}, \tag{E.2}$$

where the coefficients b_i's are a sequence of real numbers chosen so that $\psi(h_*) = O(n)$. The heuristic of this expansion is as follows: the first term corresponds to the solution of $e^{2\gamma/h} = n$. The second term is added to compensate the factor h^{r-1} in (E.1) evaluated with the previous bandwidth, and the third term aims at compensating the factor e^{2a/h^r}. Notice that $r - 1 < 0$ and that the definition of k implies that $r > r + (r-1) > \cdots > r + k(r-1) > 0 > r + (k+1)(r-1)$. This explains the range of the index i in the sum of the right hand side of (E.2).

Plugging (E.2) into (E.1),

$$\begin{aligned}
\psi(h_*) =& n\big(\log n\big)^{r-1} \exp\Big(\sum_{i=0}^{k} b_i (\log n)^{r+i(r-1)}\Big) \\
& \times \exp\Big(\frac{2a}{(2\gamma)^r}\big(\log n\big)^r \big(1 + \frac{(r-1)\log\log(n) + \sum_{i=0}^{k} b_i(\log n)^{r+i(r-1)}}{\log n}\big)^r\Big) \\
& \times (2\gamma)^{r-1}\big(\log n\big)^{-(r-1)}\Big(1 + \frac{(r-1)\log\log(n) + \sum_{i=0}^{k} b_i(\log n)^{r+i(r-1)}}{\log n}\Big)^{-(r-1)} \\
=& (2\gamma)^{r-1} n (1+v_n)^{1-r} \exp\Big(\sum_{i=0}^{k} b_i (\log n)^{r+i(r-1)}\Big) \\
& \times \exp\Big(\frac{2a}{(2\gamma)^r}\big(\log n\big)^r \Big[1 + \sum_{j=0}^{k} \frac{r(r-1)\cdots(r-j)}{(j+1)!} v_n^{j+1} + o(v_n^{k+1})\Big]\Big)
\end{aligned}$$

where

$$v_n = \frac{(r-1)\log\log(n) + \sum_{i=0}^{k} b_i(\log n)^{r+i(r-1)}}{\log n} = (r-1)\frac{\log\log(n)}{\log n} + \sum_{i=0}^{k} b_i (\log n)^{(i+1)(r-1)}$$

converges to zero when $n \to +\infty$. We note that

$$\begin{aligned}
v_n^{j+1} &= \sum_{i=0}^{k-j-1} \sum_{p_0+\cdots p_j = i} b_{p_0}\cdots b_{p_j} (\log n)^{(i+j+1)(r-1)} + O\Big(\big(\log n\big)^{(k+1)(r-1)}\Big) \\
&= \sum_{\ell=j+1}^{k} \sum_{p_0+\cdots p_j = \ell-j-1} b_{p_0}\cdots b_{p_j} (\log n)^{\ell(r-1)} + O\Big(\big(\log n\big)^{(k+1)(r-1)}\Big).
\end{aligned}$$

So

$$\begin{aligned}
\psi(h_*) = & (2\gamma)^{r-1} n (1+v_n)^{1-r} \exp\Big(\sum_{i=0}^{k} b_i (\log n)^{r+i(r-1)}\Big) \\
& \times \exp\Big\{\frac{2a}{(2\gamma)^r}(\log n)^r \\
& \qquad + \frac{2a}{(2\gamma)^r} \sum_{\ell=1}^{k}\sum_{j=0}^{\ell-1} \Big[\frac{r(r-1)\cdots(r-j)}{(j+1)!} \sum_{p_0+\cdots p_j = \ell-j-1} b_{p_0}\cdots b_{p_j}\Big] (\log n)^{r+\ell(r-1)}
\end{aligned}$$

$$+ O\Big(\big(\log n \big)^{(k+1)(r-1)} \Big) \Big\}$$

$$= (2\gamma)^{r-1} n (1+v_n)^{1-r} \exp\Big(\sum_{i=0}^{k} M_i (\log n)^{i(r-1)+r} + o(1) \Big).$$

The condition $\psi(h_*) = O(n)$ implies the following choices of constants M_i's:

$$M_0 = b_0 + \frac{2a}{(2\gamma)^r}, \qquad \forall i > 0,\ M_i = b_i + \frac{2a}{(2\gamma)^r} \sum_{j=0}^{i-1} \frac{r(r-1)\cdots(r-j)}{(j+1)!} \sum_{p_0+\cdots p_j = i-j-1} b_{p_0} \cdots b_{p_j}.$$

Since h_* solves (E.1) if all the $M_i = 0$ for $i \in \{0, \cdots k\}$, the above system provides equation by equation the proper coefficients b_i^*.

$$b_0^* = -\frac{2a}{(2\gamma)^r}, \qquad b_i^* = -\frac{2a}{(2\gamma)^r} \sum_{j=0}^{i-1} \frac{r(r-1)\cdots(r-j)}{(j+1)!} \sum_{p_0+\cdots p_j = i-j-1} b_{p_0}^* \cdots b_{p_j}^*. \tag{E.3}$$

Replacing in (4.6), we get:

$$MISE = O\Big(\exp\Big\{ -\frac{2a}{(2\gamma)^r} \Big[\log n + (r-1)\log\log n + \sum_{i=0}^{k} b_i^* (\log n)^{r+i(r-1)} \Big]^r \Big\} \Big).$$

Case $r > 1$.

Here, let us denote by k the integer such that $\frac{k}{k+1} < \frac{1}{r} \le \frac{k+1}{k+2}$. We look here for a bandwidth of the form:

$$h_*^r = 2a\Big(\log n + \frac{r-1}{r} \log\log(n) + \sum_{i=0}^{k} d_i (\log n)^{\frac{1}{r} - i\frac{r-1}{r}} \Big)^{-1}, \tag{E.4}$$

where the coefficients d_i's will be chosen so that $\psi(h_*) = O(n)$.

Similar computations as for the case $r < 1$ provide that:

$$\psi(h_*) = (2a)^{\frac{r-1}{r}} n (1+v_n)^{-\frac{r-1}{r}} \times \exp\Big(\sum_{i=0}^{k} d_i (\log n)^{\frac{1}{r} - i\frac{r-1}{r}} \Big)$$

$$\times \exp\Big(\frac{2\gamma}{(2a)^{1/r}} (\log n)^{1/r} \Big[1 +$$

$$\sum_{\ell=1}^{k} \sum_{j=0}^{\ell-1} \sum_{p_0+\cdots p_j = \ell-j-1} \frac{\frac{1}{r}\big(\frac{1}{r}-1\big)\cdots\big(\frac{1}{r}-j\big)}{(j+1)!} d_{p_0}\cdots d_{p_j} (\log n)^{\ell\frac{1-r}{r}} + O\big((\log n)^{k\frac{1-r}{r}} \big) \Big] \Big)$$

$$= (2a)^{\frac{r-1}{r}} n (1+v_n)^{-\frac{r-1}{r}} \exp\Big(\sum_{i=0}^{k} M_i (\log n)^{\frac{1}{r} - i\frac{r-1}{r}} + o(1) \Big)$$

where here

$$v_n = \frac{\frac{r-1}{r}\log\log(n) + \sum_{i=0}^{k} d_i (\log n)^{\frac{1}{r} - i\frac{r-1}{r}}}{\log n},$$

and

$$M_0 = d_0 + \frac{2\gamma}{(2a)^{1/r}}, \qquad \forall i > 0,\ M_i = d_i + \frac{2\gamma}{(2a)^{1/r}} \sum_{j=0}^{i-1} \sum_{p_0+\cdots p_j = i-j-1} \frac{\frac{1}{r}\left(\frac{1}{r}-1\right)\cdots\left(\frac{1}{r}-j\right)}{(j+1)!} d_{p_0}\cdots d_{p_j} \tag{E.5}$$

Solving $M_0 = \cdots = M_k = 0$ provides the coefficients d_i^* so that (E.1) is satisfied.

Plugging the bandwidth h_* with the coefficients d_i^* into (4.6), we obtain:

$$MISE = O\Big(\frac{1}{n}\exp\Big\{\frac{2\gamma}{(2a)^{1/r}}\Big[\log n + \frac{r-1}{r}\log\log n + \sum_{i=0}^{k} d_i^* (\log n)^{\frac{1}{r}-i\frac{r-1}{r}}\Big]^{1/r}\Big\}\Big).$$

This concludes the proof of Corollary 4.3.