Bayesian Analysis (2015)

Supplementary Material of "Posterior Concentration Rates for Counting Processes with Aalen Multiplicative Intensities"

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Appendix A

A.1 Lemma 2

Lemma 2. *Under condition* (5)*, there exist constants* ξ, $K > 0$*, only depending on* M_{λ_0}, α, m_1 *and* m_2*, such that, for any non-negative function* λ_1*, there exists a test* ϕ_{λ_1} *so that*

$$\mathbb{E}_{\lambda_0}^{(n)}[\mathbf{1}_{\Gamma_n}\phi_{\lambda_1}] \le 2\exp\left(-Kn\|\lambda_1-\lambda_0\|_1 \times \min\{\|\lambda_1-\lambda_0\|_1,\, m_1\}\right)$$

and

$$\sup_{\lambda:\ \|\lambda-\lambda_1\|_1<\xi\|\lambda_1-\lambda_0\|_1} \mathbb{E}_{\lambda}^{(n)}[\mathbf{1}_{\Gamma_n}(1-\phi_{\lambda_1})] \le 2\exp\left(-Kn\|\lambda_1-\lambda_0\|_1 \times \min\{\|\lambda_1-\lambda_0\|_1,\, m_1\}\right).$$

Proof of Lemma 2. For any λ, we denote by $\mathbb{E}_{\lambda,\Gamma_n}^{(n)}[\cdot] = \mathbb{E}_{\lambda}^{(n)}[\mathbf{1}_{\Gamma_n}\times\cdot]$. For any λ, λ', we define

$$\|\lambda-\lambda'\|_{\tilde{\mu}_n} := \int_\Omega |\lambda(t)-\lambda'(t)|\tilde{\mu}_n(t)\mathrm{d}t.$$

We have

$$m_1\|\lambda-\lambda_0\|_1 \le \|\lambda-\lambda_0\|_{\tilde{\mu}_n} \le m_2\|\lambda-\lambda_0\|_1. \tag{27}$$

The main tool for building convenient tests is Theorem 3 (and its proof) of Hansen et al. (2015) applied to the univariate setting. By mimicking the proof of this theorem from inequality (24) to inequality (26), if H is a deterministic function bounded by b, we have that, for any $u \ge 0$,

$$\mathbb{P}_{\lambda}^{(n)}\left(\left|\int_0^T H_t\mathrm{d}N_t - \int_0^T H_t\lambda(t)Y_t\mathrm{d}t\right| \ge \sqrt{2\sigma_n^2 u} + \frac{bu}{3} \text{ and } \Gamma_n\right) \le 2e^{-u}, \tag{28}$$

where σ_n^2 is a deterministic constant such that, on Γ_n,

$$\int_0^T H_t^2\lambda(t)Y_t\mathrm{d}t \le \sigma_n^2 \quad \text{almost surely.}$$

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Typically, σ_n^2 increases linearly with n. For any non-negative function λ_1, we define the sets

$$A := \{t \in \Omega : \ \lambda_1(t) \geq \lambda_0(t)\} \quad \text{and} \quad A^c := \{t \in \Omega : \ \lambda_1(t) < \lambda_0(t)\},$$

together with the following pseudo-metrics

$$d_A(\lambda_1, \lambda_0) := \int_A [\lambda_1(t) - \lambda_0(t)]\tilde{\mu}_n(t)\mathrm{d}t \quad \text{and} \quad d_{A^c}(\lambda_1, \lambda_0) := \int_{A^c} [\lambda_0(t) - \lambda_1(t)]\tilde{\mu}_n(t)\mathrm{d}t.$$

Note that $\|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n} = d_A(\lambda_1, \lambda_0) + d_{A^c}(\lambda_1, \lambda_0)$. For $u > 0$, if $d_A(\lambda_1, \lambda_0) \geq d_{A^c}(\lambda_1, \lambda_0)$, define the test

$$\phi_{\lambda_1, A}(u) := \mathbf{1}\left\{N(A) - \int_A \lambda_0(t) Y_t \mathrm{d}t \geq \rho_n(u)\right\}, \qquad \text{with } \rho_n(u) := \sqrt{2nv(\lambda_0)u} + \frac{u}{3},$$

where, for any non-negative function λ,

$$v(\lambda) := (1 + \alpha) \int_\Omega \lambda(t)\tilde{\mu}_n(t)\mathrm{d}t. \tag{29}$$

Similarly, if $d_A(\lambda_1, \lambda_0) < d_{A^c}(\lambda_1, \lambda_0)$, define

$$\phi_{\lambda_1, A^c}(u) := \mathbf{1}\left\{N(A^c) - \int_{A^c} \lambda_0(t) Y_t \mathrm{d}t \leq -\rho_n(u)\right\}.$$

Since for any non-negative function λ, on Γ_n, by (19),

$$(1 - \alpha) \int_\Omega \lambda(t)\tilde{\mu}_n(t)\mathrm{d}t \leq \int_\Omega \lambda(t)\frac{Y_t}{n}\mathrm{d}t \leq (1 + \alpha) \int_\Omega \lambda(t)\tilde{\mu}_n(t)\mathrm{d}t, \tag{30}$$

inequality (28) applied with $H = \mathbf{1}_A$ or $H = \mathbf{1}_{A^c}$, $b = 1$ and $\sigma_n^2 = nv(\lambda_0)$ implies that, for any $u > 0$,

$$\mathbb{E}_{\lambda_0, \Gamma_n}^{(n)}[\phi_{\lambda_1, A}(u)] \leq 2e^{-u} \quad \text{and} \quad \mathbb{E}_{\lambda_0, \Gamma_n}^{(n)}[\phi_{\lambda_1, A^c}(u)] \leq 2e^{-u}. \tag{31}$$

Note that, by (27), if $d_A(\lambda_1, \lambda_0) \geq d_{A^c}(\lambda_1, \lambda_0)$, then

$$d_A(\lambda_1, \lambda_0) \geq \frac{1}{2}\|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n} \geq \frac{m_1}{2}\|\lambda_1 - \lambda_0\|_1, \tag{32}$$

and, by virtue of Lemma 3,

$$\begin{aligned} u_A &\geq \min\{u_{0A} n d_A^2(\lambda_1, \lambda_0), \, u_{1A} n d_A(\lambda_1, \lambda_0)\} \\ &\geq n d_A(\lambda_1, \lambda_0) \times \min\{u_{0A} d_A(\lambda_1, \lambda_0), \, u_{1A}\} \\ &\geq \frac{1}{2} n m_1 \|\lambda_1 - \lambda_0\|_1 \times \min\left\{\frac{1}{2} u_{0A} m_1 \|\lambda_1 - \lambda_0\|_1, \, u_{1A}\right\} \\ &\geq K_A n \|\lambda_1 - \lambda_0\|_1 \times \min\{\|\lambda_1 - \lambda_0\|_1, \, m_1\}, \end{aligned}$$

for K_A a positive constant small enough only depending on α, M_{λ_0}, m_1 and m_2. Similarly, if $d_A(\lambda_1, \lambda_0) < d_{A^c}(\lambda_1, \lambda_0)$,

$$
\begin{aligned}
u_{A^c} &\geq \frac{1}{2} n m_1 \|\lambda_1 - \lambda_0\|_1 \times \min\left\{\frac{1}{2} u_{0A^c} m_1 \|\lambda_1 - \lambda_0\|_1,\, u_{1A^c}\right\} \\
&\geq K_{A^c} n \|\lambda_1 - \lambda_0\|_1 \times \min\{\|\lambda_1 - \lambda_0\|_1,\, m_1\},
\end{aligned}
$$

for K_{A^c} a positive constant small enough only depending on α, M_{λ_0}, m_1 and m_2. Now, we set

$$
\phi_{\lambda_1} = \phi_{\lambda_1,A}(u_A)\mathbf{1}_{\{d_A(\lambda_1,\,\lambda_0)\geq d_{A^c}(\lambda_1,\,\lambda_0)\}} + \phi_{\lambda_1,A^c}(u_{A^c})\mathbf{1}_{\{d_A(\lambda_1,\,\lambda_0)< d_{A^c}(\lambda_1,\,\lambda_0)\}},
$$

so that, with $K = \min\{K_A,\, K_{A^c}\}$,

$$
\min(u_A, u_{A^c}) \geq K n \|\lambda_1 - \lambda_0\|_1 \times \min\{\|\lambda_1 - \lambda_0\|_1,\, m_1\}
$$

and, by using (31),

$$
\begin{aligned}
\mathbb{E}^{(n)}_{\lambda_0,\Gamma_n}[\phi_{\lambda_1}] &= \mathbb{E}^{(n)}_{\lambda_0,\Gamma_n}[\phi_{\lambda_1,A}(u_A)]\mathbf{1}_{\{d_A(\lambda_1,\,\lambda_0)\geq d_{A^c}(\lambda_1,\,\lambda_0)\}} \\
&\qquad\qquad + \mathbb{E}^{(n)}_{\lambda_0,\Gamma_n}[\phi_{\lambda_1,A^c}(u_{A^c})]\mathbf{1}_{\{d_A(\lambda_1,\,\lambda_0)< d_{A^c}(\lambda_1,\,\lambda_0)\}} \\
&\leq 2e^{-u_A}1_{\{d_A(\lambda_1,\,\lambda_0)\geq d_{A^c}(\lambda_1,\,\lambda_0)\}} + 2e^{-u_{A^c}}1_{\{d_A(\lambda_1,\,\lambda_0)< d_{A^c}(\lambda_1,\,\lambda_0)\}} \\
&\leq 2\exp\left(-Kn\|\lambda_1 - \lambda_0\|_1 \times \min\{\|\lambda_1 - \lambda_0\|_1,\, m_1\}\right).
\end{aligned}
$$

Now, we set

$$
\xi = \frac{m_1(1-\alpha)}{4m_2(1+\alpha)}.
$$

Let λ be fixed. If $\|\lambda - \lambda_1\|_1 < \xi\|\lambda_1 - \lambda_0\|_1$, then

$$
\|\lambda - \lambda_1\|_{\tilde{\mu}_n} \leq \frac{1-\alpha}{4(1+\alpha)}\|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n}
$$

and Lemma 3 shows that

$$
\begin{aligned}
\mathbb{E}^{(n)}_{\lambda,\Gamma_n}[1-\phi_{\lambda_1}] &\leq 2e^{-u_A}\mathbf{1}_{\{d_A(\lambda_1,\,\lambda_0)\geq d_{A^c}(\lambda_1,\,\lambda_0)\}} + 2e^{-u_{A^c}}\mathbf{1}_{\{d_A(\lambda_1,\,\lambda_0)< d_{A^c}(\lambda_1,\,\lambda_0)\}} \\
&\leq 2\exp\left(-Kn\|\lambda_1 - \lambda_0\|_1 \times \min\{\|\lambda_1 - \lambda_0\|_1,\, m_1\}\right),
\end{aligned}
$$

which completes the proof of Lemma 2. □

The following lemma is used in the proof of Lemma 2.

Lemma 3. *Assume condition* (5) *is verified. Let λ be a non-negative function. Assume that*

$$
\|\lambda - \lambda_1\|_{\tilde{\mu}_n} \leq \frac{1-\alpha}{4(1+\alpha)}\|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n}.
$$

We set $\tilde{M}_n(\lambda_0) = \int_\Omega \lambda_0(t)\tilde{\mu}_n(t)\mathrm{d}t$ and we distinguish two cases.

1. *Assume that $d_A(\lambda_1,\, \lambda_0) \geq d_{A^c}(\lambda_1,\, \lambda_0)$. Then,*

$$
\mathbb{E}^{(n)}_{\lambda,\Gamma_n}[1-\phi_{\lambda_1,A}(u_A)] \leq 2\exp(-u_A),
$$

where

$$u_A = \begin{cases} u_{0A} n d_A^2(\lambda_1, \lambda_0), & \text{if } \|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n} \le 2\tilde{M}_n(\lambda_0), \\ u_{1A} n d_A(\lambda_1, \lambda_0), & \text{if } \|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n} > 2\tilde{M}_n(\lambda_0), \end{cases}$$

and u_{0A}, u_{1A} are two constants only depending on α, M_{λ_0}, m_1 and m_2.

2. *Assume that $d_A(\lambda_1, \lambda_0) < d_{A^c}(\lambda_1, \lambda_0)$. Then,*

$$\mathbb{E}^{(n)}_{\lambda,\Gamma_n}[1 - \phi_{\lambda_1,A^c}(u_{A^c})] \le 2\exp(-u_{A^c}),$$

where

$$u_{A^c} = \begin{cases} u_{0A^c} n d_{A^c}^2(\lambda_1, \lambda_0), & \text{if } \|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n} \le 2\tilde{M}_n(\lambda_0), \\ u_{1A^c} n d_{A^c}(\lambda_1, \lambda_0), & \text{if } \|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n} > 2\tilde{M}_n(\lambda_0), \end{cases}$$

and u_{0A^c}, u_{1A^c} are two constants only depending on α, M_{λ_0}, m_1 and m_2.

Proof of Lemma 3. We only consider the case where $d_A(\lambda_1, \lambda_0) \ge d_{A^c}(\lambda_1, \lambda_0)$. The case $d_A(\lambda_1, \lambda_0) < d_{A^c}(\lambda_1, \lambda_0)$ can be dealt with by using similar arguments. So, we assume that $d_A(\lambda_1, \lambda_0) \ge d_{A^c}(\lambda_1, \lambda_0)$. On Γ_n, by using (19) and (32), we have

$$\begin{aligned}\int_A [\lambda_1(t) - \lambda_0(t)] Y_t \mathrm{d}t &\ge n(1-\alpha) \int_A [\lambda_1(t) - \lambda_0(t)] \tilde{\mu}_n(t) \mathrm{d}t \\ &\ge \frac{n(1-\alpha)}{2} \|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n} \\ &\ge 2n(1+\alpha) \|\lambda - \lambda_1\|_{\tilde{\mu}_n} \\ &\ge 2n(1+\alpha) \int_A |\lambda(t) - \lambda_1(t)| \tilde{\mu}_n(t) \mathrm{d}t \\ &\ge 2 \int_A |\lambda(t) - \lambda_1(t)| Y_t \mathrm{d}t.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}^{(n)}_{\lambda,\Gamma_n}[1 - \phi_{\lambda_1,A}(u_A)] \\ &\quad = \mathbb{P}^{(n)}_{\lambda,\Gamma_n}\left(N(A) - \int_A \lambda(t) Y_t \mathrm{d}t < \rho_n(u_A) + \int_A (\lambda_0 - \lambda)(t) Y_t \mathrm{d}t\right) \\ &\quad = \mathbb{P}^{(n)}_{\lambda,\Gamma_n}\left(N(A) - \int_A \lambda(t) Y_t \mathrm{d}t < \rho_n(u_A) - \int_A (\lambda_1 - \lambda_0)(t) Y_t \mathrm{d}t \right. \\ &\qquad\qquad\qquad \left. + \int_A (\lambda_1 - \lambda)(t) Y_t \mathrm{d}t\right) \\ &\quad \le \mathbb{P}^{(n)}_{\lambda,\Gamma_n}\left(N(A) - \int_A \lambda(t) Y_t \mathrm{d}t < \rho_n(u_A) - \frac{1}{2} \int_A (\lambda_1 - \lambda_0)(t) Y_t \mathrm{d}t\right).\end{aligned} \tag{33}$$

Assume that $\|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n} \le 2\tilde{M}_n(\lambda_0)$. This assumption implies that

$$d_A(\lambda_1, \lambda_0) \le \|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n} \le 2\tilde{M}_n(\lambda_0) \le 2m_2 M_{\lambda_0}.$$

Since $v(\lambda_0) = (1+\alpha)\tilde{M}_n(\lambda_0)$, with $u_A = u_{0A} n d_A^2(\lambda_1, \lambda_0)$, where $u_{0A} \le 1$ is a constant depending on α, m_1 and m_2 chosen later, we have

$$\begin{aligned}
\rho_n(u_A) &= \sqrt{2nv(\lambda_0)u_A} + \frac{u_A}{3} \\
&\leq nd_A(\lambda_1,\, \lambda_0)\sqrt{2u_{0A}(1+\alpha)\tilde{M}_n(\lambda_0)} + \frac{u_{0A}nd_A^2(\lambda_1,\, \lambda_0)}{3} \\
&\leq nd_A(\lambda_1,\, \lambda_0)\sqrt{2u_{0A}(1+\alpha)\tilde{M}_n(\lambda_0)} + \frac{u_{0A}nd_A(\lambda_1,\, \lambda_0)\times 2\tilde{M}_n(\lambda_0)}{3} \\
&\leq K_1\sqrt{u_{0A}}nd_A(\lambda_1,\, \lambda_0),
\end{aligned}$$

with

$$K_1 = \sqrt{2(1+\alpha)\tilde{M}_n(\lambda_0)} + \frac{2\tilde{M}_n(\lambda_0)\sqrt{u_{0A}}}{3}.$$

Note that the definition of $v(\lambda)$ in (29) gives

$$\begin{aligned}
v(\lambda) &= (1+\alpha)\int_\Omega \lambda_0(t)\tilde{\mu}_n(t)\mathrm{d}t + (1+\alpha)\int_\Omega [\lambda(t)-\lambda_0(t)]\tilde{\mu}_n(t)\mathrm{d}t \\
&\leq v(\lambda_0) + (1+\alpha)\|\lambda-\lambda_0\|_{\tilde{\mu}_n} \\
&\leq v(\lambda_0) + (1+\alpha)\left[\|\lambda-\lambda_1\|_{\tilde{\mu}_n} + \|\lambda_1-\lambda_0\|_{\tilde{\mu}_n}\right] \\
&\leq v(\lambda_0) + (1+\alpha)\left[1 + \frac{1-\alpha}{4(1+\alpha)}\right]\|\lambda_1-\lambda_0\|_{\tilde{\mu}_n} \\
&\leq v(\lambda_0) + \frac{5+3\alpha}{2}\tilde{M}_n(\lambda_0) \leq C_1,
\end{aligned}$$

where C_1 only depends on α, M_{λ_0}, m_1 and m_2. Combined with (30), this implies that, on Γ_n, if

$$K_1 \leq \frac{(1-\alpha)}{4\sqrt{u_{0A}}},$$

which is true for u_{0A} small enough only depending on α, M_{λ_0}, m_1 and m_2, then

$$\begin{aligned}
\frac{1}{2}\int_A(\lambda_1-\lambda_0)(t)Y_t\mathrm{d}t - \rho_n(u_A) &\geq \frac{(1-\alpha)n}{2}d_A(\lambda_1,\lambda_0)\left(1-\frac{2K_1\sqrt{u_{0A}}}{1-\alpha}\right) \\
&\geq \frac{(1-\alpha)n}{4}d_A(\lambda_1,\lambda_0) \geq \sqrt{2nC_1r} + \frac{r}{3} \geq \sqrt{2nv(\lambda)r} + \frac{r}{3},
\end{aligned}$$

with

$$r = n\min\left\{\frac{(1-\alpha)^2}{128C_1}d_A^2(\lambda_1,\, \lambda_0),\ \frac{3(1-\alpha)}{8}d_A(\lambda_1,\, \lambda_0)\right\}.$$

Since on Γ_n,

$$\int_A \lambda(t)Y_t\mathrm{d}t \leq \int_A n(1+\alpha)\lambda(t)\tilde{\mu}_n(t)\mathrm{d}t \leq nv(\lambda),$$

inequality (28) with $H = \mathbf{1}_A$, $b=1$ and $\sigma_n^2 = nv(\lambda)$, combined with (33), leads to

$$\mathbb{E}^{(n)}_{\lambda,\Gamma_n}[1-\phi_{\lambda_1,A}(u_A)] \leq 2e^{-r}.$$

Since $r \geq u_A$ for u_{0A} small enough, then

$$\mathbb{E}^{(n)}_{\lambda,\Gamma_n}[1-\phi_{\lambda_1,A}(u)] \leq 2e^{-u_A}.$$

Assume that $\|\lambda_1-\lambda_0\|_{\tilde{\mu}_n} > 2\tilde{M}_n(\lambda_0)$. We take $u_A = u_{1A}nd_A(\lambda_1,\, \lambda_0)$, where $u_{1A}\leq 1$ is a constant depending on α chosen later. We still consider the same test $\phi_{\lambda_1,A}(u_A)$.

Observe now that, since

$$d_A(\lambda_1, \lambda_0) \geq \frac{1}{2}\|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n} \geq \tilde{M}_n(\lambda_0),$$

$v(\lambda_0) = (1+\alpha)\tilde{M}_n(\lambda_0)$ and $u_{1A} \leq 1$,

$$\begin{aligned}\rho_n(u_A) &= \sqrt{2nu_A v(\lambda_0)} + \frac{u_A}{3} \\ &\leq n\sqrt{2(1+\alpha)u_{1A}\tilde{M}_n(\lambda_0)d_A(\lambda_1, \lambda_0)} + \frac{nu_{1A}}{3}d_A(\lambda_1, \lambda_0) \\ &\leq \left(\sqrt{2(1+\alpha)} + \frac{1}{3}\right) n\sqrt{u_{1A}}d_A(\lambda_1, \lambda_0)\end{aligned}$$

and, under the assumptions of the lemma,

$$v(\lambda) \leq (1+\alpha)\tilde{M}_n(\lambda_0) + (1+\alpha)\left[\|\lambda - \lambda_1\|_{\tilde{\mu}_n} + \|\lambda_1 - \lambda_0\|_{\tilde{\mu}_n}\right] \leq C_2 d_A(\lambda_1, \lambda_0), \quad (34)$$

where C_2 only depends on α. Therefore,

$$\begin{aligned}&\frac{1}{2}\int_A (\lambda_1 - \lambda_0)(t)Y_t \mathrm{d}t - \rho_n(u_A) \\ &\qquad \geq \frac{n(1-\alpha)}{2}\int_A [\lambda_1(t) - \lambda_0(t)]\tilde{\mu}_n(t)\mathrm{d}t - \left(\sqrt{2(1+\alpha)} + \frac{1}{3}\right)\sqrt{u_{1A}}nd_A(\lambda_1, \lambda_0) \\ &\qquad \geq \left[\frac{1-\alpha}{2} - \left(\sqrt{2(1+\alpha)} + \frac{1}{3}\right)\sqrt{u_{1A}}\right]nd_A(\lambda_1, \lambda_0) \\ &\qquad \geq \frac{1-\alpha}{4}nd_A(\lambda_1, \lambda_0),\end{aligned}$$

where the last inequality is true for u_{1A} small enough depending only on α. Finally, using (34), since $u_A = u_{1A}nd_A(\lambda_1, \lambda_0)$, we have

$$\begin{aligned}\frac{1-\alpha}{4}nd_A(\lambda_1, \lambda_0) &\geq \sqrt{2nC_2 d_A(\lambda_1, \lambda_0)u_{1A}nd_A(\lambda_1, \lambda_0)} + \frac{1}{3}u_{1A}nd_A(\lambda_1, \lambda_0) \\ &\geq \sqrt{2nv(\lambda)u_A} + \frac{u_A}{3}\end{aligned}$$

for u_{1A} small enough depending only on α. By using the same arguments as before, we then obtain

$$\mathbb{E}^{(n)}_{\lambda,\Gamma_n}[1 - \phi_{\lambda_1,A}(u_A)] \leq 2e^{-u_A},$$

which completes the proof. □

A.2 Algorithm

Assuming that the current state is $(M_\lambda, \mathbf{u}, \mathbf{c}, \boldsymbol{\theta}, \mathbf{v})$, the algorithm consists in simulating iteratively:

[1.] Sampling from $M_\lambda \mid \mathbf{u}, \mathbf{c}, \boldsymbol{\theta}, \mathbf{v}, \mathbf{Z}, \delta$

From (17) and the prior distribution, we have

$$M_\lambda \sim \text{Gamma}\left(a_M + n^\star,\, b_M + \sum_{k=1}^{K_t} w_k H(\theta_k)\right),$$

where H has been defined in (15).

[2.] Sampling from $\boldsymbol{\theta} \mid \mathbf{c}, \mathbf{u}, \mathbf{v}, \mathbf{Z}, \delta$

For $k = 1, \ldots, K^\star$,

$$p(\theta_k \mid \mathbf{c}, \mathbf{u}, \mathbf{v}, \mathbf{Z}, \delta) \propto G(\theta_k) \frac{\mathbf{1}_{[1-\min\{Z_i,\, i \in \mathcal{O} | c_i = k\},\, 1]}}{\theta_k^{n_k}} \exp(-M_\lambda w_k H(\theta_k) \mathbf{1}_{k \le K_t}).$$

So, we implement

- Propose $\theta'_k \sim \mathcal{U}_{[1-\min\{Z_i,\, i \in \mathcal{O} | c_i = k\},\, 1]}$,
- Compute
$$\rho(\theta'_k, \theta_k) = \min\left\{1, \frac{G(\theta'_k)}{G(\theta_k)} \left(\frac{\theta_k}{\theta'_k}\right)^{n_k} \exp\left(-M_\lambda w_k \left[H(\theta'_k) - H(\theta_k)\right] \mathbf{1}_{k \le K_t}\right)\right\},$$
- Set $\theta_k := \theta'_k$ with probability $\rho(\theta'_k, \theta_k)$.

[3.] Sampling from $\mathbf{u} \mid \mathbf{c}, \boldsymbol{\theta}, \mathbf{v}, \mathbf{Z}, \delta$

From (17), we get

$$u_i \sim \mathcal{U}_{[0,\, w_{c_i}]}, \quad \forall\, i \in \mathcal{O}.$$

Set $u^* = \min\{u_i,\, i \in \mathcal{O}\}$.

[4.] Sampling from $\mathbf{v} \mid \mathbf{u}, \boldsymbol{\theta}, \mathbf{c}, \mathbf{Z}, \delta$

From (17) and Walker (2007), we derive the following conditional distributions: $\forall\, j = 1, \ldots, K^\star$,

$$\pi(v_j \mid \mathbf{v}_{-j}, \mathbf{u}, \boldsymbol{\theta}, \mathbf{c}, \mathbf{Z}, \delta) \propto (1 - v_j)^{A-1} \mathbf{1}_{[A_j,\, B_j]} \exp\left(-M_\lambda \sum_{k=1}^{K_t} w_k H(\theta_k)\right),$$

where

$$A_j = \frac{\max\{u_i,\, i \in \mathcal{O} \mid c_i = j\}}{\prod_{l<j}(1 - v_l)}, \quad B_j = 1 - \max\left\{\frac{u_i}{v_{c_i} \prod_{l=1, l \neq j}^{c_i - 1}(1 - v_l)},\, i \in \mathcal{O} \mid c_i > j\right\},$$

with

$$A_j = 0, \quad \text{if} \quad \{i \in \mathcal{O} \mid c_i = j\} = \emptyset,$$
$$B_j = 1, \quad \text{if} \quad \{i \in \mathcal{O} \mid c_i > j\} = \emptyset.$$

For $j = 1, \ldots, K^\star$,

- Simulate v'_j with a truncated beta distribution: $v'_j \sim \text{Beta}(1, A)|_{[A_j,\, B_j]}$,
- From v'_j, compute $\mathbf{w}'$ following the stick-breaking formula (13),
- Set $v_j := v'_j$ with probability

$$\rho(v_j', v_j) = \min\left\{1, \exp\left(-M_\lambda \sum_{k=1}^{K_t} (w_k' - w_k) H(\theta_k)\right)\right\}.$$

[5.] Adjust the vectors $\boldsymbol{\theta}$, v

We have to check if we have all the required components and add components, if necessary.

While $1 - \sum_{j=1}^{K^\star} w_j \le u^\star$*:*

[•] simulate $\theta_{K^\star+1} \sim H(\cdot)$ [•] simulate $v_{K^\star+1} \sim \text{Beta}(1, 1)$ [•]$K^\star = K^\star + 1$

[6.] Sampling from $\mathbf{c} \mid \mathbf{u}, \boldsymbol{\theta}, \mathbf{v}, \mathbf{Z}, \delta$

$\forall\, i \in \mathcal{O}$, compute

$$P(c_i = k \mid \mathbf{u}, \boldsymbol{\theta}, \mathbf{v}, \mathbf{Z}, \delta) = w_{i,k} \propto \frac{\mathbf{1}_{[1-\theta_k, 1]}(Z_i)}{\theta_k} \mathbf{1}_{[0, w_k]}(u_i), \quad \forall\, k = 1, \ldots, K^\star$$

and generate

$$c_i \sim \sum_{k=1}^{K^\star} w_{i,k} \delta_{\{k\}}.$$

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