

Either all invariant measures have finite total mass, or all have infinite total mass:

$$\text{But } \mu_0(\mathbb{X}) = \sum_{x \in \mathbb{X}} \mu_0(x) = \sum_{x \in \mathbb{X}} \mathbb{E}_0 \left[ \sum_{n=0}^{\mathcal{C}_0^+ - 1} 1_{X_n=x} \right] = \mathbb{E}_0[\mathcal{C}_0^+]$$

so finiteness/infiniteness of  $\mathbb{E}_0[\mathcal{C}_0^+]$  does not depend on  $o \in \mathbb{X}$ .

1)  $\mathbb{E}_0[\mathcal{C}_0^+] < \infty$ , then we can normalize  $\mu_0$  to get a probability measure:

$$\text{and } \pi(o) = \frac{\mu_0(o)}{\mu_0(\mathbb{X})} = \frac{1}{\mathbb{E}_0[\mathcal{C}_0^+]}$$

if  $\mathbb{E}_0[\mathcal{C}_0^+] = \infty$ , then  $\mu_0(\mathbb{X}) = \infty$  and there is no invariant law  $\square$

Remark:  $o$  is "transient" if  $P_o(\mathcal{C}_0^+ = \infty) > 0$

$o$  is "null-recurrent"  $\mathbb{E}_0[\mathcal{C}_0^+] = \infty$

$o$  is "positive-recurrent"  $\mathbb{E}_0[\mathcal{C}_0^+] < \infty$

Example: SRW on  $\mathbb{Z}^d$  : transient for  $d \geq 3$   
 : recurrent for  $d = 1$  or  $d = 2$   $\rightarrow$  null-recurrent or positive recurrent?

## ⑥ Reversibility

As any matrix,  $T$  can be viewed as an operator that acts on <sup>bounded</sup> functions:

$$Tf(x) := \sum_{y \in \mathbb{X}} T_{(x,y)} f(y)$$

$T$  is a contraction on  $L^\infty(\mathbb{X})$ , in the sense that  $\|Tf\|_\infty \leq \|f\|_\infty \quad \forall f \in C_b(\mathbb{X})$ .

More generally,  $T$  is a contraction on  $L^p(\mathbb{X}, \mu)$  for any  $p \geq 1$ , where  $\mu$  is any fully-supported invariant measure: indeed, by Jensen:

$$\|Tf\|_p^p = \sum_x \mu(x) |f(x)|^p = \sum_x \mu(x) \left| \sum_y T_{(x,y)} f(y) \right|^p$$

$$\begin{aligned} \text{Jensen} \quad \curvearrowright & \leq \sum_x \mu(x) \sum_y T(x,y) |f(y)|^p \\ \mu T = \mu \quad \curvearrowright & = \sum_y \mu(y) |f(y)|^p = \|f\|_p^p \end{aligned}$$

In particular,  $T$  is an operator from  $L^2(X, \mu)$  to  $L^2(X, \mu)$ , so it has an adjoint  $T^*$ , abstractly defined by the duality relation:

$$\langle T^* f, g \rangle_{L^2(X, \mu)} = \langle f, T g \rangle_{L^2(X, \mu)} \text{ for all } f, g \in L^2(X, \mu).$$

Taking  $f = 1_{\{y\}}$ ,  $g = 1_{\{x\}}$  gives:  $T^*(x, y) = \frac{\mu(y) T(y, x)}{\mu(x)}.$

Remark:  $T^*$  is also a stochastic matrix, by invariance of  $\mu$ .  
When  $\mu$  is an invariant law,  $T^*$  has the interpretation of time-reversal.

Exercise: suppose that  $\mu$  is an invariant law.

$$\left[ \begin{array}{l} \text{Let } X := (X_n)_{n \geq 0} \sim \text{DTMC}(\mu, T) \\ \text{Let } X^* := (X_n^*)_{n \geq 0} \sim \text{DTMC}(\mu, T^*) \\ \text{Then } (X_0^*, X_{n+1}^*, \dots, X_n^*) \stackrel{d}{=} (X_n, X_{n+1}, \dots, X_0) \text{ for all } n \geq 1. \end{array} \right.$$

def: A chain  $T$  such that  $T^* = T$  is called reversible.

This is equivalent to:  $\mu(x) T(x, y) \stackrel{(*)}{=} \mu(y) T(y, x) \quad \forall x, y \in X$

Remark: the equation  $(*)$  can be viewed as a local strengthening of the invariant property  $\mu T = \mu$ , which is obtained by summing  $(*)$  over all  $x \in X$ .

⚠ The converse is not true: there are measures  $\mu$  which are invariant, but not reversible.

Example: consider SRW on an undirected graph  $G = (V, E)$ .

Then  $p(x) := \deg(x)$  is reversible, hence invariant.

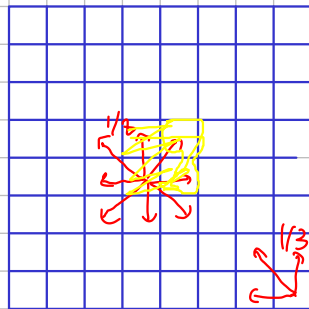
Indeed: 
$$T(x, y) = \begin{cases} 1/\deg(x) & \text{if } \{x, y\} \in E \\ 0 & \text{else} \end{cases}$$

so 
$$p(x)T(x, y) = \begin{cases} 1 & \text{if } \{x, y\} \in E \\ 0 & \text{else} \end{cases}$$

$E$  is symmetric  $\nearrow = p(y)T(y, x)$

If  $G$  is  $\begin{cases} \text{finite} \\ \text{connected} \\ \text{undirected} \end{cases}$ , then  $\pi(x) := \frac{\deg(x)}{2|E|}$  is the unique invariant law.

Exercise:



King's RW's.

- 1) starting at a corner, how long will take for the king to come back to its initial position, on average?
- 2) during this excursion, how many times will the king spend in the center of the board, on average?
- 3) same question for a queen, a knight, a rook, a bishop.

## ⑦ The ergodic theorem

For a measure  $\mu$  on  $\mathcal{X}$ , and a function  $f \in L^1(\mathcal{X}, \mu)$ , define

$$\mu f := \sum_{\mathbf{x}} \mu(\mathbf{x}) f(\mathbf{x}) = \int f d\mu$$

### Theorem (Ergodic theorem for Markov chains)

Let  $(X_n)_{n \geq 0}$  be irreducible and recurrent, with invariant measure  $\mu$ .

Then for any  $f \in L^1(\mathcal{X}, \mu)$ , we have

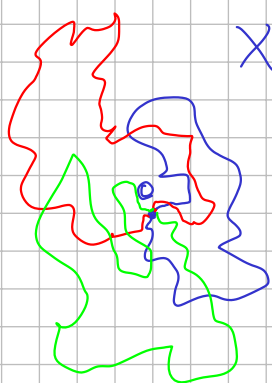
$$\left[ \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \frac{\mu f}{\mu(\mathcal{X})} = \begin{cases} 0 & \text{if the chain is null-recurrent} \\ \pi f & \text{if the chain is positive-recurrent} \end{cases} \right]$$

$\Rightarrow$  MCTC reduction, page-rank's algorithm by Google.

proof: upon replacing  $f$  by  $f_+$ ,  $f_-$ , we can assume wlog that  $f \geq 0$ .

• upon decomposing  $P$  as  $P = \sum_{\mathbf{x}} P(X_0 = \mathbf{x}) P_{\mathbf{x}}(\cdot)$ , we can assume wlog that

$X_0 = \mathbf{o}$  for some fixed  $\mathbf{o} \in \mathcal{X}$ .



For each  $k \geq 0$ , let  $\tau_k$  be the  $k$ -th visit to  $\mathbf{o}$ .

More precisely,  $\tau_0 = 0$ ,  $\tau_{k+1} = \min\{n \geq \tau_k + 1 \mid X_n = \mathbf{o}\}$

Note that each  $\tau_k$  is almost-surely finite (the chain is recurrent) and that  $\tau_k$  is a stopping time. Define:

$$Z_k := \sum_{\ell=\tau_k}^{\tau_{k+1}-1} f(X_\ell)$$

Then by the strong Markov property,  $Z_0, Z_1, \dots$  are iid and non-negative.

So by the SLLN, we know that

$$\underbrace{Z_0 + \dots + Z_{k-1}}_k \xrightarrow[k \rightarrow \infty]{\text{a.s.}} \mathbb{E}[Z_0] \in [0, \infty]$$

In other words,  $\frac{1}{n} \sum_{k=0}^{T_n-1} f(X_k) \xrightarrow{\text{a.s.}} \sum_{x \in \mathcal{X}} f(x) \cdot \underbrace{\mathbb{P}_0 \left[ \sum_{k=0}^{T_n-1} \mathbb{1}_{\{X_k=x\}} \right]}_{p_0(x)} = \text{pof}$

Now set  $N_n := \sum_{k=1}^n \mathbb{1}_{\{X_k=0\}}$  so that

$$T_{N_n} \leq n \leq T_{N_n+1}$$

Since  $f$  is non-negative, we have

$$\underbrace{\frac{1}{N_n} \sum_{k=0}^{T_{N_n}-1} f(X_k)}_{\text{pof}} \leq \frac{1}{N_n} \sum_{k=0}^{n-1} f(X_k) \leq \underbrace{\left( \sum_{k=0}^{T_{N_n+1}-1} f(X_k) \right)}_{\text{pof}} \cdot \underbrace{\frac{1}{N_n+1} \left( 1 + \frac{1}{N_n} \right)}_1$$

Note that  $N_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \infty$  because the chain is recurrent

so we know that  $\frac{1}{N_n} \sum_{k=0}^{T_{N_n}-1} f(X_k) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \text{pof}$  and same with  $N_n+1$  instead of  $N_n$ .

Therefore, 
$$\frac{1}{N_n} \sum_{k=0}^{n-1} f(X_k) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \text{pof} \quad (1)$$

But this is true for any  $f \geq 0$ .

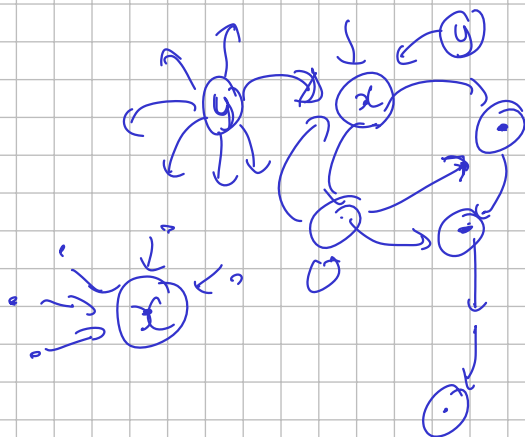
Taking  $f = \mathbb{1}$  shows that 
$$\frac{n}{N_n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} p_0(\mathcal{X}) \quad (2)$$

Dividing (1) by (2) gives exactly the desired result  $\square$ .

Corollary: If  $(X_n)_{n \geq 0}$  is irreducible and positive recurrent, then for all  $x \in \mathcal{X}$ .

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k=x\}} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \pi(x) \quad \text{where } \pi \text{ is the unique invariant law}$$

## Application: PageRank's algorithm:

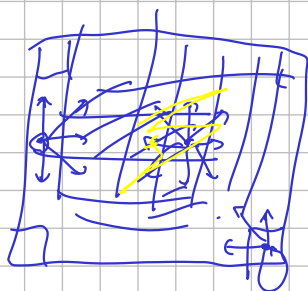


$\pi(x) :=$  "importance of  $x$ ".

$$\pi(x) = \sum_{y \in \mathcal{X}} \pi(y) \frac{1_{(y \rightarrow x)}}{\deg(y)}$$

$$\pi = \pi T$$

## Exercise:



$\deg = 14$

SRW on an undirected, connected finite graph

→ irreducible, positive-recurrent.

→  $\exists! \pi \in \mathcal{P}(\mathcal{X})$  s.t.  $\pi T = \pi$ .

we know that  $\pi(x) := \frac{\deg(x)}{\sum_x \deg(x)}$  (by reversibility)

$$\Rightarrow \mathbb{E}_x[\bar{G}_x^+] = \frac{\sum_x \deg(x)}{\deg(x)} = \frac{4 \times 3 + 24 \times 5 + 36 \times 8}{3} = [\dots] \rightarrow \text{question 1)}$$

start in a corner

$$\Rightarrow \text{On the other hand, we know that } p_0(x) := \mathbb{E}_0 \left[ \sum_{n=0}^{\bar{G}_0^+ - 1} 1_{(X_n = x)} \right] = \pi(x) \cdot p_0(\bar{G}) = \pi(x) \cdot \mathbb{E}_x[\bar{G}_x^+]$$

→ question 2)

## Exercise: positive/null recurrent for SRW on $\mathbb{Z}^d$ for $d=1,2$ .

the chain is irreducible, recurrent  $\leadsto$  unique inv. measure up to prop.

but  $\mu(x) := 1$  satisfies  $\boxed{\mu T = \mu}$

$$\text{because } \mu T(x) = \sum_y \mu(y) T(y, x) \stackrel{\mu(y)=1}{=} \sum_y T(y, x)$$

$$\stackrel{\text{SRW}}{=} \sum_y \frac{1}{\deg(y)} 1_{\{y, x \in E\}}$$

$\mu(\mathcal{X}) = \infty \Rightarrow$  null-recurrent case.

