

8) Convergence to equilibrium

Recall that, by the ergodic theorem, if $X = (X_n)_{n \geq 0}$ is an i.r.d. positive recurrent chain, then:

$$\forall x \in \mathcal{X}, \quad \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k=x\}} \xrightarrow[n \rightarrow \infty]{a.s.} \pi(x)$$

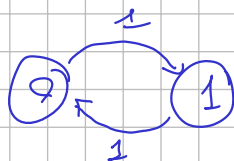
where π is the unique invariant law.

Taking expectations, we obtain: $\frac{1}{n} \sum_{k=0}^{n-1} P(X_k=x) \xrightarrow[n \rightarrow \infty]{} \pi(x) \quad \forall x \in \mathcal{X}$

In other words, $\frac{1}{n} \sum_{k=0}^{n-1} \text{Law}(X_k) \xrightarrow[n \rightarrow \infty]{} \pi$ in $\mathcal{P}(\mathcal{X})$.

Question: is it true that $X_n \xrightarrow[n \rightarrow \infty]{d} \pi(\cdot)$???

Counter-example:



→ irreducible ✓

→ finite, so positive-recurrent ✓

But (*) fails for certain initial conditions:

Indeed: $P_0(X_n=1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$ is not converging as $n \rightarrow \infty$.

def: An irreducible transition matrix T is aperiodic if $\forall (x,y) \in \mathcal{X}^2, \exists m_0 \in \mathbb{N}$ s.t. that $T^n(x,y) > 0$ for all $n \geq m_0$.

⚠ this is strictly stronger than irreducibility, as the above example shows.

Remark: (simplification). By irreducibility, given any origin $o \in \mathcal{X}$, and any $x,y \in \mathcal{X}^2$

there is $\begin{cases} k \geq 0 \\ \ell \geq 0 \end{cases}$ s.t. $\begin{cases} T^k(x,o) > 0 \\ T^\ell(o,y) > 0 \end{cases}$, and moreover

$$T^{k+\ell+m}(x,y) \geq \underbrace{T^k(x,o)}_{>0} T^m(o,o) \underbrace{T^\ell(o,y)}_{>0} \quad \text{so it is enough to}$$

check the periodicity condition at the pair $(0,0)$.

Moreover, $I := \{n \geq 1 : T^n(0,0) > 0\}$ is closed under addition.

Exercise: (discrete semi-groups): prove that any non-empty set $I \subseteq \{1, 2, \dots\}$ which is closed under addition satisfies:

$$d\mathbb{N} \cap [n, \infty[\subseteq I \subseteq d\mathbb{N} \quad (\leadsto I = d\mathbb{N} \text{ up to finitely many exceptions})$$

for some $n \in \mathbb{N}$ and $d \geq 1$. Of course, d is then the gcd of I .

Consequence: an irred. chain is aperiodic if and only if, for some $x \in \mathcal{X}$, the $\gcd(n \geq 1 : T^n(x,x) > 0) = 1$.

Theorem: any irreducible, aperiodic, positive-recurrent chain converges to the unique invariant law, i.e. $X_n \xrightarrow[n \rightarrow \infty]{d} \pi$.

Proof: let $\underline{X} = (X_n)_{n \geq 0}$ be a DTMC (μ, T) .

let $\underline{Y} = (Y_n)_{n \geq 0}$ be a DTMC (π, T) , independent of $\underline{X}$.

Then $((X_n, Y_n))_{n \geq 0}$ is a DTMC on $\mathcal{X}^2$, with transition matrix:

$$K^n(\underbrace{(x,y)}_{>0}, \underbrace{(x',y')}_{>0}) = \underbrace{T^n(x,x')}_{>0} \cdot \underbrace{T^n(y,y')}_{>0} \quad (\text{by independence})$$

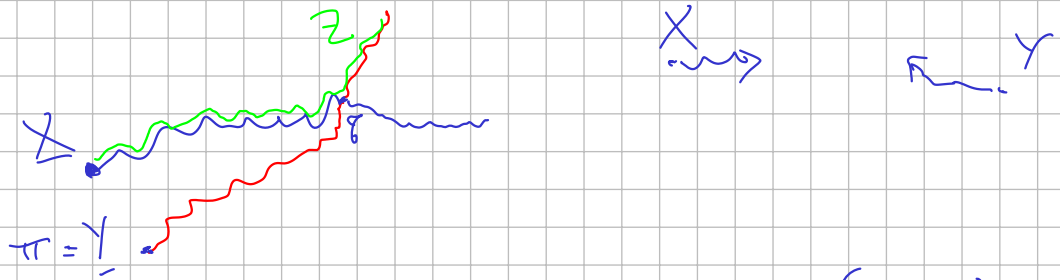
Because T is irred + aperiodic, the right-hand side is > 0 for n large enough so K is irreducible (in fact also aperiodic).

and K is ^{positive} recurrent, because $\hat{\pi}(x,y) := \pi(x)\pi(y)$ is an invariant law for K .

In particular, $((X_n, Y_n))_{n \geq 0}$ will almost-surely visit every state infinitely often.

In particular, it will almost surely visit the set:

$$\Delta := \{(x, x) : x \in \mathcal{X}\}$$



Let σ be the hitting time of Δ by the process $(X_n, Y_n)_{n \geq 1}$.
(it's a stopping time w.r.t. filtration $\mathcal{F}_n = \sigma(X_0, Y_0, \dots, X_n, Y_n)$)

Define a new process $Z = (Z_0, Z_1, \dots)$ as follows:

$$\begin{aligned} Z &:= (X_0, X_1, \dots, X_\sigma, Y_{\sigma+1}, Y_{\sigma+2}, \dots) \\ X &:= (\underbrace{X_0, X_1, \dots, X_\sigma}_{\text{past}}, \underbrace{X_{\sigma+1}, X_{\sigma+2}, \dots}_{\text{future}}) \end{aligned}$$

Claim: $Z \stackrel{d}{=} X$ because the past up to time σ is the same and the law of the future, given $\mathcal{F}_\sigma$ is the same by the Strong Markov Property at time σ .

Finally, take any $A \subseteq \mathcal{X}$ and write $\begin{cases} X_n \stackrel{d}{=} Z_n \\ Y_n \stackrel{d}{=} \pi \end{cases}$

$$\begin{aligned} \mathbb{P}(X_n \in A) - \pi(A) &= \mathbb{P}(Z_n \in A) - \mathbb{P}(Y_n \in A) \\ &\leq \mathbb{P}(Z_n \in A) - \mathbb{P}(Y_n \in A, Z_n \in A) \\ &= \mathbb{P}(Z_n \in A, Y_n \notin A) \\ &\leq \mathbb{P}(Z_n \neq Y_n) = \mathbb{P}(\sigma > n) \end{aligned}$$

Replacing A by A^c shows that

$$\sup_{A \in \mathcal{F}} |P(X_n \in A) - \pi(A)| \leq P(\mathcal{G} > n) \xrightarrow{n \rightarrow \infty} P(\mathcal{G} = \infty) \stackrel{K \text{ is recurrent}}{=} 0$$

Take $n \rightarrow \infty$ $d_n(\text{Law}(X_n, \pi)) \leq \frac{\mathbb{E}[\mathcal{G}]}{n}$

Remarks: In practice, the convergence to equilibrium

$$\text{Law}(X_n) \xrightarrow{n \rightarrow \infty} \pi \quad (*)$$

is often very important (think of MCMC or card shuffles)
but needs to be quantified:

How fast is the convergence $(*)$?

→ theory of mixing-times

Here we used a very naive "product" coupling.

Polya's theorem: SRW on $\mathbb{Z}^d$ is $\begin{cases} \text{recurrent} & \text{if } d = 1 \text{ or } 2 \\ \text{transient} & \text{if } d \geq 3 \end{cases}$ $\square$

$$\text{recurrent} \Leftrightarrow \sum_{n=0}^{\infty} T^n(0,0) = \infty$$

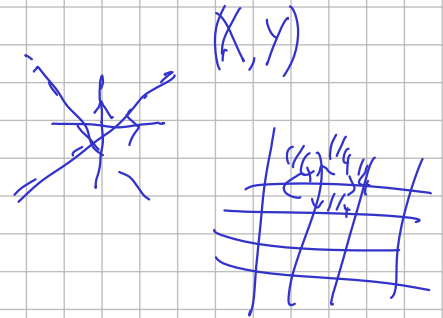


$$T^n(0,0) = 0 \text{ if } n \text{ is odd}$$

$$T^{2n}(0,0) = \binom{2n}{n} \cdot \left(\frac{1}{2}\right)^{2n} \underset{n \rightarrow \infty}{\sim} \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{4\pi n}}{\left(\left(\frac{n}{e}\right)^n \sqrt{\pi n}\right)^2} \times \frac{1}{\cancel{(n)}} \sim \frac{1}{\sqrt{\pi n}}$$

$\rightarrow \rightarrow \leftarrow \rightarrow \rightarrow \leftarrow \leftarrow \leftarrow$

$\Rightarrow$ not summable $\Rightarrow$ recurrent.



$d=2$: $T^n(0,0) = 0$ if n is odd

$$T^{2n}(0,0) = \sum_{k=0}^n \frac{(2n)!}{k! k! (n-k)! (n-k)!} \cdot \left(\frac{1}{4}\right)^{2n}$$



$$= \frac{1}{16^n} \sum_{k=0}^n \binom{n}{k} \binom{n}{n-k} = \left[\frac{1}{4^n} \binom{2n}{n} \right]^2$$

= degree n coefficient of $\left[(1+x)^n\right]^2$ $\underset{n \rightarrow \infty}{\sim} \frac{1}{\pi n}$ not summable
 = $\binom{2n}{n}$ true in continuous-time $\rightarrow$ recurrent.

$d \geq 3$: Natural guess: $T^{2n}(0,0) \leq C \cdot \left[\frac{1}{4^n} \binom{2n}{n} \right]^d \underset{n \rightarrow \infty}{\sim} \left(\frac{1}{\sqrt{\pi n}} \right)^d$
 summable if $d > 2$ $\rightarrow$ transient.

$T^n(0,0) = 0$ if n odd.

$$T^{2n}(0,0) = \sum_{k_1+k_2+\dots+k_d=2n} \frac{(2n)!}{k_1! k_2! k_3! k_4! \dots k_d! k_d!} \left(\frac{1}{2d}\right)^{2n}$$



$$= \left(\frac{1}{2d}\right)^{2n} \binom{2n}{n} \sum_{k_1+\dots+k_d=n} \left(\frac{n!}{k_1! \dots k_d!}\right)^2$$

Remark: Newton multinomial formula:

$$(x_1 + \dots + x_d)^n = \sum_{k_1+\dots+k_d=n} \frac{n!}{k_1! \dots k_d!} x_1^{k_1} \dots x_d^{k_d}$$

so taking $x_1 = x_2 = \dots = x_d = 1$ gives $\sum_{k_1+\dots+k_d=n} \frac{n!}{k_1! \dots k_d!} = d^n$.

$$\text{so } T^{2n}(0,0) \leq \left(\frac{1}{2d}\right)^{2n} \binom{2n}{n} \cdot d^n \cdot \underbrace{\max_{k_1+\dots+k_d=n} \frac{n!}{k_1! \dots k_d!}}_{d/m \stackrel{?}{\sim} \frac{n!}{\frac{n}{d}! \dots \frac{n}{d}!}} \sim \frac{\left(\frac{n}{d}\right)^m \sqrt{2\pi n}}{\left(\frac{n}{d}\right)^{\frac{n}{d}} \sqrt{\pi \frac{n}{d}}}^d$$

$$\leq C_d \cdot \frac{1}{\sqrt{2\pi n}} \cdot \frac{\sqrt{2\pi n}}{(\sqrt{n})^d} \asymp \frac{C_d}{(\sqrt{n})^d} \underset{n \rightarrow \infty}{\sim} d^n \cdot \frac{C_d \sqrt{n}}{(\sqrt{n})^d}$$