$X = (X_t)_{t \geq 0}$ CTMC (μ, T, q)

$P_t(x,y) := \mathbb{P}_x(X_t = y)$ transition semi-group : $P_0 = Id$, $P_{t+s} = P_t \cdot P_s$

$\overset{?}{\Rightarrow} P_t = e^{tQ}$, where $Q := \lim_{t \to 0^+} \frac{P_t - P_0}{t}$

Guess: $Q(x,y) := q(x)\big(T(x,y) - Id(x,y)\big)$

Big assumption: bounded rates $q_{max} := \sup_{x \in \mathcal{X}} q(x) < +\infty$.

Fact (last time): $Q = \lim_{t \to 0+} \frac{P_t - Id}{t}$ in the strong sense:

i.e w.r.t $\|A\| := \sup_x \sum_y |A(x,y)|$

In other words, the semi-group $(P_t)_{t \geq 0}$ is (strongly) right-differentiable at $t = 0+$.

In fact, much more is true:

Theorem (Kolmogorov equations): $(P_t)_{t \geq 0}$ is (strongly) differentiable everywhere, and

its derivative is $\dot{P}_t := \frac{d}{dt} P_t = \underbrace{QP_t}_{\uparrow} = \underbrace{P_t Q}_{\uparrow}$

backward Kolmogorov eq. ; forward Kolmogorov eq.

Proof: we already know this at $t = 0+$, i.e

$$\|P_h - P_0 - hQ\| = o(h) \qquad h \geq 0.$$

Now, for any $t, h \geq 0$, $P_{t+h} - P_t - hQP_t = (P_h - P_0 - hQ)P_t$

$\Rightarrow \|P_{t+h} - P_t - hQP_t\| \overset{(*)}{\leq} \|P_h - P_0 - hQ\| \cdot \|P_t\|$

$= o(h)$ (ii) ; $= 1$ (i)

so $t \mapsto P_t$ is right-diff. at time t with derivative $\dot{P}_t^+ = QP_t = P_t Q$.

$0 \le h \le t \qquad \| P_{t-h} - P_t + hQP_t \| \overset{??}{=} o(h)$

take $(*)$ with t replaced $t-h$: $\| P_t - P_{t-h} - hQP_{t-h} \| = o(h)$

This is enough, because $\| hQP_{t-h} - hQP_t \| = o(h)$

indeed, $\| QP_{t-h} - QP_t \| = o(1)$

$\| QP_{t-h}(Id - P_h) \| \le \underbrace{\|Q\|}_{\le 2\|q\|_{max}} \cdot \underbrace{\|P_{t-h}\|}_{1} \cdot \underbrace{\|Id - P_h\|}_{=o(1)}$ □

Practical use of Kolmogorov eq:

1) recall that $\mu_t := \text{law}(X_t)$ satisfies $\mu_t = \mu_0 P_t$

(indeed $\mu_t(y) = \mathbb{P}(X_t = y) = \sum_x \mathbb{P}(X_0 = x) P_t(x,y) = (\mu_0 P_t)(y)$]

So the forward Kolmogorov equation gives:

$$\dot{\mu}_t = \mu_0 \cdot \dot{P}_t = \underbrace{\mu_0 P_t}_{\mu_t} Q = \mu_t Q$$

2) Also, given $f \in L^\infty(\mathbb{X})$, the function $f_t(x) := \mathbb{E}_x[f(X_t)] = \sum_y f(y) P_t(x,y)$

$= (P_t f)(x)$

So $f_t = P_t \cdot f_0$. Therefore, the backward Kolmogorov eq gives:

$$\dot{f}_t = QP_t f_0 = Q \cdot f_t$$

Exercise: Let $X=(X_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda \in (0,\infty)$.

Write down the forward Kolmogorov equation and use it to find $\mu_t = \text{law}(X_t)$.

$\rightarrow$ X is $CTMC(\mu_0, T, q)$ where
- $\mathcal{I} = \mathbb{N} = \{0,1,2,\dots\}$
- $q(n) = \lambda \quad \forall n \in \mathbb{N}$
- $T(n,n+1) = 1$ and $T(n,k) = 0$ if $k \neq n+1$.
- $\mu_0 = \delta_0$

$\rightarrow$ $Q(x,y) = q(x)\big(T(x,y) - Id(x,y)\big) = \begin{cases} \lambda & \text{if } y = x+1 \\ -\lambda & \text{if } y = x \\ 0 & \text{else.} \end{cases}$

$\rightarrow$ $\dot{\mu}_t = \mu_t Q$ so $\dot{\mu}_t(n) = \sum_k \mu_t(k) Q(k,n)$

$= \lambda(\mu_t(n-1) - \mu_t(n))$ with $(\mu(-1) := 0)$

Define $\nu_t : e^{\lambda t}\mu_t$, then $\dot{\nu}_t = e^{\lambda t}(\lambda \mu_t + \dot{\mu}_t)$

so $\dot{\nu}_t(n) = e^{\lambda t}\left(\lambda \mu_t(n) + \lambda \mu_t(n-1) - \lambda \mu_t(n)\right)$

ie $\dot{\nu}_t(n) = \lambda \cdot \nu_t(n-1)$

ie $\nu_t(n) = \underbrace{\nu_0(n)}_{\mathbb{1}_0(n)} + \lambda \int_0^t \nu_s(n-1)\,ds$

$\rightarrow$ $\nu_t(0) = 1$

$\rightarrow$ $\nu_t(1) = \lambda t$

$$\rightarrow \quad \nu_t(2) = \lambda^2 \frac{t^2}{2}$$

$$\vdots$$

$$\rightarrow \quad \nu_t(n) = \lambda^n \frac{t^n}{n!}$$

$$\rightarrow \quad \mu_t(n) = e^{-\lambda t} \nu_t(n) = e^{-\lambda t} \frac{\lambda^n t^n}{n!}$$

$$\Rightarrow X_t \overset{d}{=} \text{Poisson}(\lambda t). \qquad \square$$

Question: can we make sense of "$P_t = e^{tQ}$"?

If $A \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}}$ satisfies $\|A\| < \infty$, then the limit

$$\lim_{N \to \infty} \underbrace{\sum_{n=0}^{N} \frac{A^n}{n!}}_{S_N}$$

exists in the strong sense, and we call it e^A. In other words,

$$\boxed{e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}}$$

Indeed, $\|S_{N+M} - S_N\| = \left\| \sum_{n=N+1}^{N+M} \frac{A^n}{n!} \right\| \overset{\text{triangle ineq. for } \|\cdot\|}{\leq} \sum_{n=N+1}^{N+M} \frac{\|A^n\|}{n!}$

$$\overset{\text{sub-mult. of } \|\cdot\|}{\leq} \sum_{n=N+1}^{N+M} \frac{\|A\|^n}{n!}$$

$$\Rightarrow \sup_{M \geq 0} \|S_{N+M} - S_N\| \leq \sum_{n=N+1}^{\infty} \frac{\|A\|^n}{n!} \xrightarrow[N \to \infty]{} 0$$

So $(S_N)_{N\geq 0}$ is Cauchy wrt $\|\cdot\|$, hence convergent.

Lemma: $e^{A+B} = e^A \cdot e^B$ for any A, B with $\|A\|, \|B\| < \infty$, provided $AB = BA$.

proof: $e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!}$

Newton's binomial formula, $AB = BA$ $\Rightarrow = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$ $\quad k \leq n$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{A^k}{k!} \frac{B^{n-k}}{(n-k)!}$$

$$= \sum_{k=0}^{\infty} \frac{A^k}{k!} \sum_{n=0}^{\infty} \frac{B^n}{n!}$$

$$= e^A \cdot e^B \qquad \square$$

$\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \left\| \frac{A^k}{k!} \frac{B^{n-k}}{(n-k)!} \right\| < \infty$

$\leq e^{\|A\|} \cdot e^{\|B\|} < \infty$

In particular, e^A is inevible, with inverse e^{-A}.

Lemma: if $\|A\| < \infty$, then $H_t := e^{tA}$ $(t \in \mathbb{R})$ solves the equations $\begin{cases} \dot{H}_t = A H_t = H_t A, & \text{(in the strong sense)} \\ H_0 = Id. \end{cases}$

Proof: $\|H_h - H_0 - hA\| = \|H_h - Id - hA\|$

$$= \left\| \sum_{n=2}^{\infty} \frac{h^n A^n}{n!} \right\|$$

$$\leq \sum_{n=2}^{\infty} \frac{|h|^n \|A\|^n}{n!} = o(h)$$

Moreover, we have $H_{t+h} = e^{(t+h)A} = e^{tA} \cdot e^{hA} = H_t \cdot H_h \quad \forall t,h \in \mathbb{R}$

so $\| H_{t+h} - H_t - hAH_t \| = \| (H_h - H_0 - hA) H_t \|$

$\leq \underbrace{\| H_h - H_0 - hA \|}_{=o(h)} \cdot \| H_t \| = o(h)$ □

Theorem: we have $P_t = e^{tQ}$ for all $t \geq 0$

proof: $M_t := P_t e^{-tQ}$ goal: $M_t = Id$ for all $t \geq 0$.

we know that $(P_t)_{t\geq 0}$ and $(e^{tQ})_{t\geq 0}$ are strongly differentiable,

so their product also is, and we have:

$$\dot{M}_t = \dot{P}_t e^{-tQ} + P_t \left(e^{-tQ}\right)^{\cdot}$$

$$= P_t Q e^{-tQ} - P_t Q e^{-tQ}$$

$$= 0$$

so $M_t = M_0 = Id$ □

Corollary: $(P_t)_{t\geq 0}$ is determined by Q

so $(\text{law}((X_t)_{t\geq 0})$ is determined by $\begin{cases} \text{law}(X_0) =: \mu \\ Q \end{cases}$

$\Rightarrow X \overset{d}{=} CTMC(\mu, Q)$

Example: the Poisson process with intensity $\lambda \in (0,\infty)$.

$$Q(x,y) = q(x)\left(T(x,y) - Id(x,y)\right) = \lambda\left(1_{y=x+1} - 1_{y=x}\right)$$

$$\Rightarrow Q = \lambda(T - Id)$$

$$\sum_{n=0}^{\infty} \frac{(-t)^n}{n!} Id^n = \left(e^{-t}\right) Id.$$

$$\Rightarrow P_t = e^{tQ} = e^{\lambda t(T-Id)} = e^{\lambda tT} \cdot e^{-\lambda t Id} = e^{-\lambda t} e^{\lambda tT}$$

$$\Rightarrow P_t(x,y) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n T^n(x,y)}{n!}$$

But $T(x,y) = 1_{y=x+1}$

so $T^n(x,y) = 1_{y=x+n}$

$$\Rightarrow P_t(x,y) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n 1_{y=n+x}}{n!} = \begin{cases} 0 & \text{if } y < x \\ e^{-t} \dfrac{t^{y-x}}{(y-x)!} \end{cases}$$

$$\boxed{P_t(x,y) = \mu_t(y-x)} \quad \text{where } \mu_t = \text{Poisson}(\lambda t) \qquad \square$$



except that all of this uses $q_{max} < \infty$.

(5') Unbounded rates

Hope: $P_t = e^{tQ}$ where $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ is defined entry-wise ??

Problem: A^2 may even not be well defined: $A^2(x,y) = \sum_z A(x,z)A(z,y)$

requires $\sum_z |A(x,z)|\,|A(z,y)| < \infty$.

in particular, Q^2 would require $\sum_{z \neq x,y} \underbrace{|Q(x,z)|}_{q(x)T(x,z)} \underbrace{|Q(z,y)|}_{q(z)T(z,y)} < \infty$

Theorem (Backward Kolmogorov equation): The semi-group $(P_t)_{t\geq 0}$ is entry-wise differentiable, and $\dot{P}_t \overset{(*)}{=} QP_t$

Moreover, it is the only entry-wise differentiable family of stochastic matrices satisfying $(*)$ and $P_0 = Id$.

Lemma (Strong Markov Property at the first jump time): Let $\underline{X} = \psi(\underline{Y}, \underline{E})$ be a CTMC (μ, T, q). Then, the shifted process $\widetilde{\underline{X}} := (X_{t+E_0})_{t\geq 0}$ satisfies: $\text{law}(\widetilde{\underline{X}} \mid E_0, Y_0, Y_1) = \text{CTMC}(\delta_{Y_1}, T, q)$.

Proof: we have $\widetilde{X} = \psi(\widetilde{Y}, \widetilde{E})$ where $\begin{cases} \widetilde{Y} = (Y_1, Y_2, \dots) \\ \widetilde{E} = (E_1, E_2, \dots) \end{cases}$

Moreover,
- $\text{Law}(\widetilde{Y} \mid E_0, Y_0, Y_1) = \text{DTMC}(\delta_{Y_1}, T)$ ✓ Markov Property at time 1 for $(Y_n)_{n\geq 0}$ wrt to the filtration $\sigma(Y_0, \dots, Y_n, E_0, \dots, E_n)$
- $\text{Law}(\widetilde{E} \mid E_0, Y) = \overset{(\times)}{=} \bigotimes_{n\geq 1} \text{Exp}(q(Y_n))$
- $\sum_{n=0}^{\infty} \widetilde{E}_n = \infty$ ✓

$(\times)$: trivial because $\text{Law}(E \mid Y) = \bigotimes_{n\geq 1} \text{Exp}\, q(Y_n)$

Proof of the theorem: Fix $t \geq 0$, $x, y \in \mathcal{X}$

$$P_t(x,y) = \mathbb{P}_x(X_t = y) = \mathbb{P}_x(X_t = y, E_0 > t) + \mathbb{P}_x(X_t = y, E_0 \leq t)$$

use the previous lemma to compute the second term by conditioning on (Y_1, E_0) ($Y_1 = z$, $E_0 = u$)

$$= e^{-q(x)t} \cdot \mathrm{Id}(x,y) + \sum_{z \in \mathcal{X}} \int_0^t \underbrace{q(x) e^{-q(x)u}}_{\text{“}\mathbb{P}(E_0 = u)\text{”}} \cdot \underbrace{T(x,z)}_{\text{“}\mathbb{P}(Y_1 = z)\text{”}} P_{t-u}(z,y)\, du$$

$$\Rightarrow P_t(x,y) = e^{-q(x)t} \mathrm{Id}(x,y) + \int_0^t q(x) e^{-q(x)u} (T P_{t-u})(x,y)\, du$$

$u \mapsto t-u$

$$= e^{-q(x)t} \left(\mathrm{Id}(x,y) + \int_0^t q(x) e^{+q(x)u} T P_u(x,y)\, du \right)$$

$$\Rightarrow e^{q(x)t} P_t(x,y) = \mathrm{Id}(x,y) + \int_0^t q(x) e^{+q(x)u} T P_u(x,y)\, du$$

So $t \mapsto P_t(x,y)$ is differentiable, and its derivative satisfies:

$$e^{q(x)t} \left[q(x) P_t(x,y) + \dot{P}_t(x,y) \right] = q(x) e^{q(x)t} T P_t(x,y)$$

$$\Rightarrow \dot{P}_t = Q P_t \qquad \left(\text{because } Q(x,y) = q(x)(T(x,y) - \mathrm{Id}(x,y)) \right)$$

So $(P_t)_{t \geq 0}$ solves the Backward Kolmogorov equation !

Proof of uniqueness: Set $S_n := E_0 + E_1 + \cdots + E_{n-1}$ $(S_0 = 0)$

define $P_t^{(n)}(x,y) := \mathbb{P}_x(X_t = y, S_n > t)$

note that $P_t^{(0)}(x,y) \equiv 0$, and that by the same computation as before

$$P_t^{(n+1)}(x,y) = \overbrace{e^{-q(x)t}\,\mathrm{Id}(x,y)}^{\text{contribution from } \{E_0 > t\}} + \overbrace{\sum_z \int_0^t q(x)\, e^{-q(x)u}\, T(x,z)\, P_{t-u}^{(n)}(z,y)\, du}^{\text{contribution from } \{E_0 \le t\}}$$

Now assume that $(\hat{P}_t)_{t\ge 0}$ also solves the FKE. $\hat{P}_t(x,y) = Q\hat{P}_t(x,y)$

this means that $e^{q(x)t}\hat{P}_t(x,y) = \mathrm{Id}(x,y) + \int_0^t q(x)\, e^{+q(x)u}\, T\hat{P}_u(x,y)\, du$

and we have seen that $e^{q(x)t}P_t^{(n+1)}(x,y) = \mathrm{Id}(x,y) + \int_0^t q(x) e^{q(x)u}\, T P_u^{(n)}(x,y)\, du$

Therefore: $\hat{P}_t(x,y) \ge P_t^{(n)}(x,y)$

(trivial $n=0$, + induction over n)

taking $n\to\infty$ shows that $\hat{P}_t(x,y) \ge P_t(x,y)$

but both matrices are stochastic, so we must have equality! □

Remark: in the explosive case, (X_t) is only defined for $t < \sum_{n=0}^{\infty} E_n$

and $P_t^{(n)}(x,y) := P_x(X_t = y,\ S_n \ge t) \xrightarrow[n\to\infty]{} \underbrace{P(X_t = y,\ \sum_{n=0}^{\infty} E_n > t)}_{\text{def } P_t(x,y)}$

So the above argument shows that (P_t) still solves FKE, and that it is the smallest solution to it.

Corollary: $(P_t)_{t\geq 0}$ is determined by Q, allowing us to write

$$X \overset{d}{=} CTMC(\mu, Q)$$

Theorem (Forward Kolmogorov equation): $(P_t)_{t\geq 0}$ is entrywise differentiable, and $\begin{cases} \dot{P}_t = P_t Q \\ P_0 = Id \end{cases}$

$\dot{P}_t = QP_t = P_t Q$

$\dot{P}_t = \underline{TP_t} = \underline{P_t T}$

Moreover, $(P_t)_{t\geq 0}$ is the only family of stochastic matrices with those properties.

Lemma (time-reversal identity): $S_n := E_0 + \cdots + E_{n-1}$ $(S_0 = 0)$

$$q(y_n)\,\mathbb{P}\left(\underbrace{S_n \leq t < S_{n+1}}_{t\in[S_n, S_{n+1}[} \mid Y_0 = y_0, Y_1 = y_1, \dots, Y_n = y_n\right)$$

$$= q(y_0)\,\mathbb{P}\left(S_n \leq t < S_{n+1} \mid Y_0 = y_n, Y_1 = y_{n-1}, \dots, Y_n = y_0\right).$$

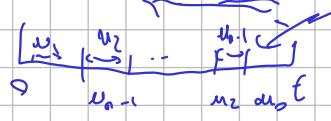
Proof: Given Y, $E = (E_0, E_1, \dots)$ are indep Exp with rates $(q(Y_n))_{n\geq 0}$

So $q(Y_n)\,\mathbb{P}(S_n \leq t < S_{n+1} \mid Y_0, Y_1, \dots, Y_n)$

$$= \int_A \left(\prod_{i=0}^{n-1} e^{-q(Y_i)u_i}\right) e^{-q(Y_n)\left(t - \sum_{i=0}^{n-1} u_i\right)} \cdot q(Y_0)\cdots q(Y_n)\, du_0 du_1 \cdots du_{n-1}$$

$$A = \left\{(u_0, \dots, u_{n-1}) \geq 0 : u_0 + \cdots + u_{n-1} \leq t\right\}.$$

This function is symmetric under reversing $(Y_0, Y_1, \dots, Y_n)$ because of the change of variable $(u_0, \dots, u_{n-1}) \mapsto t \longrightarrow \left(t - \sum_{i=0}^{n-1} u_i, u_{n-1}, u_{n-2}, \dots, u_1\right)$



Proof of the theorem

$$X = \varphi(Y, E)$$

$$P_t(x,y) = \mathbb{P}_x(X_t = y) = \sum_{n=0}^{\infty} \mathbb{P}_x\left(t \in [S_n, S_{n+1}[,\ Y_n = y\right)$$

$$= \underbrace{e^{-q(x)t}\,Id(x,y)}_{\text{contribution from } E_0 > t \text{ (term } n=0)} + \sum_z \sum_{n=1}^{\infty} \mathbb{P}_x\left(t \in [S_n, S_{n+1}[,\ Y_{n-1} = z, Y_n = y\right)$$

$$= T(z,y)$$

Thanks to the previous lemma, we have

$$\mathbb{P}_x\left(t \in [S_n, S_{n+1}[\,\middle|\, Y_{n-1} = z, Y_n = y\right) = \frac{q(x)}{q(y)} \mathbb{P}_y\left(t \in [S_n, S_{n+1}[\,\middle|\, Y_1 = z, Y_n = x\right)$$

$$= \frac{q(x)}{q(y)} \int_0^t q(y) e^{-q(y)u}\, \mathbb{P}_z\left(t \in [u + S_{n-1}, u + S_n[\,\middle|\, Y_{n-1} = x\right) du$$

$$S_{n-1} \le t-u \le S_n$$

Strong Markov Property at the first jump "$E_0 = u$"

$$= \int_0^t e^{-uq(y)}\, q(z)\, \mathbb{P}_x\left(S_{n-1} \le t-u \le S_n \,\middle|\, Y_{n-1} = z\right) du$$

use again the time reversal lemma at time $t-u$ instead of t and with $n-1$ instead of n

Insert this back into the previous computation:

$$P_t(x,y) = e^{-q(x)t} Id(x,y) + \sum_{z} \sum_{n=1}^{\infty} q(z) T(z,y) \int_0^t e^{-uq(y)}\, \mathbb{P}_x\left(S_{n-1} \le t-u \le S_n,\ \overbrace{Y_{n-1}}^{\hat{\imath}} = z\right) du \qquad X_{t-u} = z$$

$$= e^{-q(x)t} Id(x,y) + \sum_z q(x) T(z,y) \int_0^t e^{-uq(y)} P_{t-u}(x,z)\, du \qquad n-1 \mapsto n$$

$$= e^{-q(y)t} \left[Id(x,y) + \sum_z q(z) T(z,y) \int_0^t e^{uq(y)} P_u(x,z)\, du \right] \qquad u \mapsto t-u$$

$\rightarrow (P_t)_{t\geq 0}$ is differentiable entrywise, and

$$e^{q(x)t}\left[\dot{P}_t(x,y) + q(x)P_t(x,y)\right] = \sum_z q(z)T(z,y)\,e^{tq(y)}P_t(x,z)$$

$$\text{ie}\quad \dot{P}_t(x,y) = \left(\sum_z q(z)T(z,y)P_t(x,z)\right) - q(x)P_t(x,z)$$

$$= P_t \cdot Q\,(x,y)$$

$$Q(x,y) = q(x)\big(T(x,y) - \mathrm{Id}(x,y)\big)$$

Proof of uniqueness as before, introduce $P_t^{(n)}(x,y) = P_x(X_t = y, S_n > t)$

and the same computation gives $p^{(n)} = F(p^{(n-1)})$

while FKE is $\hat{P} = F(\hat{P})$

$\Rightarrow$ by induction $\hat{P}_t \geq P_t^{(n)}$

send $n \to \infty$ $\Rightarrow$ $\hat{P}_t \geq P_t$

+ stochasticity $\Rightarrow$ $\hat{P}_t = P_t$ □

We have the same conclusions as in the bounded-rate case

except that 1) entrywise differentiability instead of strong.

2) $\boxed{P_t = e^{tQ}}$ does not make sense!

except if we define $\boxed{H_t = e^{tQ}}$ as the unique solution

to $H_0 = \mathrm{Id}$, $\dot{H}_t = QH_t$ or H_tQ.