(6) Markov chains with filtrations

Let Q be a rate-matrix (i.e $Q(x,y) = q(x)\left(\Pi(x,y) - \mathrm{Id}_{(y,y)}\right)$ for some (q,Π))

i.e: Q has non-negative off-diagonal entries and negative diagonal entries and zero-sum rows

We have seen yesterday that the Kolmogorov equation defines a unique minimal family $(P_t)_{t\geq 0}$ of sub-stochastic matrices.

We call $(P_t)_{t\geq 0}$ the semi-group associated with Q, and we call Q non-explosive if $(P_t)_{t\geq 0}$ are stochastic.

From now on, $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ i.e each $\mathcal{F}_t$ is a σ-field included in $\mathcal{F}$, and $s\leq t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t$.

def: A stochastic process $X = (X_t)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-Markov chain with generator Q if it is càlàg, adapted and for all $s,t\geq 0$, for all $f \in L^\infty(\mathcal{X})$:

$$\mathbb{E}\left[f(X_{t+s}) \,|\, \mathcal{F}_s\right] = P_t f(X_s)$$

Remark: 1) Any CTMC (ν, Q) in the previous sense is a $(\mathcal{F}_t)_{t\geq 0}$-Markov chain in this new sense, with $\mathcal{F}_t = \sigma(X_s : s\in[0,t])$

Indeed, by the weak Markov property, $\mathbb{E}\left[f(X_{t+s}) \,|\, (X_u)_{u\in[0,s]}\right] = P_t f(X_s)$.

2) Conversely, any $(\mathcal{F}_t)_{t\geq 0}$-Markov chain X according to the above def. is a Markov chain in the previous sense. Indeed, for $\begin{cases} t_0 = 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \\ x_0, \dots, x_n \\ n\in\mathbb{N} \end{cases}$

$$\mathbb{P}(X_{t_0}=x_0, X_{t_1}=x_1, \dots, X_{t_n}=x_n)$$

$$\overset{\text{first condition on } \mathcal{F}_{t_{n-1}}}{=} \mathbb{E}\left[\mathbb{1}_{(X_{t_0}=x_0,\dots,X_{t_{n-1}}=x_{n-1})}\, \mathbb{E}\left[\mathbb{1}_{(X_{t_n}=x_n)} \,|\, \mathcal{F}_{t_{n-1}}\right]\right]$$

$$\mathbb{E}\left[\mathbb{1}_{(X_{t_n}=x_n)} \,|\, \mathcal{F}_{t_{n-1}}\right] \hookrightarrow P_{t_n - t_{n-1}} f(X_{n-1}) \quad \text{where } f = \mathbb{1}_{\{x_n\}}$$

$$\hookrightarrow = P_{t_n - t_{n-1}}(x_{n-1}, x_n)$$

$$= \mathbb{P}(X_{t_0}=x_0, \dots, X_{t_{n-1}}=x_{n-1})\, P_{t_n - t_{n-1}}(x_{n-1}, x_n)$$

$$= \dots =$$

$= \mu_0(x_0) \cdot P_{t_1-t_0}(x_0,x_1)\, P_{t_2-t_1}(x_1,x_2) \cdots P_{t_n-t_{n-1}}(x_{n-1},x_n)$ where $\mu_0 = \text{law}(X_0)$

these are exactly the finite-dim marginals of a CTMC$(\mu_0, (P_t)_{t\geq 0})$.

$(\mathcal{F}_t)_{t\geq 0}$

def: (martingale). A martingale $(M_t)_{t\geq 0}$ is a real-valued, adapted, integrable stochastic process such that $\mathbb{E}[M_{t+s} \mid \mathcal{F}_s] = M_s \quad \forall s,t \geq 0$.

Theorem: The following are equivalent: For any càdlàg adapted process X

1) X is a $(\mathcal{F}_t)_{t\geq 0}$ Markov chain with generator Q.

2) For any finitely supported $f : \mathcal{X} \to \mathbb{R}$, the process $(\pi_t^f)_{t\geq 0}$ defined by

$$\pi_t^f \overset{(*)}{:=} f(X_t) - f(X_0) - \int_0^t Qf(X_u)\,du$$

is a $(\mathcal{F}_t)_{t\geq 0}$-martingale.

proof: the martingale property of $(\pi_t^f)_{t\geq 0}$ is equivalent to:

$$\mathbb{E}[\pi_{t+s}^f - \pi_s^f \mid \mathcal{F}_s] = 0 \qquad \forall s,t \geq 0$$

$\mathbb{P}(A) > 0$

$$\Leftrightarrow \quad \mathbb{E}\Big[f(X_{t+s}) - f(X_s) - \int_0^t Qf(X_{u+s})\,du \;\Big|\; A \Big] = 0 \quad \forall A \in \mathcal{F}_s,\ \forall s,t \geq 0.$$

$A = \bigcup_x A \cap \{X_s = x\}$ $\quad \Leftrightarrow \quad$ $\mathbb{P}(A \cap \{X_s = x\}) > 0$

$$\mathbb{E}\Big[\underbrace{f(X_{t+s})}_{\mu_t(y)} - \underbrace{f(X_s)}_{\mu_s(y)} - \int_0^t \underbrace{Qf(X_{u+s})}_{\mu_u Q(y)}\,du \;\Big|\; A \cap \{X_s = x\} \Big] = 0$$

By linearity, true for all finitely supported f $\Leftrightarrow$ true for $f = 1_{\{y\}}$, $y \in \mathcal{X}$

$$\Leftrightarrow \quad \mu_t = \delta_x + \int_0^t \mu_u Q\,du \quad \text{where } \mu_t := \text{law}(X_{s+t} \mid \overbrace{A \cap \{X_s = x\}})$$

$$\Leftrightarrow \quad \begin{cases} \dot{\mu}_t = \mu_t Q \\ \mu_0 = \delta_x \end{cases}$$

By uniqueness in the FKE, this is $\Leftrightarrow$ $\{ \mu_t(y) = P_t(x,y) \quad \forall t \geq 0 \ \forall x,y.$

$$\Leftrightarrow \mathbb{P}(X_{s+t}=y \mid A \cap \{X_s=x\}) = P_t(x,y) \qquad \forall A \in \mathcal{F}_s,\ \forall x$$

$$\Leftrightarrow \mathbb{P}(X_{s+t}=y \mid \mathcal{F}_s) = P_t(X_s,y)$$

$$\Leftrightarrow \mathbb{E}[f(X_{s+t}) \mid \mathcal{F}_s] = P_t f(X_s) \qquad f = \mathbb{1}_y \qquad \square$$

Remark: The formula $(\Pi_t^f)_{t\geq 0}$ makes sense for any $f \in L^\infty(\mathcal{X})$. Moreover,

when $q_{max} < \infty$, $|\Pi_t^f| \leq 2\|f\|_\infty + t \underbrace{\|Q\|}_{\leq 2q_{max}} \|f\|_\infty < \infty$

so $(\Pi_t^f)_{t\geq 0}$ is still a martingale.

In the general case, $\Pi_t^f)_{t\geq 0}$ is not necessarily integrable, but it is always a local martingales, as witnessed by the localizing sequence

$$\tau_n := \inf\{t \geq 0 : q(X_t) > n\}.$$

Example: $(X_t)_{t\geq 0}$ Poisson process with intensity λ, take $f(x) = x$:

$$\Rightarrow \underbrace{X_t - X_0 - \int_0^t 1 \, du}_{\text{"0"}} \text{ is a martingale} \qquad \text{local}$$

$$(X_t - \lambda t)_{t\geq 0}.$$

$$Q(x,y) = \begin{cases} \lambda & \text{if } y = x+1 \\ -\lambda & \text{if } y = x \\ 0 & \text{else} \end{cases}$$

$$\rightarrow Qf(x) = \lambda\big(f(x+1) - f(x)\big) = \lambda.$$

$(\mathcal{F}_t)_{t\geq 0}$

Def: a stopping-time is a function $\tau : \Omega \to [0,\infty]$ such that

$$\forall t \geq 0, \quad \{\tau \leq t\} \in \mathcal{F}_t$$

Def: Given a stopping time τ, define $\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$

Exercise: 1) $\mathcal{F}_\tau$ is a σ-field

2) If $\tau \leq \tau'$ then $\mathcal{F}_\tau \subset \mathcal{F}_{\tau'}$

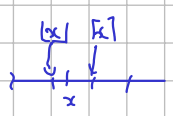
3) if τ is constant equal to t, then $\mathcal{F}_\tau = \mathcal{F}_t$.

4) τ is $\mathcal{F}_\tau$-measurable, and so is X_τ whenever $\tau<\infty$ and X is adapted, cadlag.

5) if τ is a stopping time, then so is $\tau_n := \frac{\lceil n\tau \rceil}{n}$, $n \in \mathbb{N}$

Proof of 5): Fix $t \geq 0$: want $\{\tau_n \leq t\} \in \mathcal{F}_t$?

$$\{\tau_n \leq t\} = \{\lceil n\tau \rceil \leq nt\}$$



$$= \{\lceil n\tau \rceil \leq \lfloor nt \rfloor\}$$

$$= \{n\tau \leq \lfloor nt \rfloor\} \qquad \frac{\lfloor nt \rfloor}{n} \leq t$$

$$= \{\tau \leq \tfrac{\lfloor nt \rfloor}{n}\} \in \mathcal{F}_{\frac{\lfloor nt \rfloor}{n}} \subseteq \mathcal{F}_t \qquad \square$$

Theorem (Strong Markov Property): Let $X = (X_t)_{t\geq 0}$ be a $(\mathcal{F}_t)_{t\geq 0}$-MC with generator Q, and let τ be a $(\mathcal{F}_t)_{t\geq 0}$ stopping time. Then on the event $\{\tau<\infty\}$,

$$\text{Law}\left((X_{t+\tau})_{t\geq 0} \mid \mathcal{F}_\tau\right) = \text{CTMC}(\delta_{X_\tau}, Q)$$

Proof: by successive conditionings, it is enough to prove that on $\{\tau<\infty\}$

$$\mathbb{P}(X_{t+\tau} = y \mid \mathcal{F}_\tau) = P_t(X_\tau, y)$$

Equivalently, we want $\forall A \in \mathcal{F}_\tau$, $\forall x \in \mathcal{X}$ $\qquad \mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$.

$$\mathbb{P}\left(X_{t+\tau} = y \mid A \cap \{\tau<\infty\} \cap \{X_\tau = x\}\right) = P_t(x,y)$$

i.e. $$\mathbb{P}\left(\{X_{t+\tau} = y, \tau<\infty, X_\tau = x\} \cap A\right) \overset{*}{=} P_t(x,y)\,\mathbb{P}\left(A \cap \{\tau<\infty\} \cap \{X_\tau = x\}\right)$$

Let's prove this first assuming that τ takes values in $\mathbb{Q}_+ \cup \{+\infty\}$.

$$\mathbb{P}\left(X_{t+\tau} = y, \tau<\infty, X_\tau = x\right\} \cap A\right)$$

$$= \sum_{s\in\mathbb{Q}_+} \mathbb{P}\left(\{\underbrace{\tau = s}_{\text{past}}, \underbrace{X_{t+s} = y}_{\text{future}}, \underbrace{X_s = x}_{\text{past}}\} \cap \underbrace{A}_{\text{past}}\right)$$

$$= \sum_{s\in\mathbb{Q}_+} \mathbb{P}\left(\{\tau = s, X_s = x\} \cap A\right) P_t(x,y)$$

(MP at time s)

$\{\tau = s, X_s = x\} \cap A \in \mathcal{F}_s$

$$= \sum_{s \in \mathbb{Q}_+} \mathbb{P}\left(\{\tau = s, X_\tau = x\} \cap A\right) \cdot P_t(x,y)$$

$$= \mathbb{P}\left(\{\tau < \infty, X_\tau = x\} \cap A\right) P_t(x,y), \text{ which is } (*).$$

Now, let's come back to the general case where τ takes values in $[0,\infty]$.

We of course introduce the approximation $\boxed{\tau_n := \frac{\lceil n\tau \rceil}{n} \geq \tau}$

τ_n is a stopping time, and it is $\mathbb{Q}_+ \cup \{\infty\}$-valued.

Since $A \in \mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_n}$ ($\tau \leq \tau_n$), we can apply the first part of the proof to obtain:

$$\mathbb{P}\left(A \cap \{\tau_n < \infty, X_{\tau_n} = x, X_{t+\tau_n} = y\}\right) \overset{(*) \text{ for } \tau_n}{=} \mathbb{P}\left(A \cap \{\tau_n < \infty, X_{\tau_n} = x\}\right) P_t(x,y)$$

Now take $n \to \infty$, and use $\begin{cases} \tau_n \overset{n\to\infty}{\Rightarrow} \tau \\ \tau_n \geq \tau \end{cases}$, and X càdlàg so $X_{\tau_n} \overset{n\to\infty}{\to} X_\tau$

$$\Rightarrow \mathbb{P}\left(A \cap \{\tau < \infty, X_\tau = x, X_{t+\tau} = y\}\right) = \mathbb{P}\left(A \cap \{\tau < \infty, X_\tau = x\}\right) P_t(x,y)$$

which is the desired property! □

□ Remark: SMP $\Rightarrow$ WMP (because $\tau \equiv t$ is a stopping time and $\mathcal{F}_\tau = \mathcal{F}_t$)