Habilitation à Diriger les Recherches

École Doctorale de Sciences Mathématiques de Paris Centre

SPARSE RANDOM GRAPHS

Structure & Dynamics

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Overview

This manuscript presents a synthesis of my research works – listed on the next page – over the past few years, without entering into the technical details of the proofs. It discusses my contribution to the study of large random graphs and random processes defined thereon, in the so-called *sparse regime*, where the number of vertices tends to infinity while the average degree remains bounded.

- Chapter 1 (*Framework*) sets the stage by introducing the main random graph model that will be considered throughout the manuscript (Section 1.1), as well as the unimodular Galton-Watson trees that describe its asymptotic local geometry (Section 1.2). This brief chapter is not meant as a comprehensive account on the subject; it is only aimed at presenting in a succinct way the main objects and tools used in the sequel.

- Chapter 2 (*Structural Aspects*) focuses on some fundamental properties of random graphs themselves. In particular, Section 2.1 describes their spectrum [P4, P10], Sections 2.2 and 2.3 investigate the structure of their subgraphs [P5, P6, P12], and Section 2.4 discusses the generic convergence of a broad class of graph parameters [P13].

- Chapter 3 (*Dynamical Aspects*) moves "up" from the analysis of random graphs themselves to the study of random processes defined on these graphs. In particular, Section 3.1 studies the typical rate of convergence of a popular message-passing algorithm [P1, P2, P8], Section 3.2 studies the typical distances in first-passage percolation [P7], and Sections 3.3 and 3.4 deal with mixing times and the cutoff phenomenon for random walks [P14, P15].

This thematic classification leaves aside my works [P9, P11, P16], which I will not discuss. The works [P17, P18] were completed after sending the present manuscript to the reviewers, and hence will be ignored too. The works [P4, P5, P6] were part of my PhD manuscript [P3].

Publications and preprints

All papers are available on my webpage: http://www.lpma-paris.fr/pageperso/salez/

[P1] J. Salez and D. Shah. Optimality of Belief Propagation for random assignment problem. *Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pp. 187–196, 2009. DOI: 10.1137/1.9781611973068.22.

[P2] J. Salez and D. Shah. Belief propagation: an asymptotically optimal algorithm for the random assignment problem. *Mathematics of Operations Research* 34, pp. 468–480, 2009. DOI: 10.1287/moor.1090.0380.

[P3] J. Salez. "Some implications of local weak convergence for sparse random graphs". Theses. Université Pierre et Marie Curie - Paris VI ; Ecole Normale Supérieure de Paris - ENS Paris, July 2011.

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[P6] J. Salez. Weighted enumeration of spanning subgraphs in locally tree-like graphs. *Random Structures & Algorithms* 43, pp. 377–397, 2013. DOI: 10.1002/rsa.20436.

[P7] J. Salez. Joint distribution of distances in large random regular networks. *Journal of Applied Probability* 50, pp. 861–870, 2013. DOI: 10.1239/jap/1378401241.

[P8] J. Salez. The Mézard-Parisi equation for matchings in pseudo-dimension $d > 1$. *Electronic Communications in Probability* 20, no. 13, 7, 2015. DOI: 10.1214/ECP.v20-3791.

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[P12] V. Anantharam and J. Salez. The densest subgraph problem in sparse random graphs. *The Annals of Applied Probability* 26, pp. 305–327, 2016. DOI: 10.1214/14-AAP1091.

[P13] J. Salez. The interpolation method for random graphs with prescribed degrees. *Combinatorics, Probability and Computing* 25, pp. 436–447, 2016. DOI: 10.1017/S0963548315000139.

[P14] A. Ben-Hamou and J. Salez. Cutoff for non-backtracking random walks on sparse random graphs. *The Annals of Probability* (to appear), 2015. arXiv: 1504.02429 [math.PR].

[P15] C. Bordenave, P. Caputo, and J. Salez. Random walk on sparse random digraphs. *ArXiv e-prints*, 2015. arXiv: 1508.06600 [math.PR].

[P16] A. Ben-Hamou, Y. Peres, and J. Salez. Weighted sampling without replacement. *ArXiv e-prints*, 2016. arXiv: 1603.06556 [math.PR].

[P17] J. Salez. Spectral atoms of unimodular random trees. *ArXiv e-prints*, 2016. arXiv: 1609.09374 [math.PR].

[P18] C. Bordenave, P. Caputo, and J. Salez. Cutoff at the "entropic time" for sparse Markov chains. *ArXiv e-prints*, 2016. arXiv: 1611.01079 [math.PR].

Table of Contents

CHAPTER 1

Framework

1.1 The configuration model

Among the rich variety of random graph models that have been proposed over the past years, the *configuration model* [30, 111, 112] certainly stands out as one of the simplest and most versatile choices, as it offers the (priceless) possibility to freely specify the exact degree of each vertex, while keeping as much randomness as possible in the overall structure. For the sake of clarity, we have chosen to focus our attention on it throughout the manuscript, except for Section 3.1. As explained below, our results can then be systematically transferred to other classical random graphs appearing in the literature, including the popular Erdős-Rényi model and its multiple variants. We only briefly recall the model and refer to Chapter 7 of the excellent book [84] for a detailed account.

The model. Given a sequence $\mathbf{d} = (d_1, \ldots, d_n)$ of non-negative integers such that

$$m \;:=\; \frac{1}{2}\sum_{i=1}^{n} d_i \tag{1.1}$$

is an integer, we construct a random graph $G_{\mathbf{d}}$ with vertex set $[n] := \{1, \ldots, n\}$ as follows: we first equip each vertex $i \in [n]$ with a set of d_i *half-edges*, say $\mathcal{X}_i := \{(i,1), \ldots, (i,d_i)\}$. We then pick uniformly at random a *pairing* ω on $\mathcal{X}_1 \cup \ldots \cup \mathcal{X}_n$ (i.e., an involution without fixed points), and interpret every pair of matched half-edges as an edge between the corresponding vertices (see Figure 1.1). More formally, the number of edges between i and j is declared to be $A_{ij} = |\omega(\mathcal{X}_i) \cap \mathcal{X}_j| = A_{ji}$. Note that the degree of i is then precisely d_i, and that the total number of edges is m.

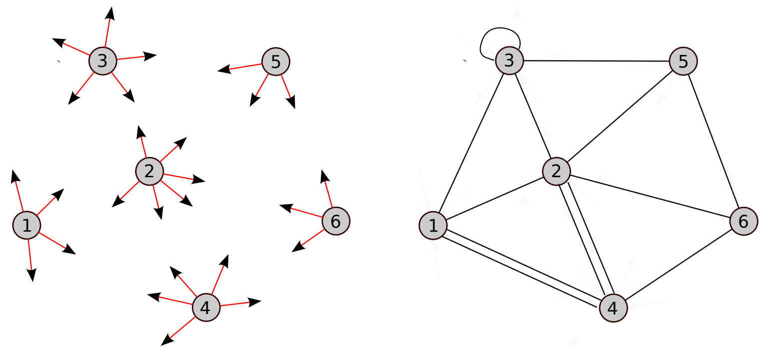


Figure 1.1: The set of half-edges and a possible realization of $G_{\mathbf{d}}$ in the case $\mathbf{d} = (4, 6, 5, 5, 3, 3)$

Sparse regime. In order to study large-size asymptotics, we will always let our parameter $\mathbf{d}$ depend on n, and consider the limit as $n \to \infty$. To keep the notation simple, we refrain from making the dependence on n explicit. We shall be interested in the so-called *sparse* regime, where *most degrees remain of order* 1 as $n \to \infty$. The precise quantification will depend on the context, but a typical assumption might look like

$$\sum_{i \in [n]} d_i^2 \quad = \quad \mathcal{O}(n). \tag{1.2}$$

It will often be convenient to further assume some kind of consistency of $\mathbf{d}$ as n varies, by requiring that the *empirical degree distribution* $\frac{1}{n} \sum_{i \in [n]} \delta_{d_i}$ approaches a limit law $\pi = (\pi_k)_{k \geq 0}$ on $\mathbb{Z}_+$:

$$\forall k \in \mathbb{Z}_+, \qquad \frac{1}{n} \sum_{i \in [n]} \mathbf{1}_{(d_i = k)} \quad \xrightarrow[n \to \infty]{} \quad \pi_k. \tag{1.3}$$

Simple graphs. As visible on Figure 1.1, the random graph $G_{\mathbf{d}}$ is not necessarily *simple*: self-loops and multiple edges are allowed. In fact, without extra assumptions, it is not even guaranteed that a simple graph with degrees $\mathbf{d}$ exists at all. However, in the regime (1.2), a result by Janson [88, 89] asserts that $\liminf_{n \to \infty} \mathbb{P}(G_{\mathbf{d}} \text{ is simple}) > 0$. Moreover, conditionally on being simple, $G_{\mathbf{d}}$ is a uniform simple graph with degrees $\mathbf{d}$. Consequently, any property that holds with high probability (w.h.p.) for $G_{\mathbf{d}}$ also holds w.h.p. under the uniform simple graph model.

Random degrees. Many classical random graphs, including the popular Erdős-Rényi random graph or the generalized random graph model are actually mixtures of the above model: the vertex degrees $d_1, \ldots, d_n$ themselves are random, but two simple graphs with the same degrees occur with the same probability. Thus, results established under the uniform simple graph model apply *conditionally*, with $\mathbf{d}$ being random. Note that in the Erdős-Rényi case, the random sequence $\mathbf{d}$ satisfies (1.2) and (1.3) almost-surely, with π being Poisson. See [84, Section 6] for details.

1.2 Local weak convergence

In the sparse regime (1.3), there is a sense in which the random graph $G_{\mathbf{d}}$, viewed from a uniformly chosen vertex, resembles "locally" a certain Galton-Watson tree. We only briefly recall the necessary framework, and refer to the seminal paper [21] or the surveys [9, 10, 34] for details.

The space $\mathcal{G}_\star$. A *rooted graph* (G, o) is a graph $G = (V, E)$ together with a distinguished vertex $o \in V$, called the *root*. An isomorphism between two rooted graphs is a graph isomorphism which additionally maps the root to the root. We let $\mathcal{G}_\star$ denote the set of all countable, locally finite, connected rooted graphs considered up to isomorphism. We turn $\mathcal{G}_\star$ into a Polish space by equipping it with the metric

$$d_{\text{LOC}}\left((G,o),(G',o')\right) \quad := \quad \frac{1}{\inf\{r \geq 1\colon \mathcal{B}_r(G,o) \neq \mathcal{B}_r(G',o')\}}. \tag{1.4}$$

Here, $\mathcal{B}_r(G, o)$ denotes the ball of radius r around o in G, viewed as an element of $\mathcal{G}_\star$. In words, two rooted graphs are close to each other if the local structure around their roots is similar.

Local weak limits. *Uniform rooting* is a natural way of turning a finite, non-rooted, deterministic graph $G = (V, E)$ into a random element of $\mathcal{G}_\star$: we simply declare a uniformly chosen vertex $o \in V$ as the root, restrict the graph to the corresponding connected component, and forget the labels. We let $\mathcal{U}(G) \in \mathcal{P}(\mathcal{G}_\star)$ denote the law of the resulting random variable. Thus, $\mathcal{U}(G)$ is the empirical distribution of all possible rootings of G (see Figure 1.2). If $(G_n)_{n\geq 1}$ is a sequence of finite graphs such that $(\mathcal{U}(G_n))_{n\geq 1}$ admits a limit $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\star)$ in the usual weak sense for Borel probability measures on Polish spaces, we call $\mathcal{L}$ the *local weak limit* of $(G_n)_{n\geq 1}$ and write

$$G_n \quad \xrightarrow[n\to\infty]{} \quad \mathcal{L}.$$

In short, $\mathcal{L}$ describes the asymptotic local structure of G_n, viewed from a uniformly chosen vertex. Not every probability measure on $\mathcal{G}_\star$ can arise as a local weak limit: uniform rooting confers to $\mathcal{L}$ a powerful distributional invariance called *unimodularity*, which we now discuss.

Unimodularity. Let $\mathcal{G}_{\star\star}$ denote the set of locally finite connected graphs with a distinguished oriented edge, taken up to the natural isomorphism relation and equipped with the natural analogue of (1.4). With any function $f\colon \mathcal{G}_{\star\star} \to \mathbb{R}$ is naturally associated a function $\partial f\colon \mathcal{G}_\star \to \mathbb{R}$, defined by

$$\partial f(G,o) \quad = \quad \sum_{i\sim o} f(G,i,o),$$

where $\sim$ denotes adjacency in the graph. One may think of $f(G, i, o)$ as an amount of mass sent from i to o in G, and of $\partial f(G, o)$ as the total mass received by the root. A measure $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\star)$

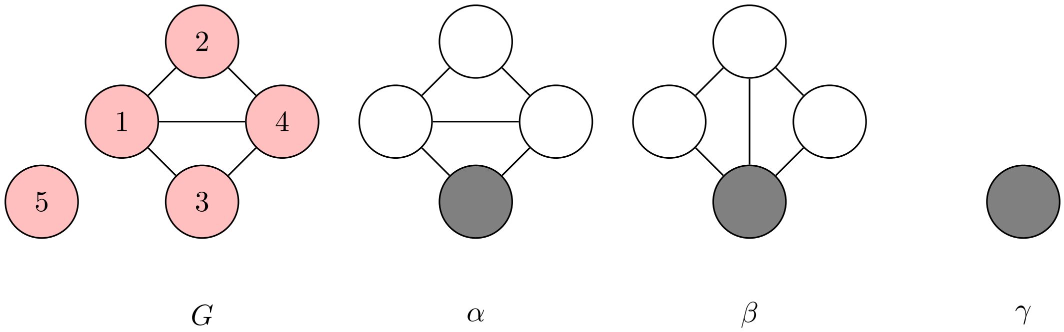


Figure 1.2: A graph G and its possible rootings $\alpha, \beta, \gamma \in \mathcal{G}_\star$. Here, $\mathcal{U}(G) = \frac{2}{5}\delta_\alpha + \frac{2}{5}\delta_\beta + \frac{1}{5}\delta_\gamma$.

is called *unimodular* if it obeys the so-called *Mass Transport Principle*: for any Borel function $f\colon \mathcal{G}_{\star\star} \to [0,\infty)$, the expected amounts of mass received and sent by the root coincide, i.e.,

$$\int_{\mathcal{G}_\star} (\partial f)\, d\mathcal{L} \quad = \quad \int_{\mathcal{G}_\star} (\partial f^*)\, d\mathcal{L},$$

where $f^*(G,i,o) = f(G,o,i)$ denotes the *reversal* of f. It is not hard to see that the local weak limit of any sequence of finite graphs is unimodular (see, e.g., [9]). Whether the converse also holds constitutes an important open problem, with deep implications (see [9, 62]). The set of unimodular measures will appear a few times in the manuscript, and will be denoted by $\mathcal{U}$.

Unimodular Galton-Watson trees. Let $\pi = (\pi_k)_{k\geq 0}$ be a probability distribution on $\mathbb{Z}_+$ with finite, non-zero mean. A *unimodular Galton-Watson tree* with degree distribution π is a random rooted tree obtained by a Galton-Watson branching process where the root has offspring distribution π and all descendants have the size-biased offspring distribution $\widehat{\pi} = (\widehat{\pi}_k)_{k\geq 0}$, where

$$\widehat{\pi}_k \quad = \quad \frac{(k+1)\pi_{k+1}}{\sum_i i\pi_i}. \tag{1.5}$$

The law of this random rooted tree is easily seen to be unimodular, and is denoted by $\mathrm{UGWT}(\pi)$. Such trees play a distinguished role, as they are the local weak limits of many natural sequences of graphs (see, e.g., [34]). In particular, in the regime (1.2)-(1.3), our random graph $G_{\mathbf{d}}$ satisfies

$$G_{\mathbf{d}} \quad \xrightarrow[n\to\infty]{} \quad \mathrm{UGWT}(\pi), \tag{1.6}$$

almost-surely under any coupling of the underlying random graphs (complete convergence). This now-classical fact will be used several times in the course of our investigation.

CHAPTER 2

Structural aspects

2.1 Eigenvalues [P4, P10]

Empirical spectral distribution. A graph G on $[n]$ can be equivalently represented by its *adjacency matrix* A_G, the $n \times n$ matrix whose entry (i,j) counts the number of edges between i and j. It is customary to encode the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ of this symmetric matrix into a probability measure on $\mathbb{R}$, called the *empirical spectral distribution* of G:

$$\mu_G \quad = \quad \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}. \tag{2.1}$$

This fundamental graph invariant happens to capture a considerable amount of structural information (see, e.g., [52]), as can be felt already from the following elementary identity relating the kth moment of μ_G to the number $\gamma_k(G,o)$ of paths of length k from o to o in G:

$$\int_{\mathbb{R}} \lambda^k \, d\mu_G(\lambda) \quad = \quad \frac{1}{n}\sum_{o\in[n]} \gamma_k(G,o). \tag{2.2}$$

It is natural to ask about the *typical* behavior of μ_G when G is a sparse graph. An elegant answer can be given using the framework of local weak convergence. The first part of the following result is already present in [36, P4], and the Kolmogorov-Smirnov refinement can be found in [1, 35].

Theorem 1. *If $(G_n)_{n\geq 1}$ admits a local weak limit $\mathcal{L}$, then $(\mu_{G_n})_{n\geq 1}$ admits a weak limit $\mu_{\mathcal{L}} \in \mathcal{P}(\mathbb{R})$. Moreover, the convergence holds in the Kolmogorov-Smirnov metric:*

$$\sup_{\lambda\in\mathbb{R}} \left|\mu_{G_n}\left((-\infty,\lambda]\right) - \mu_{\mathcal{L}}\left((-\infty,\lambda]\right)\right| \quad \xrightarrow[n\to\infty]{} \quad 0.$$

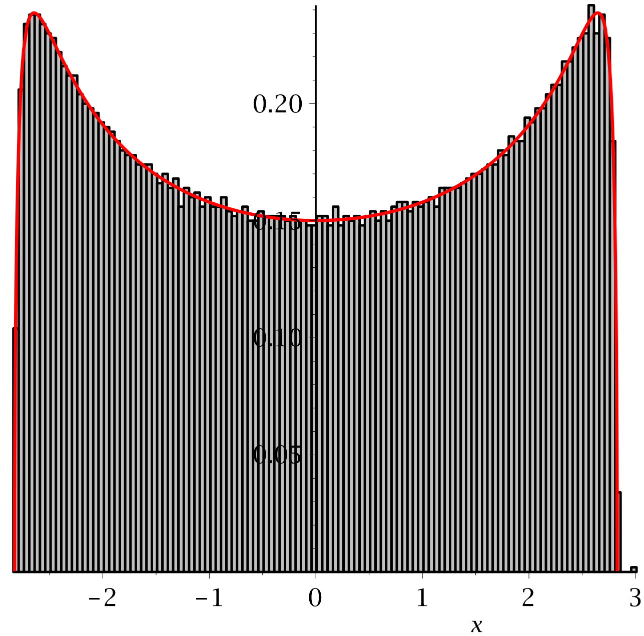


Figure 2.1: Histogram of the eigenvalues of a uniform 3−regular random graph on $n = 10^4$ vertices (in gray), compared with the Kesten McKay density (2.6) (in red).

The interest of this surprising result is that it allows one to replace the asymptotic spectral analysis of sparse graphs by the direct study of the spectra of their local weak limits. Before we start to implement this program, let us make a few comments about the limit $\mu_{\mathcal{L}}$, which we shall naturally call the *empirical spectral distribution* of $\mathcal{L}$. When the degrees in $(G_n)_{n\geq 1}$ are uniformly bounded, we may safely take the limit in (2.2) to obtain the following moment characterization:

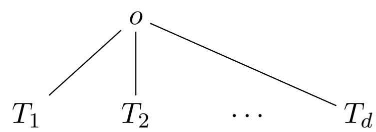
$$\int_{\mathbb{R}} \lambda^k \, d\mu_{\mathcal{L}}(\lambda) \quad = \quad \int_{\mathcal{G}_\star} \gamma_k(G,o) \, d\mathcal{L}(G,o). \tag{2.3}$$

In the case of unbounded degrees however, there might be several probability measures with moments given by (2.3) or, even worse, the right-hand side could very well be infinite. The correct way to define $\mu_{\mathcal{L}}$ in general turns out to be via its *Cauchy-Stieltjes transform*: for every $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{1}{\lambda - z} \, d\mu_{\mathcal{L}}(\lambda) \quad = \quad \int_{\mathcal{G}_\star} (A_G - z)^{-1}_{oo} \, d\mathcal{L}(G,o). \tag{2.4}$$

Note that the definition of the resolvent matrix $(A_G - z)^{-1}$ on the right-hand side is not immediate, as the adjacency operator A_G of a infinite graph G may fail to be essentially self-adjoint. Examples of such pathological graphs can be found in, e.g., [113]. Nevertheless, they can be shown to have zero-measure under any $\mathcal{L} \in \mathcal{U}$, so that (2.4) makes perfect sense. See [35] for details.

A distributional fixed point equation. Computing the crucial quantity $f_{(G,o)}(z) := (A_G - z)^{-1}_{oo}$ appearing in (2.4) happens to be easy when (G, o) is a finite rooted tree T: let us remove the root o to create $d = \deg(T, o)$ smaller rooted trees $T_1, \ldots, T_d$, as depicted in the following diagram:



Then an easy application of the Schur complement formula yields the following recursive identity:

$$f_T(z) \quad = \quad \frac{-1}{z + \sum_{i=1}^{d} f_{T_i}(z)}. \tag{2.5}$$

With some effort, this recursion can be extended to infinite trees (see [P3] for details). In particular, one may consider the case where T is a random Galton-Watson tree. Conditionally on the root-degree $d = \deg(T, o)$, the subtrees $T_1, \ldots, T_d$ are independent with the same law as T. Consequently, (2.5) yields a distributional fixed-point equation for the law of the random variable $f_T(z)$. The solution can be shown to be unique, and plugging it into (2.4) gives access to the Cauchy-Stieltjes transform of the spectral measure. Since this transform is injective, we have, in principle, reduced the spectral analysis of unimodular Galton-Watson trees to the study of the distributional fixed point equation (2.5). In the degenerate case where the degree distribution is a Dirac mass $\pi = \delta_d$, the resolution is trivial and we recover the Kesten-McKay density [107] (see Figure 2.1):

$$\mu_{\mathrm{UGWT}(\delta_d)}(d\lambda) \quad = \quad \frac{d\sqrt{4(d-1) - \lambda^2}}{2\pi(d^2 - \lambda^2)} \mathbf{1}_{(-2\sqrt{d-1}, 2\sqrt{d-1})}(\lambda)\, d\lambda. \tag{2.6}$$

For non-degenerate degrees, such as the case $\pi = \mathrm{Poisson}(c)$ describing the $n \to \infty$ limit of the Erdős-Rényi model with n vertices and edge probability $\frac{c}{n}$ (see Figure 2.2), extracting any information from (2.5) turns out to be a daunting task, and the full understanding of $\mu_{\mathrm{UGWT}}(\pi)$ seems out of reach at present. We describe two results that we have obtained in this direction.

Mass of the atom at zero. The main result of my work [P4] with Bordenave and Lelarge is the explicit resolution of the equation (2.5) on a general unimodular Galton-Watson tree in the limit where $z \to 0$ along the imaginary axis, with the following consequence:

Theorem 2 (Mass at zero, [P4])**.** *Consider an arbitrary degree distribution $\pi \in \mathcal{P}(\mathbb{Z}_+)$ with finite mean, and let $\phi(x) = \sum_k \pi_k x^k$ denote its generating series. Then,*

$$\mu_{\mathrm{UGWT}(\pi)}(\{0\}) \quad = \quad \max_{x \in [0,1]} \left\{ \phi'(1) x \overline{x} + \phi(1 - x) + \phi(1 - \overline{x}) - 1 \right\}, \quad \text{where} \quad \overline{x} = \frac{\phi'(1 - x)}{\phi'(1)}.$$

Moreover, the above maximum may be restricted to those $x \in [0, 1]$ solving $x = \overline{\overline{x}}$.

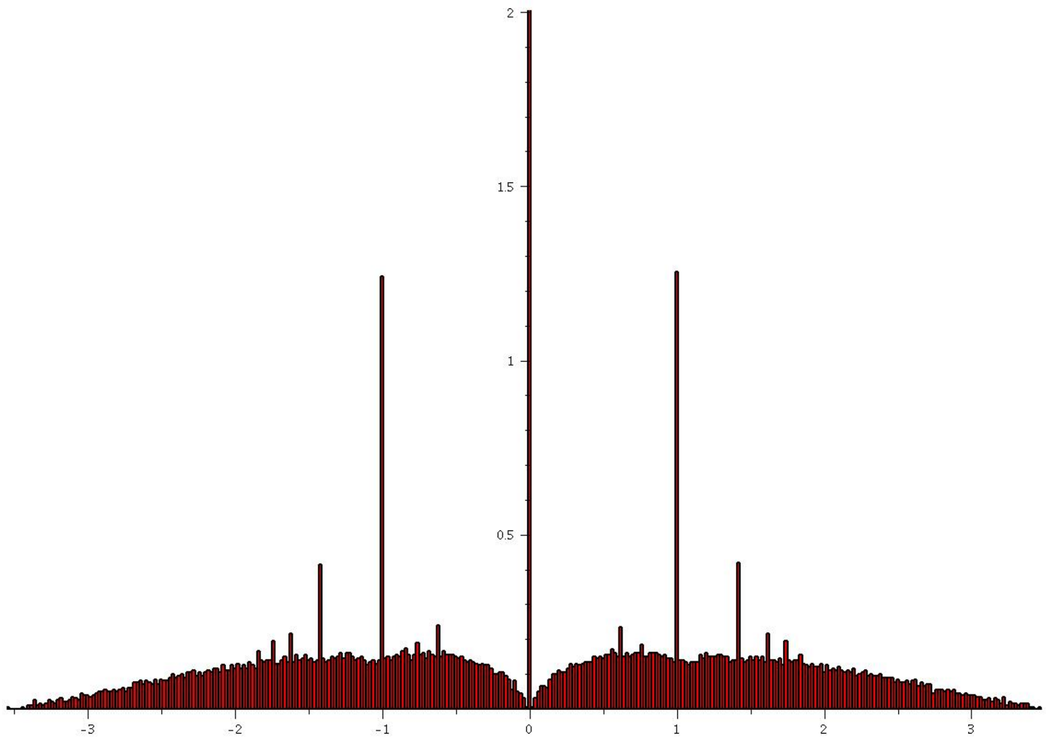


Figure 2.2: Histogram of the eigenvalues of the Erdős-Rényi random graph with $n = 10^4, p = 2/n$

In particular, in light of Theorem 1, Theorem 2 describes the asymptotic co-rank of the adjacency matrix of the random graph $G_{\mathbf{d}}$ as $\mathbf{d}$ approaches π in the sense of (1.3). In the special case $\pi = \text{Poisson}(c)$ corresponding to the Erdős-Rényi model, the formula evaluates to $x_* + e^{-cx_*} + cx_* e^{-cx_*} - 1$, where $x_* \in (0,1)$ denotes the smallest root of $x = e^{-ce^{-cx}}$: this settles a conjecture of Bauer and Golinelli [17] and answers a question of Costello and Vu [50].

Support of the atomic part. More generally, one may wonder about the value of $\mu_{\mathrm{UGWT}(\pi)}(\{\lambda\})$ at other locations $\lambda \in \mathbb{R}$ (the *spikes* visible on Figure 2.2). During the 2010 AIM Workshop on Random Matrices, Ben Arous [115, Problem 14] raised the question of determining the set of atoms

$$\Sigma(\mu) \quad := \quad \{\lambda \in \mathbb{R} \colon \mu(\{\lambda\}) > 0\}$$

of the measure $\mu = \mu_{\mathrm{UGWT}(\mathrm{Poisson}(c))}$ corresponding to the Erdős-Rényi model. My paper [P10] answers the question completely, as follows. A trivial outer-bound is obtained by observing that any eigenvalue of a finite graph must, by definition, belong to the ring $\mathbb{A}$ of *totally real algebraic integers*, i.e. roots of real-rooted monic polynomials with integer coefficients: in view of Theorem 1, any local weak limit immediately inherits from the same property. On the other side, a crude inner-bound is given by the fact that any finite tree has positive probability under the Poisson-Galton-Watson law. This leaves us with the inclusions

$$\bigcup_{T\colon \text{finite tree}} \Sigma(\mu_T) \;\subseteq\; \Sigma(\mu_{\mathrm{UGWT(Poisson}(c))}) \;\subseteq\; \mathbb{A}.$$

Perhaps surprisingly, the inner and outer-bound turn out to coincide, as I established in [P10].

Theorem 3 ([P10])**.** *Every totally real algebraic integer is an eigenvalue of some finite tree.*

The weaker statement obtained by replacing *tree* by *graph* was conjectured forty years ago by Hoffman [81], and established seventeen years later by Estes [64] using deep algebraic number theory, see also [16]. My work [P10] provides an elementary and constructive proof of a stronger statement, namely that the graph may actually be chosen to be a tree. The recursion (2.5) plays a crucial role in the proof: for a fixed $\lambda \in \mathbb{R}\setminus\{0\}$, it implies that the set

$$\mathbb{Q}_\lambda \;:=\; \left\{1+\frac{1}{\lambda f_T(\lambda)} : T \text{ finite tree}\right\}$$

is stable under internal addition and under the map $x \mapsto \frac{1}{\lambda^2(1-x)}$. When $\lambda \in \mathbb{A}$, these properties happen to turn $\mathbb{Q}_\lambda$ into a field. In particular, it must contain 1, which means that $f_T(\lambda) = \infty$ for some tree finite T, as desired. As a by-product, Theorem 3 solves Ben Arous' question:

Corollary 4 (Atomic support, [P10])**.** *Assume that $\pi_k > 0$ for each $k \geq 1$. Then, the atoms of $\mu_{\mathrm{UGWT}(\pi)}$ are precisely the totally real algebraic integers, i.e., $\Sigma(\mu_{\mathrm{UGWT}(\pi)}) = \mathbb{A}$.*

Perspectives. In contrast, if $\pi_1 = 0$, it is conjectured that the atomic support $\Sigma(\mu_{\mathrm{UGWT}(\pi)})$ is actually finite [**Question 4.5.**, 35], and I am currently working on this problem. Many interesting questions regarding the spectral measure $\mu = \mu_{\mathrm{UGWT}(\pi)}$ remain unanswered, even in the simple case $\pi = \mathrm{Poisson}(c)$ corresponding to the Erdős-Rényi model. For example, Corollary 4 implies the existence of many atoms, but nothing is known about their masses apart from the one at 0. Answering this would require the understanding of the distributional equation 2.5 as z approaches the real axis, which seems non-trivial. Also, all the above results concern the atomic part of the spectrum only. Bordenave, Sen and Viràg [32] recently proved the emergence of a non-atomic part (i.e., $\sum_{\lambda\in\Sigma(\mu)} \mu(\{\lambda\}) < 1$) as soon as $\mathrm{UGWT}(\pi)$ is super-critical (i.e., $\sum_k k\widehat{\pi}_k > 1$). However, the exact nature of this non-atomic part and, in particular, the existence of an absolutely continuous component, is still far from being understood. We refer to [35, Section 4] for several fascinating questions along this line.

2.2 Subgraphs with degree constraints [P5, P6]

Matchings. Understanding the structure of a graph $G = (V, E)$ inevitably passes through the analysis of its various subgraphs. Lyons [105] used local weak convergence to count *spanning trees* in large graphs. We investigate here the case of *matchings*, i.e. subsets of edges $F \subseteq E$ such that in the resulting subgraph $G' = (V, F)$, every vertex has degree 0 or 1 (see Figure 2.2). This basic structure plays a central role in graph theory, and we refer to the monographs [73, 102] for details. The polynomial that enumerates the sizes of all matchings on G is the *matching polynomial*:

$$P_G(z) = \sum_{F \subseteq E} \mu(F) z^{|F|} \qquad \text{where} \qquad \mu(F) = \begin{cases} 1 & \text{if F is a matching} \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

In particular, its degree is the maximum possible size of a matching on G, known as the *matching number* $\nu(G)$. The latter was analyzed on the Erdős-Rényi model by Karp and Sipser [92], and on special cases of the configuration model by Bohman and Frieze [29]. In [P5], Bordenave, Lelarge and myself determined the asymptotic behavior of the whole polynomial P_G on an arbitrary convergent graph sequence. More precisely, we showed that the elementary recursion

$$P_G(z) = P_{G-o}(z) + z \sum_{i \sim o} P_{G-o-i}(z), \tag{2.8}$$

can be considered, in an appropriate sense, directly on a unimodular measure $\mathcal{L} \in \mathcal{U}$. It determines a unique analytic function $f_{\mathcal{L}} \colon \mathbb{C} \setminus (-\infty, 0) \to \mathbb{C}$ that describes the asymptotic behavior of the matching polynomial along any graph sequence that approaches $\mathcal{L}$:

Theorem 5 (Matchings, [P5])**.** *Let $(G_n)_{n \geq 1}$ admit a local weak limit $\mathcal{L}$. Then,*

$$\frac{1}{|V_n|} \log P_{G_n} \xrightarrow[n \to \infty]{} f_{\mathcal{L}},$$

uniformly on compact subsets of $\mathbb{C} \setminus (-\infty, 0)$ and, moreover,

$$\frac{\nu(G_n)}{|V_n|} \xrightarrow[n \to \infty]{} \lim_{z \to \infty} f'_{\mathcal{L}}(z).$$

This considerably refines a result of Elek and Lippner [63]. Moreover, in the case where $\mathcal{L}$ is a unimodular Galton-Watson tree, (2.8) simplifies into a distributional fixed point equation which can be solved explicitly in the limit where $z \to \infty$. As a result, we obtain the following general formula for the matching number of our random graph $G_{\mathbf{d}}$.

Corollary 6 (Matching number, [P5])**.** *Assume that the degrees $\mathbf{d} = (d_1, \ldots, d_n)$ approach a distribution $\pi \in \mathcal{P}(\mathbb{Z}_+)$ with finite mean, in the sense of (1.3). Set $\phi(x) = \sum_k \pi_k x^k$. Then,*

$$\frac{\nu(G_{\mathbf{d}})}{n} \xrightarrow[n \to \infty]{\mathbb{P}} \frac{1}{2} \min_{x \in [0,1]} \left\{ 2 - \phi'(1) x \overline{x} - \phi(1 - x) - \phi(1 - \overline{x}) \right\}, \quad \text{where} \quad \overline{x} = \frac{\phi'(1 - x)}{\phi'(1)}.$$

In the Poisson case ($\phi(x) = \exp(cx - c)$), we recover exactly the celebrated Karp-Sipser formula, obtained in [92] by analyzing an iterative leaf-removal algorithm on the Erdős-Rényi random graph.

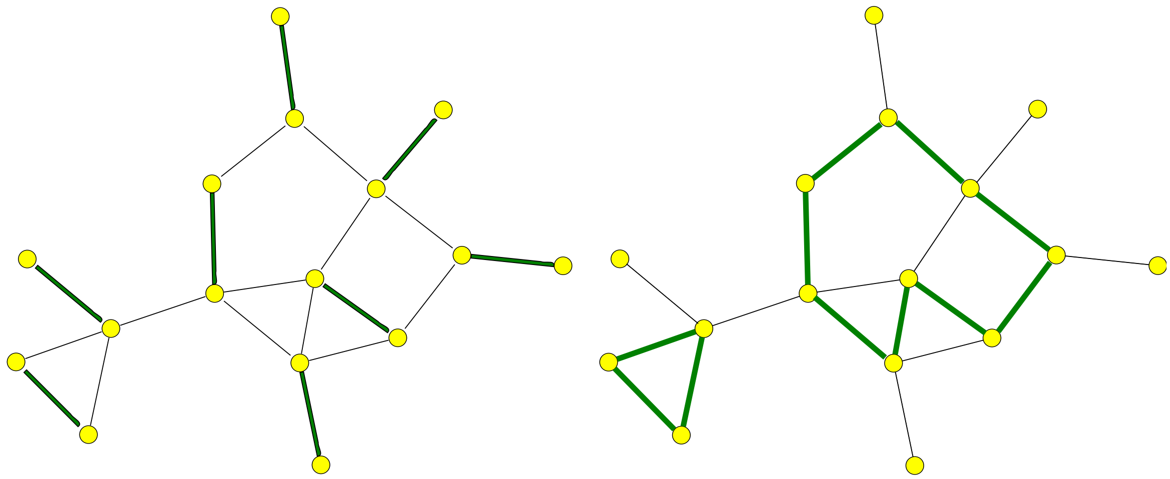


Figure 2.3: A matching (left) and a collection of disjoint cycles (right) on the same graph

General degree constraints. An interesting question arises in the more general framework where the indicator in (2.7) is replaced by a weight of the form

$$\mu(F) \quad = \quad \prod_{v \in V} \chi(\deg_F(v)),$$

where $\chi \colon \mathbb{Z}_+ \mapsto \mathbb{R}_+$ is an arbitrary function and where $\deg_F(v)$ denotes the degree of v in the graph (V, F). Matchings correspond to the case $\chi = \mathbf{1}_{\{0,1\}}$, for which Theorem 5 applies. On the contrary, the simple choice $\chi = \mathbf{1}_{\{0,2\}}$ leads to disjoint cycles (see Figure 2.2), for which such a continuity principle is doomed to fail (the counting polynomial will evaluate very differently on a path of length n and on a cycle of length n but both share the same local weak limit $(\mathbb{Z}, 0)$). What conditions shall one then impose on the local weight χ in order for the resulting structure to be "continuous" w.r.t. to local weak convergence? A corollary of the main result in my work [P6] states that the analogue of Theorem 5 will hold as soon as χ is *log-concave*, i.e.,

$$\forall d \geq 1, \qquad \chi(d+1)\chi(d-1) \quad \leq \quad (\chi(d))^2\,, \tag{2.9}$$

and without internal zeros. This condition is trivially verified for matchings ($\chi = \mathbf{1}_{\{0,1\}}$) and, more generally, for spanning subgraphs with degrees at most some given $\Delta \in \mathbb{Z}_+$ ($\chi = \mathbf{1}_{\{0,1,\ldots,\Delta\}}$). We refer to the monograph [121] for a comprehensive survey on this important graph structure. As an illustration, I obtained in [P6] an explicit formula analogous to Corollary 6 for the maximum size of such subgraphs on $G_{\mathbf{d}}$. To the best of my knowledge, the limit was only known to exist in the Erdős-Rényi case [69, Theorem 3], and could not be explicitly determined.

2.3 Density profile [P12]

We now go deeper into the analysis of subgraphs of a large sparse graph $G = (V, E)$ by exploring one of their most fundamental aspects: their *density*. In particular, the classical *densest subgraph problem* asks for the value

$$\varrho(G) \quad := \quad \max\left\{\frac{|E[H]|}{|H|} : \emptyset \neq H \subseteq V\right\}, \tag{2.10}$$

where $E[H]$ denotes the number of edges with both end-points in H. Evaluating this graph parameter on large random graphs constitutes a natural problem in probabilistic combinatorial optimization: *how dense can a subgraph of our sparse random graph* $G_{\mathbf{d}}$ *be, as* $\mathbf{d}$ *approaches some degree distribution* π? The Erdős-Rényi case $\pi = \mathrm{Poisson}(c)$ was investigated in great detail in the 90's by Hajek [79, 80], who formulated a remarkably precise conjecture regarding the limit of (2.10). To explain it, we need to make a detour through a seemingly unrelated problem.

Load balancing. Suppose each edge of G carries a unit amount of load, which must be distributed over its end-points in such a way that the total load is as balanced as possible across the graph. More formally, define an *allocation* as a map $\theta\colon \vec{E} \to [0,1]$ satisfying $\theta(i,j) + \theta(j,i) = 1$ for every $\{i,j\} \in E$, where $\vec{E}$ denotes the set of oriented edges formed by replacing each edge $\{i,j\} \in E$ with the two oriented edges (i,j) and (j,i). Such an allocation induces a *load* at every vertex $o \in V$:

$$\partial\theta(o) \quad := \quad \sum_{i \sim o} \theta(i,o).$$

The allocation θ is called *balanced* if modifying it along a single edge can not further reduce the load imbalance between its end-points, i.e. for every $(i,j) \in \vec{E}$,

$$\partial\theta(i) < \partial\theta(j) \quad \Longrightarrow \quad \theta(i,j) = 0. \tag{2.11}$$

Perhaps surprisingly, this local optimality happens to guarantee global optimality in a strong sense.

Lemma 7 (see [79])**.** *For an allocation θ on G, the following conditions are equivalent.*

(i). θ is balanced.

(ii). θ minimizes $\sum_{o \in V} \{\partial\theta(o)\}^2$.

(iii). θ minimizes $\sum_{o \in V} f(\partial\theta(o))$, for every convex function $f\colon [0,\infty) \to \mathbb{R}$.

It is clear from the quadratic formulation (ii) that balanced allocations always exist, and that they all induce the same loads $\partial\theta(o), o \in V$. The irrelevance of the convex function in (iii) confers

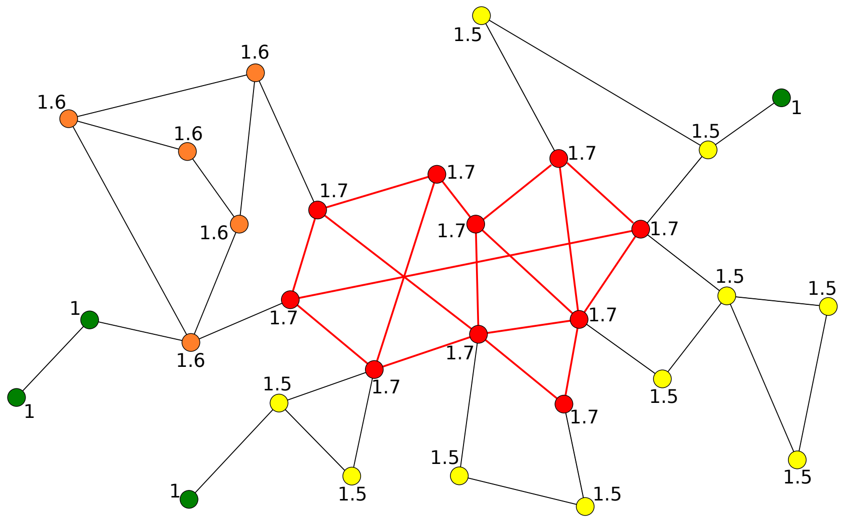


Figure 2.4: Balanced loads on a particular graph G, with the densest subgraph in red ($\varrho(G) = 1.7$).

an intrinsic value to these balanced loads: the number $\partial\theta(o)$ may safely be taken as a reliable graph-theoretical measure of the *local density* of G at o (see Figure 2.3). Indeed, it was observed in [79] that the set of maximum-load vertices $H = \operatorname{argmax} \partial\theta$ actually solves the densest subgraph problem (2.10), and that the value of the maximum load is the maximum subgraph density of G:

$$\varrho(G) \quad = \quad \max\{\partial\theta(o) \colon o \in V\}. \tag{2.12}$$

This surprising connection justifies a deeper study of balanced allocations on sparse graphs. To this end, it is convenient to encode the loads induced by some (every) balanced allocation θ on G into a probability measure on $\mathbb{R}$, which we call the *load profile* of G:

$$\Lambda_G \quad = \quad \frac{1}{|V|} \sum_{o \in V} \delta_{\partial\theta(o)}. \tag{2.13}$$

A conjecture of Hajek. Hajek [79] studied the behavior of Λ_G on the Erdős-Rényi model. In the regime where the average degree is kept fixed while the number n of vertices tend to infinity, he conjectured that Λ_{G_n} approaches a deterministic probability measure $\Lambda \in \mathcal{P}(\mathbb{R}_+)$ and that

$$\varrho(G_n) \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad \varrho \ := \ \sup\{t \geq 0 \colon \Lambda\left([t, +\infty)\right) > 0\}.$$

Using a non-rigorous analogy with the case of finite trees, Hajek even proposed a description of Λ and ϱ in terms of the solutions to a distributional fixed-point equation which will be given later. In our joint paper [P12], Anantharam and myself establish this conjecture and its extension to the more general configuration model $G_{\mathbf{d}}$, using the unifying framework of local weak convergence.

Research program. We carry out the following research program, which can be seen as a novel illustration of the general *objective method* developed by Aldous and Steele [10]:

1. Extend the notion of balanced allocations from finite graphs to their local weak limits.
2. Prove that the resulting load profile is continuous w.r.t. to local weak convergence.
3. Exploit the self-similarity of Galton-Watson trees to determine their load profile.
4. Relate the essential supremum of this profile to the asymptotic maximum subgraph density.

There is, however, a major obstacle to a direct application of the method here: the balanced load at a vertex is *not* determined by the local environment around it. For example, every ball of radius r in a $d-$regular graph with large girth is isomorphic to a $d-$regular tree with height r, but the induced load is $\frac{d}{2}$ on the $d-$regular graph, and $1-\frac{1}{(d-1)^{r-1}d}$ at the root of the tree. This long-range dependence gives rise to non-uniqueness issues when one tries to extend the notion of balanced loads to infinite graphs, making our problem non-trivial. We refer to [80] for a detailed study of this phenomenon – therein called *load percolation* – as well as several fascinating questions.

Balanced loads on unimodular graphs. Our idea to overcome this lack of correlation decay was to introduce a suitable relaxation of the balancing condition (2.11), which we call $\varepsilon-$balancing. The role of the perturbation parameter $\varepsilon>0$ is, in spirit, comparable to that of the activity z in [P5], although no Gibbs-Boltzmann measure is involved in the present case. Surprisingly, any positive value of ε suffices to annihilate the long-range dependences described above. More precisely, the mass $\Theta_\varepsilon(G,i,j)$ sent from i to j in the unique $\varepsilon-$balanced allocation on G is a continuous function of the edge-rooted graph (G,i,j). We then use unimodularity to prove that the continuous map $\Theta_\varepsilon\colon \mathcal{G}_{\star\star}\to[0,1]$ converges as $\varepsilon\to 0$, in a certain sense, to a measurable map $\Theta_0\colon \mathcal{G}_{\star\star}\to[0,1]$ with the following optimality property (the notation is that of Section 1.2):

Theorem 8 (Optimal load, [P12])**.** *For any measurable map $\Theta\colon\mathcal{G}_{\star\star}\to[0,1]$ such that $\Theta^\star+\Theta=1$, any convex function $f\colon\mathbb{R}_+\to\mathbb{R}_+$ and any measure $\mathcal{L}\in\mathcal{U}$ with $\int\deg(G,o)d\mathcal{L}(G,o)<\infty$,*

$$\int_{\mathcal{G}_\star}(f\circ\partial\Theta_0)\,d\mathcal{L}\quad\le\quad\int_{\mathcal{G}_\star}(f\circ\partial\Theta)\,d\mathcal{L}. \tag{2.14}$$

Continuity of the load profile. With our allocation scheme Θ_0 in hands, we may give a proper meaning to the load profile Λ_G when the finite graph G is replaced by a unimodular measure $\mathcal{L}\in\mathcal{U}$ with finite mean-degree: we simply define $\Lambda_{\mathcal{L}}\in\mathcal{P}(\mathbb{R}_+)$ as the push-forward of $\mathcal{L}$ through the measurable map $\partial\Theta_0$ (i.e., the law of $\partial\Theta_0(G,o)$ when $(G,o)\sim\mathcal{L}$). Note that we recover the original definition (2.13) when $\mathcal{L}=\mathcal{U}(G)$, by Lemma 7. As already noted, the map $\partial\Theta_0$ is not continuous at all points of $\mathcal{G}_\star$. However, the induced push-forward map $\mathcal{L} \mapsto \partial\Theta_0(\mathcal{L})$ can be shown to be continuous when restricted to unimodular measures with finite mean-degree:

Theorem 9 (Continuity, [P12])**.** *If $(G_n)_{n\geq 1}$ admits a local weak limit $\mathcal{L}$ with $\int \deg d\mathcal{L} < \infty$, then*

$$\Lambda_{G_n} \quad \xrightarrow[n\to\infty]{\mathcal{P}(\mathbb{R})} \quad \Lambda_{\mathcal{L}}.$$

Variational characterization. Recall that the *stop-loss transform* of a non-negative integrable random variable X (in fact, of its law Λ) is the function $\Psi_X = \Psi_\Lambda$ defined by

$$\Psi_X(t) \quad = \quad \mathbb{E}[(X-t)^+] \; = \int_{\mathbb{R}} (x-t)^+ d\Lambda(x) \qquad (t \geq 0).$$

This function plays a central role in the theory of *convex ordering* (see e.g. [122]), due to the classical equivalence between the conditions (i) $\Psi_X \leq \Psi_Y, \Psi_X(0) = \Psi_Y(0)$, and (ii) $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for every convex function $f\colon [0,\infty) \to \mathbb{R}$. In particular, Ψ_Λ determines Λ. Our next result is a variational characterization of the stop-loss transform of our load profile $\Lambda_{\mathcal{L}}$.

Theorem 10 (Variational formulation, [P12])**.** *Let $\mathcal{L} \in \mathcal{U}$ with $\int \deg d\mathcal{L} < \infty$. Then, for $t \geq 0$,*

$$\Psi_{\Lambda_{\mathcal{L}}}(t) \quad = \quad \max_{f\colon \mathcal{G}_\star \to [0,1]} \left\{ \frac{1}{2} \int_{\mathcal{G}_\star} (\partial \widehat{f})\, d\mathcal{L} - t \int_{\mathcal{G}_\star} f \, d\mathcal{L} \right\},$$

where $\widehat{f}(G,i,o) := f(G,o) \wedge f(G,i)$, and the maximum is taken over all measurable $f\colon \mathcal{G}_\star \to [0,1]$.

Resolution on Galton-Watson trees. As many graph-theoretical problems, load balancing has a simple recursive structure on trees: once the value along an edge $\{i,j\}$ has been fixed, the problem naturally decomposes into two independent sub-problems, corresponding to the two disjoint sub-trees formed by removing $\{i,j\}$. Note, however, that in the resulting sub-problems the loads of i and j must be shifted by a suitable amount to take into account the contribution of the removed edge. The precise effect of this shift on the final loads induced at i and j defines what we call the *response functions* of the two sub-trees, and those functions satisfy a recursive formula. As usual, recursions on trees give rise to distributional fixed point equations when specialized to Galton-Watson trees. In our case, the equation reads

$$Z_1 \overset{d}{=} \left(1 - \left(t - \sum_{j=1}^{\widehat{D}} Z_j\right)_+\right)_+, \tag{2.15}$$

where $\widehat{D}$ follows the size-biased distribution $\widehat{\pi}$ defined at (1.5), and where $(Z_k)_{k\geq 1}$ are i.i.d. and independent of $\widehat{D}$, with an unknown law. As conjectured by Hajek [79], the solutions to this distributional fixed point equation turn out to determine the value of $\Psi_{\Lambda_{\mathcal{L}}}(t)$. The latter can then be numerically evaluated, see [79] for detailed tables in the case where π is Poisson.

Theorem 11 (The Galton-Watson tree case, [P12]). *When $\mathcal{L} = \mathrm{UGWT}(\pi)$, we have for every $t \geq 0$:*

$$\Psi_{\Lambda_{\mathcal{L}}}(t) \quad = \quad \max\left\{\frac{\mathbb{E}[D]}{2}\mathbb{P}\left(Z_1 + Z_2 > 1\right) - t\mathbb{P}\left(Z_1 + \cdots + Z_D > t\right)\right\},$$

where $D \sim \pi$ and where the maximum range over the solutions $(Z_k)_{k\geq 1}$ to (2.15).

Back to the densest subgraph problem. Recalling the relation (2.12) between the maximum subgraph density $\varrho(G)$ and balanced allocations on G, we may finally define the maximum subgraph density $\varrho(\mathcal{L})$ of a measure $\mathcal{L} \in \mathcal{U}$ with finite mean degree as the essential supremum of the random variable $\partial\Theta_0(G, o)$ when $(G, o) \sim \mathcal{L}$. Equivalently, in terms of the stop-loss transform,

$$\varrho(\mathcal{L}) \quad := \quad \sup\{t \geq 0 \colon \Psi_{\Lambda_{\mathcal{L}}}(t) > 0\}.$$

In light of Theorem 9, it is natural to seek a continuity principle of the form

$$\left(G_n \xrightarrow[n\to\infty]{} \mathcal{L}\right) \quad \Longrightarrow \quad \left(\varrho(G_n) \xrightarrow[n\to\infty]{} \varrho(\mathcal{L})\right). \tag{2.16}$$

However, a moment of thought shows that the graph parameter $\varrho(G)$ is too sensitive to be captured by local weak convergence, due to the possible presence of extremely dense subgraphs with negligible size. On our random graph $G_{\mathbf{d}}$, we show that such a scenario is unlikely to occur as long as

$$\sum_{i=1}^{n} e^{\theta d_i} \quad = \quad \mathcal{O}(n), \tag{2.17}$$

for some fixed $\theta > 0$ (independent of n).

Theorem 12 (Maximum subgraph density of $G_{\mathbf{d}}$, [P12]). *Assume that $\mathbf{d}$ satisfies (2.17) for some $\theta > 0$, and that $\mathbf{d}$ approaches some distribution π in the sense of (1.3), with $\pi_0 + \pi_1 < 1$. Then,*

$$\varrho\left(G_{\mathbf{d}}\right) \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad \varrho(\mu_{\mathrm{UGWT}(\pi)}).$$

.

Our results apply in particular to the Erdős-Rényi model, where it settles the conjectures of [80] and validates the numerical tables for ϱ given therein. We also note that our Theorem 12 refines and extends recently obtained thresholds on the $k-$orientability of the Erdős-Rényi random graph [40, 65], since a graph G is $k-$*orientable* ($k \in \mathbb{Z}_+$) if and only if $\varrho(G) < k$.

Perspectives. We naturally expect (2.16) to fail in the regime where the degrees are heavy-tailed, due to the presence of extremely dense subgraphs with negligible size. Determining the asymptotic behavior of the maximum density in that regime certainly constitutes an interesting open problem.

2.4 Concave parameters [P13]

In this last section, we investigate the intriguing possibility of a *global* convergence theorem – much stronger than (1.6) – for our random graph $G_{\mathbf{d}}$. Let us call *graph parameter* any real-valued function defined on finite graphs and invariant under graph isomorphism. We shall restrict our attention to parameters that are *extensive*: if $\sqcup$ denotes the vertex-disjoint union, then

$$\mathfrak{f}(G \sqcup G') \quad = \quad \mathfrak{f}(G) + \mathfrak{f}(G'). \tag{2.18}$$

We will also require $\mathfrak{f}$ to be *smooth*, in the sense that its value changes by at most some constant κ upon addition of a new edge between two existing vertices i and j: for any choice of G, i, j,

$$|\mathfrak{f}(G+ij) - \mathfrak{f}(G)| \quad \leq \quad \kappa \tag{2.19}$$

These two conditions imply that $\mathfrak{f}(G) = \mathcal{O}(|V_G| + |E_G|)$; in the sparse regime, it is thus natural to normalize by the number of vertices and expect some kind of stabilization under the configuration model. An obvious requirement, without which even the most basic choice $\mathfrak{f}(G) = \#\{v\colon \deg(G,v) = k\}$ will not stabilize, is that the vertex degrees approach a limit distribution $\pi \in \mathcal{P}(\mathbb{Z}_+)$ in the sense of (1.3). In order for the trivial parameter $\mathfrak{f}(G) = |E(G)|$ to also stabilize, we should additionally require that π has a finite mean and that

$$\frac{1}{n}\sum_{i=1}^{n} d_i \quad \xrightarrow[n\to\infty]{} \quad \sum_{k=1}^{\infty} k\pi_k. \tag{2.20}$$

Note that (1.3) and (2.20) together are equivalent to requiring that the empirical degree distribution $\frac{1}{n}\sum_{i=1}^{n}\delta_{d_i}$ converges to π in the $1-$Wasserstein space $\mathcal{P}_1(\mathbb{Z}_+)$: this is the space of integrable probability measures on $\mathbb{Z}_+$, equipped with the Wasserstein (or Kantorovich–Rubinstein) distance

$$d_{\mathcal{W}}\left(\pi, \pi'\right) \quad = \quad \sum_{i=1}^{\infty}\left|\sum_{k\geq i}^{\infty}\left(\pi_k - \pi'_k\right)\right|. \tag{2.21}$$

Finding oneself unable to devise a single counter-example, one may venture the following question.

Question 13. *As* $d_{\mathcal{W}}\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{d_i}, \pi\right) \xrightarrow[n\to\infty]{} 0$, *does every extensive smooth parameter* $\mathfrak{f}$ *satisfy*

$$\frac{\mathfrak{f}(G_{\mathbf{d}})}{n} \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad \ell_{\mathfrak{f}}(\pi), \tag{2.22}$$

for some deterministic limit $\ell_{\mathfrak{f}}(\pi)$ *?*

This is widely open even in the special case where all degrees are equal to some fixed value (the regular case). An affirmative answer would constitute a far-reaching generalization of numerous convergence results discovered over the last decade, and would imply in particular the long-conjectured *right-convergence* of sparse random graphs [31, 37, 38, 70]. At a higher level, it would unveil a remarkable *structural regularity* among sparse graphs, whereby "most" graphs with the same degree distribution are close to a certain *infinite-volume limit*, in a deep and global sense. Motivated by this fascinating question, I introduced in [P13] the notion of *concavity* for graph parameters, and proved that the convergence (2.22) holds under this additional requirement. The class of concave parameters is broad enough to include several intrinsically complex invariants, including the independence number, the maximum cut size, the logarithm of the Tutte polynomial, or the free energy of the anti-ferromagnetic Ising and Potts models.

Concave parameters. A real symmetric matrix $\Delta = (\Delta_{ij})_{1\le i,j\le n}$ is called *conditionally negative semi-definite* (CND) if its restriction to zero-sum vectors is negative semi-definite, that is,

$$\left(\sum_{i=1}^n x_i = 0\right) \implies \left(\sum_{1\le i,j\le n} \Delta_{ij} x_i x_j \le 0\right).$$

This weaker form of negative semi-definiteness has been studied in detail, due to its intimate connection with the classical notion of infinite divisibility. We only mention two alternative formulations, and refer to the book [13, Chapter 4] for more details and a plethora of additional results.

Lemma 14. *For a $n\times n$ real matrix $\Delta = (\Delta_{ij})_{1\le i,j\le n}$, the following three conditions are equivalent:*

1. *The matrix Δ is* CND
2. *The matrix $(e^{-\lambda\Delta_{ij}})_{1\le i,j\le n}$ is positive definite for all $\lambda > 0$.*
3. *The matrix $(\alpha_i + \alpha_j - \Delta_{ij})_{1\le i,j\le n}$ is positive semi-definite for some $(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n$.*

Moreover, a sufficient condition for the matrix Δ to be CND *is*

4. *The matrix $(e^{\Delta_{ij}})_{1\le i,j\le n}$ is* CND.

We may now introduce our notion of concavity: we call a graph parameter $\mathfrak{f}$ *concave* if for any finite graph G, the $V_G\times V_G$ matrix $(\mathfrak{f}(G+ij)-\mathfrak{f}(G))$ is CND, where $G+ij$ is obtained from G by adding a new edge between i and j. Before giving examples, let us state the main result of [P13].

Theorem 15 ([P13])**.** *Any extensive, smooth and concave graph parameter $\mathfrak{f}$ admits an "infinite volume limit" $\ell_{\mathfrak{f}}\colon \mathcal{P}_1(\mathbb{Z}_+)\to\mathbb{R}$ in the sense that*

$$\frac{\mathfrak{f}(G_{\mathbf{d}})}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \ell_{\mathfrak{f}}(\pi).$$

for every $\pi\in\mathcal{P}_1(\mathbb{Z}_+)$ and every degree sequence $\mathbf{d}$ verifying $d_{\mathcal{W}}\left(\frac{1}{n}\sum_{i=1}^n \delta_{d_i},\pi\right)\to 0$. The convergence actually holds almost-surely under any coupling of the underlying random graphs.

The existence of $\ell_{\mathfrak{f}}(\pi)$ for any choice of the asymptotic degree distribution $\pi \in \mathcal{P}_1(\mathbb{Z}_+)$ allows us to investigate the variational properties of the infinite volume limit as a function of π. We establish the following result, which justifies our choice for the terminology.

Theorem 16 ([P13])**.** *For $\mathfrak{f}$ as in Theorem 15, the infinite volume limit $\ell_{\mathfrak{f}} \colon \mathcal{P}_1(\mathbb{Z}_+) \to \mathbb{R}$ is always*

1. *Lipschitz-continuous: for all $\pi, \pi' \in \mathcal{P}_1(\mathbb{Z}_+)$,*

$$\left|\ell_{\mathfrak{f}}(\pi) - \ell_{\mathfrak{f}}(\pi')\right| \quad \leq \quad 2\kappa \, d_{\mathcal{W}}(\pi, \pi').$$

2. *Concave: for all $\pi, \pi' \in \mathcal{P}_1(\mathbb{Z}_+)$ and $0 \leq \theta \leq 1$,*

$$\theta \ell_{\mathfrak{f}}(\pi) + (1-\theta)\ell_{\mathfrak{f}}(\pi') \quad \leq \quad \ell_{\mathfrak{f}}(\theta\pi + (1-\theta)\pi').$$

Examples. Many important graph parameters (or their negative when indicated by the symbol $\star$) happen to be extensive, smooth and concave. Here are a few examples.

- $\star$ *Number of connected components*: in this case we clearly have $\mathfrak{f}(G+ij) - \mathfrak{f}(G) = -\mathbf{1}_{i \not\leftrightarrow j}$ where $\leftrightarrow$ denotes the reachability relation in G. Since $\leftrightarrow$ is an equivalence relation, its incidence matrix $(1_{i \leftrightarrow j})$ is positive semi-definite, so the incidence matrix of its negation $(1_{i \not\leftrightarrow j})$ is CND.

- $\star$ *Decycling number*: the decycling number of a graph is the smallest possible size of a set $S \subseteq V$ whose removal makes the graph acyclic. It satisfies $\mathfrak{f}(G+ij) - \mathfrak{f}(G) = \mathbf{1}_{i \leftrightarrow j}$, where $i \leftrightarrow j$ if and only if i and j belong to the same tree in every maximum-size induced forest.

- *Independence number*: the independence number of a graph is the maximum possible size of an independent set (i.e., a subset of vertices, no two of which are neighbors). The concavity is clear since $\mathfrak{f}(G+ij) - \mathfrak{f}(G) = -\left(\mathbf{1}_S \mathbf{1}_S^\top\right)_{ij}$, where S is the intersection of all maximum independent sets on G, and $\mathbf{1}_S$ denotes its incidence vector.

- *Maximum cut size*: a cut in G is just an arbitrary bipartition $S \cup S^c$ of its vertex set, and its size is the number of cross-edges $|E(S, S^c)|$. The maximum possible size satisfies $\mathfrak{f}(G+ij) - \mathfrak{f}(G) = \mathbf{1}_{i \not\leftrightarrow j}$, where $i \leftrightarrow j$ if i and j lie on the same side of each maximum cut.

- *Free energies*: fix a finite set S (*spins*), a function $h \colon S \to (0, +\infty)$ (*potential*) and a symmetric function $J \colon S \times S \to (0, +\infty)$ (*interactions*), and consider the graph parameter

$$\mathfrak{f}(G) \quad := \quad \log\left(\sum_{\sigma \colon V \to S} w(\sigma)\right) \text{ where } w(\sigma) = \prod_{i \in V} h(\sigma_i) \prod_{ij \in E} J(\sigma_i, \sigma_j).$$

$\mathfrak{f}$ is easily seen to be extensive and smooth, and a sufficient condition for it to be concave is that the interaction matrix J itself is CND. Indeed, this implies that the matrix $(J(\sigma_i,\sigma_j))_{(i,j)\in V\times V}$ is CND for each fixed $\sigma\colon V\to S$, and hence so is the convex combination

$$e^{\mathfrak{f}(G+ij)-\mathfrak{f}(G)} \quad = \quad \frac{\sum_\sigma w(\sigma)J(\sigma_i,\sigma_j)}{\sum_\sigma w(\sigma)}.$$

We may then use the last condition in Lemma 14 to conclude. In particular, the free energy of the *Ising model* ($S=\{-1,+1\}$, $J(s,t)=e^{\beta st}$) and *Potts model* ($S=\{1,\ldots,q\}$, $J(s,t)=\mathbf{1}_{s\neq t}+e^{\beta}\mathbf{1}_{s=t}$) are concave for any $\beta\le 0$ (the so-called anti-ferromagnetic regime).

- *Tutte polynomial:* fix $x,y>0$ and consider the graph parameter

$$\mathfrak{f}(G) \quad := \quad \log\left(\sum_{A\subseteq E} x^{c(A)}y^{|A|}\right),$$

where $c(A)$ denotes the number of connected components of the graph (V,A). If $x\le 1$, then $\mathfrak{f}$ is concave, as can be seen from the expression

$$e^{\mathfrak{f}(G+ij)-\mathfrak{f}(G)} \quad = \quad \frac{\sum_{A\subseteq E} x^{c(A)}y^{|A|}\left(1+y+y\left(x^{-1}-1\right)\mathbf{1}_{i\overset{A}{\not\leftrightarrow} j}\right)}{\sum_{A\subseteq E} x^{c(A)}y^{|A|}},$$

using a similar argument as above (here, $i\overset{A}{\leftrightarrow} j$ denotes reachability in (V,A)).

- *Spanning forests:* fix $x\ge 0$ and consider the graph parameter

$$\mathfrak{f}(G) \quad := \quad \log\left(\sum_{A\subseteq E\text{ acyclic}} x^{|A|}\right),$$

where the sum runs over all $A\subseteq E$ such that (V,A) is acyclic; $\mathfrak{f}$ is concave, as above.

Near super-additivity. The main technical ingredient in [P13] is the discovery of a general super-additive property satisfied by concave graph parameters when evaluated on the configuration model. Let $\mathbf{d}=(d_1,\ldots,d_n)$ be an arbitrary degree sequence and let $[n]=A\cup B$ be a bipartition of $[n]$. Write $\mathbf{d}|A$ and $\mathbf{d}|B$ for the restricted sequences $(d_i)_{i\in A}$ and $(d_i)_{i\in B}$, respectively. Then, for any extensive, smooth and concave graph parameter $\mathfrak{f}$, we have

$$\mathbb{E}\left[\mathfrak{f}\left(G_{\mathbf{d}}\right)\right] \quad \ge \quad \mathbb{E}\left[\mathfrak{f}\left(G_{\mathbf{d}|A}\right)\right]+\mathbb{E}\left[\mathfrak{f}\left(G_{\mathbf{d}|B}\right)\right]-7\kappa\sqrt{m\ln(1+m)}. \tag{2.23}$$

where $m=\frac{1}{2}\sum_{i=1}^n d_i$ is the number of edges. Once combined with the concentration inequality

$$\mathbb{P}\left(|\mathfrak{f}(G_{\mathbf{d}})-\mathbb{E}[\mathfrak{f}(G_{\mathbf{d}})]|\ge\varepsilon\right) \quad \le \quad \exp\left(-\frac{\varepsilon^2}{8\kappa^2 m}\right), \tag{2.24}$$

valid for any $\varepsilon > 0$, this easily leads to the existence of the limit in Theorem 15. The inequality (2.24) is classical, and we refer to, e.g., [127, Theorem 2.19] for a proof in the regular case, and to [33, Corollary 3.27] for the general case. The crucial near super-additive property (2.23) on the other hand, is new, and its proof relies on a discrete interpolation between $G_{\mathbf{d}}$ and the disjoint reunion $G_{\mathbf{d}|A} \sqcup G_{\mathbf{d}|B}$. Roughly speaking, we condition $G_{\mathbf{d}}$ to have a certain number t of *cross-edges* (edges having one end-point in A and the other in B), and use the concavity assumption to control the expected change of $\mathfrak{f}$ as our interpolation parameter decreases from t to $t-1$. Integrating over t yields (2.23). This is inspired from an idea introduced in the context of spin glasses to prove the existence of an infinite volume limit for free energies of certain interacting particle systems: the so-called *interpolation method*.

The interpolation method. A decade ago, Guerra and Toninelli [76] introduced a powerful method to prove the existence of an infinite volume limit for the free energy of the classical *Sherrington–Kirkpatrick model* from statistical physics. The argument is based on an ingenious interpolation scheme which allows a system of size n to be compared with two similar but independent systems of sizes n_1 and n_2 respectively, where $n_1 + n_2 = n$. The quantity of interest turns out to be super-additive with respect to n, hence convergent once divided by n. We refer to the recent books [116, 123] for more details. This technique was then transferred from the fully-connected regime to its diluted counterpart, where each particle only interacts with a bounded number of neighbors, see [66, 67]. In a recent breakthrough [18], the interpolation method was applied to several other important models from statistical physics, including, among others, the *Ising model*, the *Potts model* and the *hard-core model*. Connections with the question of *right-convergence* of sparse random graphs (a special case of Question 13) were put forward in [37, 70]. Our proof of Theorem 15 extends the applicability of the interpolation method from the specific context of free energies of certain statistical physics models to the general class of concave graph parameters.

Perspectives. Convergence results based on super-additivity are intrinsically non-constructive. From his original interpolation scheme however, Guerra subsequently managed to extract a tight lower bound on the free energy of the Sherrington-Kirkpatrick model [75]. It would be extremely interesting to extend this to our concave graph parameters. In a different direction, the conclusion of Theorem 15 extends by linearity to any difference of two extensive, smooth and concave graph parameters. The latter clearly remain extensive and smooth, and it is natural – yet perhaps too optimistic? – to wonder whether any extensive and smooth graph parameter can actually be obtained in this way. A positive answer would settle Question 13. In any case, characterizing the parameters that may be written as such a difference is a worthy problem, in light of Theorem 15.

CHAPTER 3

Dynamical aspects

We now move "up" from the analysis of random graphs themselves to the study of random processes defined on these graphs. We first analyze a popular message-passing algorithm for solving the assignment problem with random costs. We then consider first-passage percolation and random walks. Understanding the long-term behavior of these processes on a fixed infinite graph (say, the d−dimensional lattice $\mathbb{Z}^d$) has now become a classical topic in modern probability theory. Contrastingly, very little has been said about these dynamics on large but finite random graphs. The double layer of randomness and the finiteness of the state-space give rise to a rich variety of phenomena and to questions of a different (more quantitative) nature, with direct applications to real-world networks.

3.1 The random assignment problem [P1, P2, P8]

The *assignment problem* is one of the fundamental problems in combinatorial optimization. Given a $n \times n$ real matrix $X = (X_{i,j})_{1 \le i,j \le n}$, it asks for a permutation σ of $[n]$ that achieves the minimum

$$M_n \quad := \quad \min_{\sigma} \left\{ \sum_{i=1}^{n} X_{i,\sigma(i)} \right\}. \tag{3.1}$$

One may think of $X_{i,j}$ has the *cost* of assigning a task i to an agent j, the aim being to allocate each task to a different agent in such a way that the overall cost is as small as possible. In a more graph-theoretical language, M_n is the minimum length of a perfect matching on the complete bipartite graph $K_{n,n}$ with edge-lengths $(X_{i,j})_{1 \le i,j \le n}$. The asymptotic behavior of M_n on large random instances has received considerable attention, see e.g. [49, 60, 74, 91, 124]. Using the celebrated *replica symmetry ansatz* from statistical physics, Mézard and Parisi [108, 109, 110] made the following remarkable prediction when the $X_{i,j}$ are independent, say, uniform on $[0,1]$:

$$M_n \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad \frac{\pi^2}{6}. \tag{3.2}$$

This was finally established by Aldous in [4], leading to the formalism of the *objective method* [10].

The Poisson Weighted Infinite Tree. The central idea in [4] was to observe that the graph $K_{n,n}$ equipped with the random edge-lengths $(nX_{i,j})_{1\le i,j\le n}$ converges in an appropriate, local sense to a limit called the *Poisson Weighted Infinite Tree* (PWIT), defined as follows: consider the infinite tree $\mathcal{T} = (V,E)$ with V the set of all finite words over the alphabet $\mathbb{N} = \{1,2,\ldots\}$ and $E = \{\{v,vi\}, v\in V, i\ge 1\}$. Assign a random length $\xi_{v,i}$ to each edge $\{v,vi\}\in E$ where, independently for each node $v\in V$, $\xi_{v,1}\ge\xi_{v,2}\ge\ldots$ are the ordered points of a Poisson point processes with intensity 1 on $\mathbb{R}^+$. Aldous used the convergence $K_{n,n}\to\mathcal{T}$ to replace the asymptotic study of the minimum-length perfect matching problem on $K_{n,n}$ with the direct analysis of a suitably defined optimization problem on $\mathcal{T}$. The self-similar structure of the PWIT allowed him to reduce this infinite-dimensional problem to the resolution of a single distributional fixed point equation:

$$Z \quad \overset{d}{=} \quad \min_{i\ge 1}\left\{\xi_i - Z_i\right\}, \tag{3.3}$$

where $\{\xi_i\}_{i\ge1}$ is a Poisson point process with intensity 1 on $\mathbb{R}^+$ and $(Z_i)_{i\ge1}$ are i.i.d. with the same law as Z, independent of $\{\xi_i\}_{i\ge1}$. Equivalently, $f(x) = \mathbb{P}\left(Z > x\right)$ is a fixed point of the operator

$$(Tf)(x) \quad = \quad \exp\left(-\int_{-x}^{+\infty} f(y)dy\right). \tag{3.4}$$

The logistic distribution $f(x) = (1+e^x)^{-1}$ is easily seen to be the only solution, and this fact quickly leads to (3.2). Since then, several alternative proofs have been found [97, 114, 125].

Belief-Propagation. The problem of actually computing the minimizer σ in (3.1) has been extensively studied, and its consideration laid foundations for the rich theory of network flow algorithms [61, 90]. The non-rigorous approach underlying (3.2) suggested the following simple heuristics: construct $n\times n$ matrices $M^0, M^1,\ldots$ and $N^0,N^1,\ldots$ by setting $M^0=N^0=0$ and

$$M^{t+1}_{j,k} \;:=\; \min_{i\neq k}\left\{X_{i,j} - N^t_{i,j}\right\} \quad\text{and}\quad N^{t+1}_{j,k} \;:=\; \min_{i\neq k}\left\{X_{j,i} - M^t_{i,j}\right\}. \tag{3.5}$$

Then use the matrix M^t to define an approximate solution $\sigma^t\colon [n]\to[n]$ as follows:

$$\sigma^t(j) \quad := \quad \arg\min_{i}\left\{X_{j,i} - M^t_{i,j}\right\}. \tag{3.6}$$

The number $M_{j,k}^t$ may be thought of as a *message*, or *belief* sent from the jth task to the kth agent, and conversely for $N_{j,k}^t$; these messages propagate along the edges of the bipartite graph $K_{n,n}$, using the local rule (3.5). This is an instance of the popular *Belief-Propagation* (BP) heuristic (see, e.g. [118, 128]). Here the convergence and correctness were established by Bayati, Shah and Sharma:

Theorem 17 ([19]). *If the minimum (3.1) is achieved at a unique $\sigma^\star \in \mathfrak{S}_n$, then $\sigma^t \xrightarrow[t\to\infty]{} \sigma^\star$.*

BP on the PWIT. Motivated by the remarkable empirical performances of BP, Shah and myself studied the typical rate of convergence in Theorem 17 on large random instances [P1, P2]. Exploiting the convergence $K_{n,n} \to \mathcal{T}$, we proved that for any fixed $t \geq 0$, the messages M^t, N^t converge as $n \to \infty$, in a suitable sense, to those produced by running BP directly on the PWIT. Note that on the latter, the message Z_v^t sent at time t from a vertex v to its father satisfies

$$Z_v^{t+1} = \min_{i\geq 1}\left\{\xi_{v,i} - Z_{vi}^t\right\} \quad \text{and} \quad Z_v^0 = 0.$$

Understanding the long-time behavior of this process involves the study of the dynamical system $(T^t f)_{t\geq 0}$ induced by the operator (3.4), and was listed as an open Problem by Aldous and Bandyopadhyay [6, Problem 62]. We solved it completely and deduced that the map σ^t obtained by applying the rule (3.6) on the PWIT converges, as $t \to \infty$, to the optimum $\sigma^\star$ constructed by Aldous. As a result, we obtained the following quantitative version of Theorem 17. Roughly speaking, it asserts that the typical number of iterations needed for BP to solve the $n \times n$ random assignment problem within any given precision $\varepsilon > 0$ remains of order 1 as $n \to \infty$.

Theorem 18 ([P1, P2]). *For each fixed $t \geq 0$, the fraction of wrong assignments made by BP after t iterations on independent, uniform random costs $(X_{i,j})_{1\leq i,j\leq n}$ satisfies*

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{(\sigma^t(i)\neq\sigma^\star(i))} \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad \delta(t),$$

for some (explicit) deterministic function $\delta\colon \mathbb{Z}_+ \to [0,1]$ such that $\delta(t) \to 0$ as $t \to \infty$.

Universality. The original prediction by Mézard and Parisi actually concerned the more general setting where the costs $X_{i,j}$ are i.i.d. samples from an arbitrary distribution satisfying

$$\mathbb{P}\left(X_{i,j} \leq x\right) \sim x^{\mathfrak{d}} \quad \text{as } x \to 0^+, \tag{3.7}$$

for some exponent $0 < \mathfrak{d} < \infty$ called the *pseudo-dimension*. Specifically, they conjectured that

$$\frac{M_n}{n^{1-1/\mathfrak{d}}} \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad -\mathfrak{d}\int_{\mathbb{R}} f(x)\ln f(x)dx, \tag{3.8}$$

where the function $f\colon \mathbb{R} \to [0,1]$ solves the so-called *cavity equation*:

$$f(x) \quad = \quad \exp\left(-\int_{-x}^{+\infty} \mathfrak{d}(x+y)^{\mathfrak{d}-1} f(y)dy\right). \tag{3.9}$$

In the special case $\mathfrak{d} = 1$, the term $(x+y)^{\mathfrak{d}-1}$ simplifies and (3.9) becomes exactly solvable, yielding (3.2). When $\mathfrak{d} \neq 1$ however, showing that there is a unique solution is listed as an open problem by Aldous and Bandyopadhyay [6, Problem 63]. The answer is crucial, as uniqueness is the only ingredient that misses for the proof of Aldous to carry over to pseudo-dimensions $\mathfrak{d} \neq 1$. Indeed, the graph $K_{n,n}$ equipped with edge-lengths $(n^{2-\frac{1}{\mathfrak{d}}} X_{i,j})$ now converges to a PWIT with edge-lengths given by a Poisson point process with intensity $\mathfrak{d}x^{\mathfrak{d}-1}\,dx$ on $\mathbb{R}_+$, and substituting in equation (3.3) yields exactly (3.9). In a paper published in the Annals of Mathematics, Wästlund [126] circumvented the uniqueness issue for $\mathfrak{d} > 1$ by considering the truncated equation

$$f_\lambda(x) \quad = \quad \exp\left(-\int_{-x}^{\lambda} \mathfrak{d}(x+y)^{\mathfrak{d}-1} f_\lambda(y)dy\right), \qquad 0 < \lambda < \infty. \tag{3.10}$$

Using an ingenious game-theoretical interpretation, he showed the existence of a unique, global attractive solution $f_\lambda\colon [-\lambda, \lambda] \to [0,1]$ for every $0 < \lambda < \infty$, and used this to establish that

$$\frac{M_n}{n^{1-1/\mathfrak{d}}} \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad \lim_{\lambda\to+\infty} \uparrow -\mathfrak{d}\int_{-\lambda}^{\lambda} f_\lambda(x)\ln f_\lambda(x)dx. \tag{3.11}$$

Wästlund explicitly left open the problem of completing the proof of the original Mézard-Parisi prediction by showing (i) that the original cavity equation (3.9) admits a unique solution f and (ii) that $f_\lambda \to f$ as $\lambda \to \infty$. I established this conjecture in the short work [P8].

Theorem 19 ([P8])**.** *For any dimension $\mathfrak{d} \geq 1$, the Mézard-Parisi equation (3.9) admits a unique solution $f\colon \mathbb{R} \to [0,1]$. Moreover, $f_\lambda \to f$ point-wise as $\lambda \to +\infty$, and*

$$\int_{-\lambda}^{\lambda} f_\lambda(x)\ln f_\lambda(x)dx \quad \xrightarrow[\lambda\to+\infty]{} \quad \int_{\mathbb{R}} f(x)\ln f(x)dx.$$

Consequently, the two limits in (3.8) and (3.11) coincide.

In addition, [P8] provides a short alternative proof of the crucial result of Wästlund that the truncated equation (3.10) admits a unique, attractive solution. We note that the case $\mathfrak{d} < 1$ remains completely open. Beyond the random assignment problem, *recursive distributional equations* such as (3.3) arise naturally in a variety of models from statistical physics, and the question of uniqueness of solutions plays a crucial role for the rigorous understanding of those models. We refer to the comprehensive survey [6] for more details and many open problems. In particular, [6, Section 7.4] contains a detailed discussion on equation (3.9). Distributional equations for other mean-field combinatorial optimization problems have been analyzed in e.g. [71, 94, 117].

3.2 First-passage percolation [P7]

Setting. Assigning independent exponential random lengths $(\ell_e)_{e\in E}$ to the edges of a graph $G = (V, E)$ induces a natural random metric on the vertex set V, namely

$$\mathcal{D}_{i,j} = \min \left\{ \sum_{e\in\mathfrak{p}} \ell_e : \mathfrak{p} \text{ is a path between } i \text{ and } j \right\}.$$

A now classical problem is that of understanding the asymptotic shape of a ball $\{j \in V : \mathcal{D}_{i,j} \le t\}$ as $t \to \infty$, when G is a fixed infinite graph (see e.g. [51, 93]). More recently, several authors have studied the way in which random edge-lengths affect the inherent geometry of large finite graphs and, in particular, their typical distances and diameter [7, 11, 24, 25, 26, 27, 59, 85, 86, 87]. One particularly striking result is a remarkable second-order behaviour established by Bhamidi, van der Hofstad and Hooghiemstra for the distance between two typical points on $G_{\mathbf{d}}$. Although the result holds for an arbitrary prescribed degree distribution with finite variance, we will restrict ourselves to the regular case for the sake of simplicity. Fix an integer $d \ge 3$ and, for $n \ge 1$ such that dn is even, consider the random regular graph produced by the configuration model with n equal degrees $(d, \ldots, d)$. Conditionally on the graph, assign independent unit-rate exponential random lengths to the edges. It was shown in [26] that the distance between two fixed nodes – say 1 and 2 – satisfies

$$\mathcal{D}_{1,2} - \frac{\log n}{d-2} \xrightarrow[n\to\infty]{d} W, \tag{3.12}$$

for some non-degenerate limit W with the following explicit law: $W = X + X' + Y$, where X, X', Y are independent with densities on $\mathbb{R}$ given respectively by

$$f_X(u) = f_{X'}(u) = \frac{d-2}{\Gamma(\frac{d}{d-2})} e^{-du} e^{-e^{-(d-2)u}} \tag{3.13}$$

$$f_Y(u) = 2de^{(d-2)u} e^{-\frac{1}{2d(d-2)} e^{(d-2)u}}. \tag{3.14}$$

A reasonable hope. Two-point marginals do not capture correlations between distances, and it is natural to look for a more refined description of the relative geometry of the network. In light of the results obtained by Aldous and Bhamidi on the complete graph [7, 24], a reasonable hope is that the whole array of re-centered distances converges in distribution to some infinite random array $\mathbf{W} = \{W_{i,j} : 1 \le i < j < \infty\}$ in the usual product-topology sense, i.e. for each fixed $k \ge 1$,

$$\left\{ \mathcal{D}_{i,j} - \frac{\log n}{d-2} : 1 \le i < j \le k \right\} \xrightarrow[n\to\infty]{d} \{W_{i,j} : 1 \le i < j \le k\}. \tag{3.15}$$

Note that the limit $\mathbf{W}$, if it exists, must automatically be *partially exchangeable* in the sense described at (2.2) of the survey paper [5], which discusses a much broader picture of representing complex random structures via induced substructures on randomly-chosen points. In addition, the single marginals must coincide with the one given above. A simple candidate would then be

$$W_{i,j} = X_i + X_j + Y_{i,j}, \tag{3.16}$$

where $\{X_i : i \geq 1\} \cup \{Y_{i,j} : 1 \leq i < j < \infty\}$ is a collection of mutually independent random variables, with each X_i admitting the density (3.13) and each $Y_{i,j}$ admitting the density (3.14). Note that such a simplistic decomposition would admit the following elegant interpretation, illustrated on Figure 3.1: the random variable X_i may be thought of as a "local cost" to escape (or enter) the neighborhood of node i, regardless of the destination (or source), and the random variable $Y_{i,j}$ as a "transfer cost" to switch from the peripheral neighborhood of i to that of j. One of my

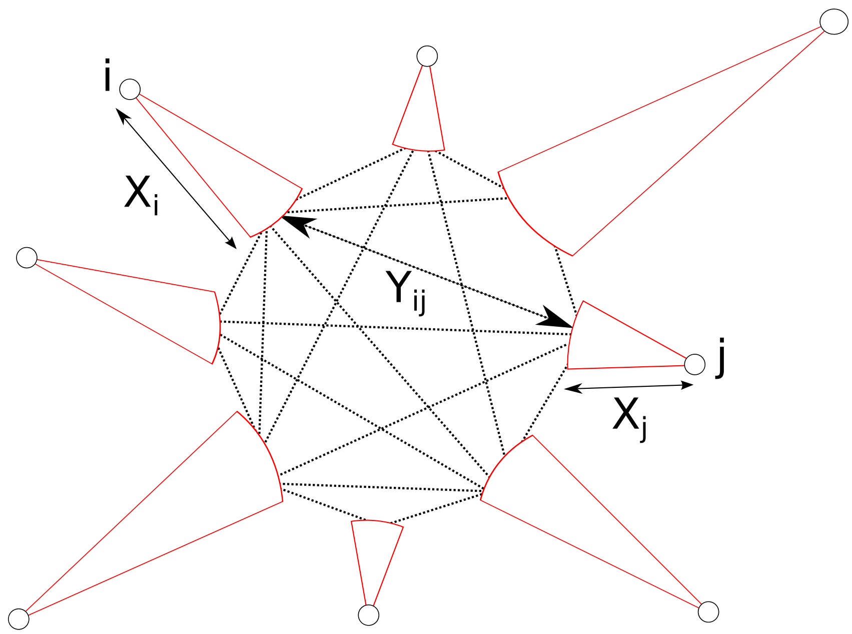


Figure 3.1: The typical relative geometry of a sparse random network

post-doctoral works was devoted to establishing this remarkably simple picture:

Theorem 20 ([P7])**.** *For the $d-$regular graph generated by the configuration model on $[n]$, with independent mean-1 exponential edge-lengths, we have*

$$\left\{\mathcal{D}_{i,j} - \frac{\log n}{d-2} : 1 \leq i < j \leq n\right\} \xrightarrow[n\to\infty]{d} \mathbf{W} = \{W_{i,j} : 1 \leq i < j < \infty\},$$

where $\mathbf{W}$ *is the partially exchangeable random array defined by (3.16).*

Simple graphs. As explained in the introduction, the configuration model was originally introduced as a device to simplify the analysis of the uniform $d-$regular simple graph on $[n]$: with probability bounded away from 0 as $n \to \infty$, the $d-$regular graph produced by the configuration model is simple, and conditioning on that event, it is uniformly distributed among all simple $d-$regular graphs on $[n]$. As a result, any event that occurs w.h.p. under the configuration model also does so under the uniform model [30]. However, this does not directly carry over to weak convergence results such as Theorem 20. Here we exploit the independence inherent to the limiting array $\mathbf{W}$ to transfer our main result to the uniform $d-$regular simple graph as well.

Corollary 21 ([P7]). *For the uniform simple $d-$regular graph with exponential edge-lengths,*

$$\left\{\mathcal{D}_{i,j} - \frac{\log n}{d-2} : 1 \le i < j \le n\right\} \xrightarrow[n\to\infty]{d} \mathbf{W} = \{W_{i,j} : 1 \le i < j < \infty\},$$

where $\mathbf{W}$ is the partially exchangeable random array described above.

The Richardson process. Our proof of Theorem 20 consists in analyzing the invasion of the network by k mutually exclusive flows emanating from different sources and propagating simultaneously at unit speed along the edges. Since the edge-lengths are independent exponential random variables, this is exactly the multi-type Richardson process introduced by Häggström and Pemantle [77] as a model for competing spatial growth. A considerable amount of work has been devoted to understanding the long-time behavior of this process on lattices and other fixed infinite graphs [53, 54, 55, 72, 78, 82, 83]. Our concern here is quite different, since we are interested in the second-order behavior of the times at which the various species collide on a large but finite random graph. More recently, a version of the Richardson process was analyzed on random regular graphs by Antunović, Dekel, Mossel and Peres [12].

Perspectives. As already mentioned, the two-point analysis conducted in [26] is by no means restricted to the regular case, and there is no doubt that the multi-point refinement established here will extend to general degree distributions. However, several technical hurdles arise in the non-regular case, and it would be pleasant to find a neat way of dealing with them without substantially lengthening the proof. More importantly, the assumption that the edge-lengths are exponentially distributed is restrictive from the point of view of modeling delays or transportation costs in real-world networks. Bhamidi, van der Hofstad and Hooghiemstra [23] recently managed to extend the results of [26] to arbitrarily distributed edge-lengths. It would be interesting to establish the joint convergence of the whole array of distances in this general setting. For recent, closely related results as well as many open problems, see [28].

3.3 Non-backtracking random walk [P14]

The cutoff phenomenon. Any Markov chain $(X_0, X_1, \ldots)$ with irreducible, aperiodic transition matrix P on a finite state space $\mathcal{X}$ approaches its unique invariant distribution $\pi = \pi P$ as the number of iterations grows, regardless of the initial condition (see, e.g., the book [96]). A natural way of quantifying this convergence is through the worst-case *total-variation distance*:

$$\mathcal{D}(t) \quad = \quad \max_{x \in \mathcal{X}} \|P^t(x, \cdot) - \pi(\cdot)\|_{\mathrm{TV}} \;\in\; [0,1]. \tag{3.17}$$

Understanding the rate at which this function of t decreases from 1 to 0 is an important theoretical problem, with applications in a broad variety of contexts (Monte-Carlo methods, Metropolis-Hastings algorithms, etc). This question is particularly meaningful when the number of states becomes large. One is thus naturally led to consider a *family* of Markov chains indexed by a size parameter n, and to investigate the asymptotic behavior of the associated distances $(\mathcal{D}_n)_{n\ge1}$ in the $n \to \infty$ limit. In certain cases, a critical time-scale $(t_n)_{n\ge1}$ can be identified, at which the chain undergoes a remarkably abrupt transition from *out-of-equilibrium* to *equilibrium*:

$$\mathcal{D}_n\left(\lfloor \lambda t_n \rfloor\right) \quad \xrightarrow[n\to\infty]{} \quad \begin{cases} 1 & \text{if } \lambda < 1 \\ 0 & \text{if } \lambda > 1. \end{cases} \tag{3.18}$$

This is the celebrated *cutoff phenomenon*, discovered by Aldous, Diaconis and Shahshahani in the context of card shuffling [3, 8, 58], and now established for several chains arising in various settings, from random walks on groups to interacting particle systems [42, 56, 120]. Despite many efforts, the general conditions underpinning this phase transition are still very far from being understood.

Window and profile. In situations where (3.18) holds, it is natural to try and *zoom-in* around the cutoff point t_n until the details of the abrupt transition from 1 to 0 become visible: one looks for a suitable scale $(w_n)_{n\ge1}$, with $w_n \ll t_n$ as $n \to \infty$, such that for each $\lambda \in \mathbb{R}$,

$$\mathcal{D}_n\left(\lfloor t_n + \lambda w_n \rfloor\right) \quad \xrightarrow[n\to\infty]{} \quad \Phi(\lambda), \tag{3.19}$$

where $\Phi\colon \mathbb{R} \to [0,1]$ decreases from $\lim_{-\infty} \Phi = 1$ to $\lim_{+\infty} \Phi = 0$. When it exists, this limit shape – called the *cutoff profile* – captures, in an extremely precise sense, the way in which the chain approaches its equilibrium. Establishing this second-order refinement of (3.18) generally requires a deep understanding of the underlying dynamics, and has so far been restricted to a very limited list of examples, typically enjoying a high enough degree of symmetry and structure to allow for an explicit evaluation of the distance (3.17) (e.g., the transition matrix P admits a simple eigen-decomposition, the equilibrium π is the uniform law on $\mathcal{X}$, the dynamics is time-reversible, etc).

Non-backtracking random walk. One important class of Markov chains for which the above questions are particularly relevant is that of random walks on graphs. The time it takes for a walk to *forget* its initial state is an excellent gauge for an array of properties of the underlying geometry: distances between vertices, local traps, global bottlenecks, isoperimetric profile, expansion, etc. Moreover, the ability of random walks to quickly approximate the equilibrium law on large graphs plays a prominent role in many modern applications to real-world networks, such as exploration [45] and ranking [43] in the World Wide Web. These practical implications have motivated the mathematical analysis of random walks on *sparse random graphs* [20, 44, 47]. A few variants have been considered, but we will here restrict our discussion to the *non-backtracking random walk* (NBRW): the states are the oriented edges of the graph (formed by replacing each edge $\{i, j\}$ with two arcs (i, j) and (j, i)), and the allowed transitions from a given arc (i, j) are the $d_j - 1$ out-going arcs (j, k), $k \neq j$, all being equally likely. The corresponding transition matrix P is easily seen to be doubly stochastic, so that the uniform distribution is always invariant.

The regular case. In a breakthrough work [104], Lubetzky and Sly proved that on "most" $d-$regular graphs with n vertices, the NBRW exhibits cutoff around time

$$t_n \quad = \quad \frac{\log(dn)}{\log(d-1)}, \tag{3.20}$$

in the precise sense that the convergence (3.18) holds in probability when the graph is chosen uniformly at random among all $d-$regular graphs on $[n]$. The degree d must be at least 3, and is allowed to grow with n as long as $d = n^{o(1)}$. Moreover, the cutoff window was shown to be of constant-order width ($w_n = \Theta(1)$). The regular case is special in many respects: first, all paths with a given origin, length and destination are equally likely to be followed by the walk, so that only their number actually matters. Second, there is an exact correspondence between the simple random walk and its non-backtracking counterpart. Third, large regular graphs typically exhibit a remarkable spectral structure that makes them optimal expanders in a very strong sense [39, 68]. The exact role played by this eigenstructure was subsequently understood by Lubetzky and Peres [103], who managed to extend the above result to all non-bipartite *Ramanujan graphs*.

General degrees. Inspired by these seminal results, I decided to explore the non-regular case along with my PhD student Anna Ben-Hamou (co-supervised with S. Boucheron). Introducing heterogeneity in the vertex degrees turns out to affect the qualitative behavior of the walk significantly, as its trajectory becomes essentially supported on an exponentially small fraction of the trajectory space, in relation with the classical *dimension drop* of harmonic measure on non-regular Galton-Watson trees [106]. In our joint paper [P14], accepted for publication in the *Annals of*

Probability, Anna and I established the cutoff phenomenon (3.18) for the NBRW on the random graph $G_{\mathbf{d}}$, provided the vertex degrees $\mathbf{d} = (d_1, \ldots, d_n)$ satisfy

$$\min_{i\in[n]} d_i \geq 3 \quad \text{and} \quad \max_{i\in[n]} d_i \leq n^{o(1)}.$$

Perhaps surprisingly, the asymptotics in this regime only depend on the degrees through two simple statistics: the mean logarithmic forward degree of an edge, and the corresponding variance

$$\mu \ := \ \frac{1}{2m}\sum_{i\in[n]} d_i \log\left(d_i - 1\right) \quad \text{and} \quad \sigma^2 \ := \ \frac{1}{2m}\sum_{i\in[n]} d_i \left\{\log\left(d_i-1\right) - \mu\right\}^2,$$

where $m = \frac{1}{2}(d_1 + \cdots + d_n)$. We shall also need some control on the third absolute moment: $\tau = \frac{1}{2m}\sum_i d_i \left|\log\left(d_i-1\right) - \mu\right|^3$. It might help the reader to think of μ, σ and τ as being fixed, or bounded away from 0 and ∞. However, we only impose the following (much weaker) condition:

$$\frac{\sigma^2}{\mu^3} \gg \frac{(\log\log n)^2}{\log n} \quad \text{and} \quad \frac{\sigma^3}{\tau\sqrt{\mu}} \gg \frac{1}{\sqrt{\log n}}.$$

Theorem 22 ([P14]). *Under the above assumptions on* $\mathbf{d}$*, the* NBRW *on* $G_{\mathbf{d}}$ *exhibits cutoff in the precise sense that the convergence (3.19) holds in probability, with*

$$t_n = \frac{\log m}{\mu}, \qquad w_n = \sqrt{\frac{\sigma^2 \log m}{\mu^3}}, \qquad \Phi(\lambda) := \frac{1}{2\pi}\int_\lambda^\infty e^{-\frac{u^2}{2}}\,\mathrm{d}u.$$

Theorem 22 partially overlaps with results obtained simultaneously by Berestycki, Lubetsky, Peres and Sly [22]: the latter concern both the simple random walk and the NBRW, but do not include the second-order refinement (3.19). Note that the limit profile Φ is universal, in the sense that it does not depend at all on the precise degrees. Intriguingly, the emergence of the Gaussian tail function inside the cutoff window has been observed in a few unrelated models, such as random walk on the $n-$dimensional hypercube [57], or the symmetric exclusion process on the circle [95].

Mixing time vs typical distance. Recall that $\mu = \mathbb{E}[\log D]$, where D denotes the forward degree of a uniformly sampled edge. By Jensen's inequality, the mixing time t_n thus satisfies

$$t_n \quad \geq \quad \frac{\log m}{\log \mathbb{E}[D]},$$

with equality if and only if the graph is regular. The right-hand side is well known to be the asymptotic distance between two typical points in G (see e.g., [84]). One notable effect of heterogeneous degrees is thus that the mixing time becomes significantly larger than the typical inter-point distance. A heuristic explanation is as follows: in the regular case, all paths of length t between two points are equally likely to be followed by the NBRW, and mixing occurs as soon as t is large enough for many such paths to exist. In the non-regular case however, different paths have very different weights, and most of them actually have a negligible chance of being seen by the walk. Consequently, one has to make t larger in order to see paths with a “reasonable” weight.

3.4 Random walks on random directed graphs [P15]

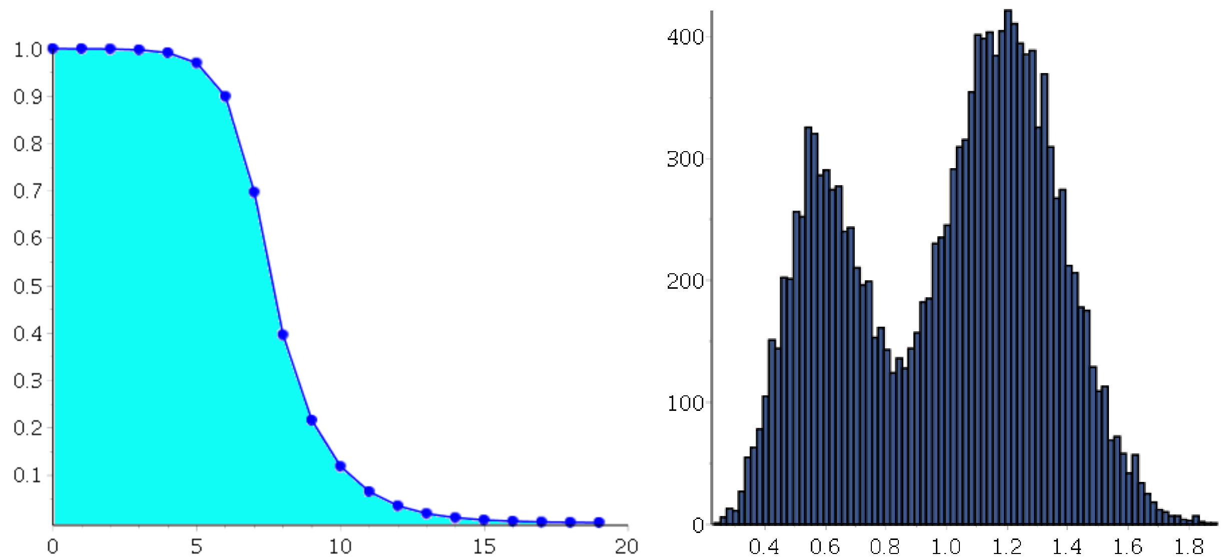


Figure 3.2: Distance to equilibrium along time (left) and histogram of the invariant masses multiplied by n (right) for the random walk on a random directed graph with $n = 5000 + 5000 + 5000$ vertices having respective degrees $(d^+, d^-) = (3, 2)$, $(3, 4)$ and $(4, 4)$.

Random walk on directed graphs. There are very few rigorous results on the cutoff phenomenon for non-reversible Markov chains. This observation was the starting point of my recent joint work with Bordenave and Caputo [P15], devoted to the analysis of random walks on large directed graphs. In sharp contrast with the undirected case, the invariant law π itself is already a highly non-trivial object, whose understanding has direct applications to link-based ranking algorithms in large databases (see, e.g., [43] and the references therein). In [48], Cooper and Frieze considered the random directed graph on $[n]$ formed by independently placing an arc between every pair of vertices with probability $p = \frac{d \log n}{n}$, where $d > 1$ is fixed while $n \to \infty$. In this regime, they prove that π is asymptotically close to the in-degree distribution. The recent work [2] by Addario-Berry, Balle and Perarnau provided precise estimates on the extrema of π in a model where all out-degrees are equal. To the best of our knowledge, our work [P15] provides the first proof of the cutoff phenomenon in this non-reversible setting. We work under the *directed configuration model* [46], allowing both the in-degrees and the out-degrees to be freely specified.

Setting. Given two sequences of positive integers $\mathbf{d}^- = (d_1^-, \ldots, d_n^-)$ and $\mathbf{d}^+ = (d_1^+, \ldots, d_n^+)$ with equal sum m, we construct a directed graph with vertex set $[n]$, in-degrees $\mathbf{d}^-$, and out-degrees $\mathbf{d}^+$ as follows. We formally equip each vertex $i \in [n]$ with a set E_i^+ of d_i^+ *tails* and a set E_i^- of d_i^- *heads*. We then simply choose a tail-to-head bijection $\omega\colon \bigcup_i E_i^+ \to \bigcup_i E_i^-$ (the *environment*), and interpret each coordinate $\omega(e) = f$ as an arc from the vertex of e to that of f (loops and multiple edges are allowed). Our interest is in the random walk on the resulting directed graph, i.e. the discrete-time Markov chain with state space $[n]$ and transition matrix

$$P(i,j) \quad := \quad \frac{1}{d_i^+} \# \left\{ e \in E_i^+ : \omega(e) \in E_j^- \right\}.$$

As long as the graph is strongly connected (in the sense that there is a directed path from i to j, for every $1 \le i, j \le n$), the invariant law π is unique and we define

$$\mathcal{D}_i(t) \quad := \quad \|P^t(i,\cdot) - \pi\|_{\mathrm{TV}}.$$

Our aim is to investigate the behavior of the measure π and of the decreasing functions $(\mathcal{D}_i)_{i\in[n]}$ when the environment ω is chosen uniformly at random among the $m!$ possible choices. This turns P, π and $(\mathcal{D}_i)_{i\in[n]}$ into random objects, parametrized by the degrees $\mathbf{d}^\pm$. We let the latter implicitly depend on n and consider the limit as $n \to \infty$. We consider the regime where

$$\delta := \min_{1\le i\le n} d_i^\pm \ge 2 \quad \text{and} \quad \Delta := \max_{1\le i\le n} d_i^\pm = O(1). \tag{3.21}$$

Note in particular that $m = \Theta(n)$. The requirement on δ ensures that the graph is strongly connected with high probability (see, e.g., [46]), so that the invariant law π is unique.

Results. Our first result is that the random walk exhibits a cutoff (visible on Figure 3.2) at time

$$t_n := \frac{\log n}{\mu}, \quad \text{where} \quad \mu := \frac{1}{m} \sum_{i=1}^n d_i^- \log d_i^+.$$

The degree statistics μ can be viewed as the average logarithmic out-degree of a directed edge.

Theorem 23 (Cutoff at time t_n [P15])**.** *For $t = \lambda t_n + o(t_n)$ with fixed $\lambda \ge 0$, we have*

$$\lambda < 1 \quad \Longrightarrow \quad \min_{i\in[n]} \mathcal{D}_i(t) \xrightarrow[n\to\infty]{\mathbb{P}} 1,$$
$$\lambda > 1 \quad \Longrightarrow \quad \max_{i\in[n]} \mathcal{D}_i(t) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Quenched concentration. Under the quenched law, every directed path of length t starting at i receives a *weight* equal to the inverse product of the out-degrees along that path. At a high level, the occurrence of a cutoff at $t_n = (\log n)/\mu$ is a consequence of the following two key results:

(i) Trajectories whose weight exceeds $\frac{1}{n}$ constitute the essential obstruction to mixing, in the precise sense that their total weight is roughly equal to the distance to equilibrium $\mathcal{D}_i(t)$.

(ii) The random walk of length t concentrates on trajectories with weight $e^{-\mu t+\mathcal{O}(\sqrt{t})}$.

Thus, the essence of the cutoff lies here in a quenched law of large numbers for the weight seen by the walk. Refining this with a quenched CLT for the fluctuations allowed us to capture the details of the transition from 1 to 0 inside the cutoff window. The appropriate scale is

$$w_n := \frac{\sigma\sqrt{\log n}}{\mu^{3/2}}, \quad \text{where} \quad \sigma^2 := \frac{1}{m}\sum_{i=1}^{n} d_i^- \left(\log d_i^+ - \mu\right)^2.$$

Remarkably, the graph of the function $t \mapsto \mathcal{D}_i(t)$ inside this window approaches a universal profile, independent of the initial position i and the precise degrees: the Gaussian tail function.

Theorem 24 (Window and profile [P15])**.** *Assume that the degree distribution is asymptotically non-degenerate in the sense that* $\sigma^2 \gg \frac{(\log\log n)^2}{\log n}$. *Then, for* $t = t_n + \lambda w_n + o(w_n)$ *with* $\lambda \in \mathbb{R}$ *fixed,*

$$\max_{i\in[n]}\left|\mathcal{D}_i(t) - \frac{1}{\sqrt{2\pi}}\int_\lambda^\infty e^{-\frac{u^2}{2}}\,du\right| \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Impact of the initial law. It is natural to wonder how the mixing time changes if one starts from a more-spread out law $\nu \in \mathcal{P}([n])$. Note that we always have, by convexity,

$$\|\nu P^t - \pi\|_{\mathrm{TV}} \le \max_{i\in[n]} \mathcal{D}_i(t).$$

We considered the case where ν is the law induced on vertices by the choice of a uniform tail:

$$\nu(i) := \frac{d_i^-}{m}, \qquad i \in [n]. \tag{3.22}$$

Our third result asserts that the convergence to equilibrium reduces from $\log n$ to constantly many steps when starting from the distribution ν. To make this precise, let us introduce the statistics

$$\varrho^2 := \frac{1}{m}\sum_{i=1}^{n}\frac{d_i^-}{d_i^+}, \qquad \gamma := \frac{1}{m}\sum_{i=1}^{n}\frac{\left(d_i^-\right)^2}{d_i^+}. \tag{3.23}$$

Note that $\gamma \ge 1$ by the Cauchy-Schwarz inequality, with equality if and only if $d_i^+ = d_i^-$ for all $i \in [n]$ (in this exceptional case known as the *Eulerian* setting, π miraculously reduces to ν).

Theorem 25 (Exponential convergence [P15])**.** *For* ν *as in (3.22) and any fixed* $t \in \mathbb{Z}_+$,

$$\|\nu P^t - \pi\|_{\mathrm{TV}} \le \frac{1}{2}\sqrt{\frac{n(\gamma-1)}{m(1-\varrho)}}\,\varrho^t + o_{\mathbb{P}}(1),$$

where $o_{\mathbb{P}}(1)$ *denotes a term that tends to* 0 *in probability as* $n \to \infty$.

Description of the invariant law. Note that $\varrho \le \frac{1}{\sqrt{\delta}}$ and that the constant in front of ϱ^t in Theorem 25 is less than $\sqrt{\Delta}$. Thus, the convergence to equilibrium occurs exponentially fast, uniformly in n. One surprising implication is that the (unknown) equilibrium mass $\pi(i)$ assigned to a typical vertex $i \in [n]$ is well approximated by $(\nu P^t)(i)$ for t large but independent of n ! In other words, $\pi(i)$ is essentially determined by the (backward) local neighborhood of i only. Since the latter is approximately described by a Galton-Watson tree, we obtain rather precise estimates on the equilibrium measure. Specifically, our last result asserts that the empirical distribution of the masses $(n\pi(i))_{i\in[n]}$ (the histogram visible on the right of Figure 3.2) concentrates around a deterministic probability measure $\Lambda \in \mathcal{P}(\mathbb{R}_+)$, explicitly identified as the law of the sum

$$\frac{n}{m}\sum_{k=1}^{d^-_{\mathcal{I}}} Z_k, \tag{3.24}$$

where $\mathcal{I}$ is uniformly distributed on $[n]$ and independent of the $(Z_k)_{k\ge1}$. The latter are i.i.d. mean-one random variables with law determined by the distributional fixed-point equation

$$Z_1 \stackrel{d}{=} \frac{1}{d^+_{\mathcal{J}}}\sum_{k=1}^{d^-_{\mathcal{J}}} Z_k, \tag{3.25}$$

where $\mathcal{J}$ has the out-degree distribution $\mathbb{P}(\mathcal{J}=i) = d_i^+/m$, and is independent of $(Z_k)_{k\ge1}$. Interestingly, this distributional fixed point equation has been extensively studied, in connection with Mandelbrot's multiplicative cascades and branching random walks. Among others, the works by Rösler [119], Liu [98, 99, 100, 101] and Barral [14, 15] provide detailed results concerning the uniqueness of the solution Z_1, its left and right tails, its positive and negative moments, its support, and even its absolute continuity w.r.t. Lebesgue's measure.

Theorem 26 (The equilibrium law [P15])**.** *Let Λ be the law of the random sum (3.24). Then,*

$$d_{\mathcal{W}}\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{n\pi(i)}, \Lambda\right) \xrightarrow[n\to\infty]{\mathbb{P}} 0,$$

where $d_{\mathcal{W}}$ denotes the $1-$Wasserstein (or Kantorovich-Rubinstein) distance on $\mathcal{P}_1(\mathbb{R})$, i.e.

$$d_{\mathcal{W}}(\Lambda_1,\Lambda_2) = \sup\left\{\left|\int_{\mathbb{R}} f\,d\Lambda_1 - \int_{\mathbb{R}} f\,d\Lambda_2\right| : f \in \mathrm{Lip}_1(\mathbb{R},\mathbb{R})\right\}.$$

Remarks. The fact that the random empirical measure $\frac{1}{n}\sum_{i=1}^n \delta_{n\pi(i)}$ approaches the deterministic law Λ in the $1-$Wasserstein sense rather than just in the usual weak sense is important, as it ensures that the first moment is conserved, i.e., that Λ has mean 1. This removes the scale-indeterminacy inherent to equation (3.25), and rules out the possibility that a non-vanishing part of the mass of π concentrates on a negligible fraction of the vertices. Note that the deterministic measure Λ actually depends on n, as did the quantities $\mu, \sigma^2, \gamma, \varrho$ appearing in the above theorems: we have chosen to express all our approximations directly in terms of the degrees $\mathbf{d}^{\pm}$ (which we view as our input parameters), rather than loosing in generality by artificially assuming the convergence of the empirical degree distribution $\frac{1}{n}\sum_{i=1}^{n}\delta_{(d_i^-,d_i^+)}$ to some $n-$independent limit. Of course, any external assumption on the $n \to \infty$ behavior of the degrees can then be "projected" onto the $n-$dependent constants provided by our results to yield *bona fide* convergence results, if needed.

Perspectives. The exponential convergence in Theorem 25 raises an interesting question. In a reversible setting such an inequality would typically follow from a uniform estimate on the modulus of the second largest eigenvalue of P. In our case however, the matrix P is not diagonalizable and the relation between Theorem 25 and the spectral gap is much less clear. We nevertheless conjecture that ϱ is indeed the asymptotic value of the spectral radius of P. This prediction is supported by the numerical simulation displayed in Figure 3.3. If true, this would constitute a first general result on the spectral gap of sparse non-hermitian random transition matrices (see [41] for results in the dense regime). I am currently working on this with my new PhD student, Simon Coste, jointly supervised with Bordenave.

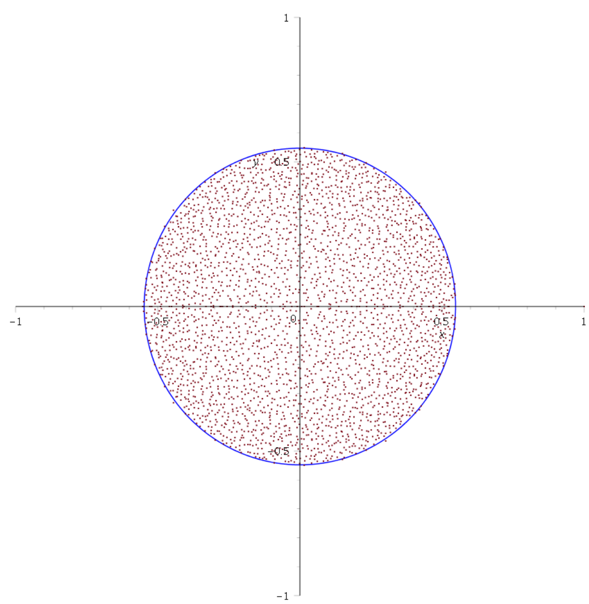


Figure 3.3: Red: eigenvalues of P for a random directed graph on $1000+1000+1000$ vertices with respective degrees $(d^+, d^-) = (3,2)$, $(3,4)$ and $(4,4)$. Blue: circle of radius ϱ, with ϱ as in (3.23).

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