Markov processes: exam 2026

The notation is that of the course. In particular, we write $\mathbb{P}_x$ for the law starting from x, and $\mathbb{E}_x, \mathrm{Var}_x$ for the corresponding expectation and variance.

Question (2 points)

State (without proof) the Hille-Yosida theorem for continuous contraction semi-groups on Banach spaces.

Exercise 1 (4 points)

A king starts in a corner of a 8×8 chessboard, and performs a discrete-time random walk by repeatedly moving to an adjacent square chosen uniformly at random (horizontally, vertically, or diagonally).

1. On average, how many moves will the king need in order to return to its initial position?
2. On average, how many times will the king spend in the four central squares during such an excursion?

The answers to both questions should be carefully justified.

Exercise 2 (6 points)

Let $(X_t)_{t\geq 0}$ be a Markov process with transition semi-group $(P_t)_{t\geq 0}$ and generator Q on a Polish space $\mathbb{X}$.

1. Prove that for any $f \in \mathrm{Dom}(Q)$, the following convergence holds uniformly on $\mathbb{X}$
$$\frac{(P_t f)^2 - f^2}{t} \quad \xrightarrow[t\to 0^+]{} \quad 2fQf,$$
2. Deduce that when $f, f^2 \in \mathrm{Dom}(Q)$, the following limit exists in $L^\infty(\mathbb{X})$, and relate it to Q and f.
$$\forall x \in \mathbb{X}, \qquad (\Gamma f)(x) \quad := \quad \lim_{t\to 0^+} \frac{\mathrm{Var}_x\left[f(X_t)\right]}{t}.$$
3. Explicitate Γf for a measure-determining class of test functions $f\colon \mathbb{X} \to \mathbb{R}$, in the special case where X is a Markov diffusion on $\mathbb{X} = \mathbb{R}^d$, i.e. a solution to a SDE of the form
$$\mathrm{d}X_t \quad = \quad b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,$$
where B is a $d-$dimensional Brownian motion and $b\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are Lipschitz continuous.
4. Explicitate Γf for all $f \in L^\infty(\mathbb{X})$ in the special case where $\mathbb{X}$ is discrete and X is a continuous-time Markov chain with transition matrix T and rate function $q\colon \mathbb{X} \to (0,\infty)$, with $\sup_{x\in\mathbb{X}} q(x) < \infty$.

Exercise 3 (8 points)

Fix a stochastic matrix T on a countable state space $\mathbb{X}$, a rate function $q\colon \mathbb{X} \to (0,\infty)$, and an initial law $\mu \in \mathcal{P}(\mathbb{X})$ with full support. Following our general construction of continuous-time chains, we consider a discrete-time process $Y = (Y_0, Y_1, \ldots)$ and a sequence of durations $E = (E_0, E_1, \ldots)$ distributed as follows:

$$\mathrm{Law}(Y) \;=\; \mathrm{DTMC}(\mu, T), \quad \text{and} \quad \mathrm{Law}(E \,|\, Y) \;= \bigotimes_{n\in\mathbb{N}} \mathrm{Exp}(q(Y_n)).$$

The goal of the present problem is to provide a spectral characterization of the non-explosion condition

$$\mathbb{P}\left(\sum_{n=0}^{\infty} E_n = \infty\right) \;=\; 1, \tag{1}$$

in terms of the generator Q associated with (T,q). We recall that Q is the $\mathbb{X}\times\mathbb{X}-$matrix with entries

$$\forall (x,y) \in \mathbb{X}^2, \quad Q(x,y) \;:=\; q(x)\left[T(x,y) - \mathbf{1}_{(y=x)}\right].$$

As usual, we view Q as an operator whose action on any bounded function $f\colon \mathbb{X}\to\mathbb{C}$ is given by

$$\forall x \in \mathbb{X}, \qquad Qf(x) \;:=\; \sum_{y\in\mathbb{X}} Q(x,y) f(y).$$

A number $\lambda \in \mathbb{C}$ is called an eigenvalue of Q if $Qf = \lambda f$ for some non-trivial bounded function $f\colon \mathbb{X}\to\mathbb{C}$.

1. From now on, we fix $\lambda \in (0,\infty)$. Compute $\mathbb{E}_x[e^{-\lambda E_0}]$ for all $x \in \mathbb{X}$.

2. Deduce that the function $f\colon x \mapsto \mathbb{E}_x\left[e^{-\lambda \sum_{n=0}^{\infty} E_n}\right]$ satisfies the identity

$$\forall x \in \mathbb{X}, \quad f(x) \;=\; \frac{q(x)}{q(x)+\lambda} Tf(x).$$

3. Check that the above identity is equivalent to $Qf = \lambda f$.

4. Conclude that if (1) fails, then every positive number is an eigenvalue of Q.

5. For each $n \in \mathbb{N}$, we now consider the function $f_n\colon \mathbb{X}\to(0,\infty)$ defined by

$$\forall x \in \mathbb{X}, \qquad f_n(x) \;:=\; \mathbb{E}_x\left[e^{-\lambda \sum_{k=0}^{n-1} E_k}\right],$$

   with the usual convention that an empty sum is zero. Find a recursion satisfied by $(f_n)_{n\in\mathbb{N}}$.

6. Deduce that any bounded function $g\colon \mathbb{X}\to\mathbb{C}$ solving the equation $Qg = \lambda g$ must satisfy

$$\forall n \in \mathbb{N}, \qquad \forall x \in \mathbb{X}, \qquad |g(x)| \;\le\; \|g\|_\infty f_n(x).$$

7. Conclude that if (1) holds, then no positive number is an eigenvalue of Q.