Spectra of sparse random graphs

Towards a limit theory

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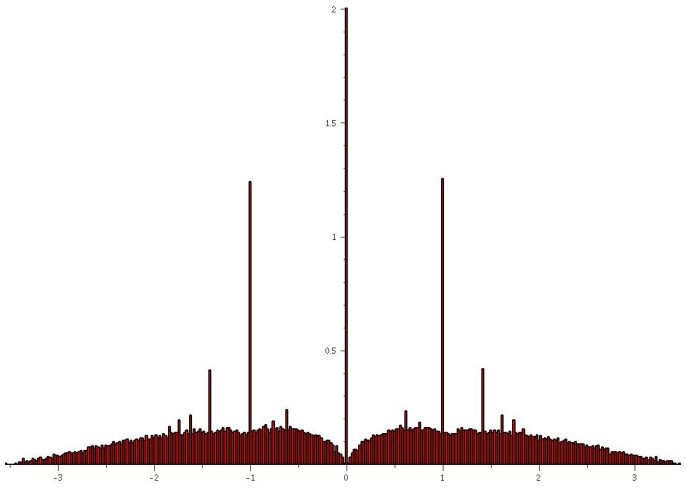


Contents

1 Introduction

Building infinite random objects which capture, in a unified way, the typical properties of large finite structures is a recurrent theme in modern probability. Let us explain this with a naive analogy. Let $(S_n)_{n\geq 0}$ denote a simple random walk on $\mathbb{Z}$, i.e. $S_0 = 0$ and for all $n \geq 1$,

$$S_n \;:=\; X_1 + \cdots + X_n, \tag{1}$$

where $(X_n)_{n\geq 1}$ are independent Rademacher variables. Consider the following basic exercise in probability: how does $\mathbb{E}\left[\max_{0\leq k\leq n} S_k\right]$ grow as $n \to \infty$? The answer is of course well-known to be

$$\mathbb{E}\left[\max_{0\leq k\leq n} S_k\right] \;\underset{n\to\infty}{\sim}\; \sqrt{\frac{2n}{\pi}}, \tag{2}$$

and our random walk is simple enough to allow for a variety of direct proofs, ranging from brute-force estimation to applications of the optional stopping theorem. A sophisticated, but much more insightful approach consists in invoking the invariance principle: the distributional convergence

$$\left(\frac{S_{\lfloor nt\rfloor}}{\sqrt{n}} : t \in \mathbb{R}_+\right) \;\xrightarrow[n\to\infty]{\text{(d)}}\; \left(B(t) : t \in \mathbb{R}_+\right), \tag{3}$$

holds with respect to the topology of uniform convergence on compact sets, with B a standard brownian motion. Since $f \mapsto \max_{t\in[0,1]} f(t)$ is continuous for this topology, we deduce that

$$\frac{1}{\sqrt{n}} \max_{0\leq k\leq n} S_k \;\xrightarrow[n\to\infty]{\text{(d)}}\; \max_{t\in[0,1]} B(t). \tag{4}$$

The right-hand side is well-known to be distributed as $|B(1)|$, and therefore

$$\mathbb{E}\left[\max_{t\in[0,1]} B(t)\right] \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{2}} \mathrm{d}x \;=\; \sqrt{\frac{2}{\pi}}. \tag{5}$$

The claim (2) readily follows from this computation and the (easy) justification that $(\frac{S_n^\star}{\sqrt{n}})_{n\geq 1}$ is uniform integrable. At a high level, we have here replaced the asymptotic estimation of the observable of interest by a direct computation on an appropriate limiting object. There is of course a price to pay in terms of formalism, but the advantages of such a *conceptual shift* are numerous:

1. The method involves exact computations rather than asymptotic approximations.
2. It applies to all observables which are continuous with respect to the underlying topology.
3. It immediately extends to all objects that converge in a similar fashion.
4. It provides a unifying, insightful view on a number of model-specific results.

The present lecture notes are based on a similar philosophy: we will demonstrate that the asymptotic spectral statistics of large sparse graphs can be studied in a systematic and unified way by working directly at the level of certain infinite random structures known as *local weak limits*.

2 The framework of local weak limits

In the sparse regime, many classical random graph models (Erdős-Rényi model, random regular graphs, configuration model, preferential attachment, recursive trees, etc.) happen to converge in the *local weak sense*, a notion introduced by Benjamini & Schramm [11] and developped further by Aldous & Steele [5] and Aldous & Lyons [4]. The limiting objects are countable random rooted graphs enjoying a form of stationarity known as *unimodularity*. They describe the internal geometry of large graphs when seen from a uniformly chosen vertex. These *local weak limits* are often much more convenient to work with than the finite-graph sequences that they approximate, and they have been shown to capture the asymptotic behavior of a number of important graph parameters, including the matching number, the number of spanning trees, or the empirical spectral distribution. The purpose of this chapter is to provide a short introduction to this modern framework.

2.1 Local weak convergence

Describing the *local geometry* of a graph is easy to do if we only care about the neighborhood of a single vertex, so we investigate this case first. Let us start by agreeing on some terminology.

Graph terminology. By a *graph*, we will always mean a pair $G = (V_G, E_G)$ where V_G is a finite or countable set whose elements are called *vertices*, and E_G is a set of unordered pairs of vertices called *edges*. We shall restrict our attention to *locally finite* graphs, meaning that the *degree*

$$\deg(G, x) \quad := \quad \sum_{y \in V_G} \mathbf{1}_{x \sim y} \tag{6}$$

is finite at every $x \in V_G$. Here and throughout the notes, we use the standard notation $x \sim y$ to mean that the vertices x and y are *neighbors* in the graph under consideration, i.e. $\{x, y\} \in E_G$. A *path* of length $\ell \geq 0$ from x to y is a sequence of $\ell + 1$ vertices $x_0, \ldots, x_\ell$ such that $x_0 = x$, $x_\ell = y$ and $x_{i-1} \sim x_i$ for $1 \leq i \leq \ell$. The *graph distance* on V_G is defined by

$$\mathrm{d}_G(x, y) \quad := \quad \inf \{\ell \geq 0 \colon \text{there is a path of length } \ell \text{ from } x \text{ to } y\}, \tag{7}$$

with the convention that $\inf(\emptyset) = +\infty$. The graph is *connected* if $\mathrm{d}_G(x, y) < \infty$ for all $(x, y) \in V_G^2$.

Rooted graphs. A *rooted graph* (G, o) is a graph G together with a distinguished vertex $o \in V_G$, called the *root*. Two rooted graphs $(G, o), (G', o')$ are *isomorphic* (written $\cong$) if there is a bijection $\phi \colon V_G \to V_{G'}$ which preserves roots ($\phi(o) = o'$) and edges ($\{x, y\} \in E_G \iff \{\phi(x), \phi(y)\} \in E_{G'}$). For a rooted graph (G, o) and an integer $r \geq 0$, we let $\mathcal{B}_r(G, o)$ denote the rooted subgraph induced

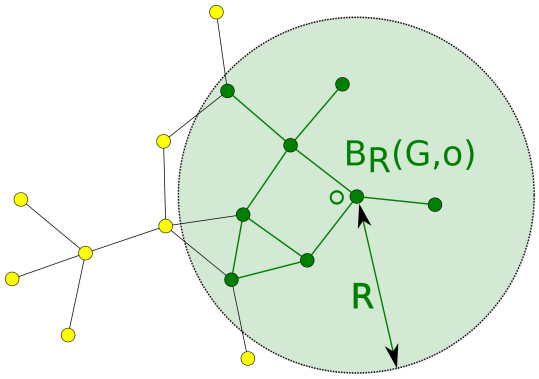


Figure 1: The ball or radius $r = 2$ around a rooted graph (G, o).

by the ball $\{x \in V_G \colon \mathrm{d}_G(o, x) \leq r\}$, see Figure 1. We let $\mathcal{G}_\star$ denote the set of all connected, locally finite rooted graphs considered up to isomorphism, and we equip it with the distance

$$\mathrm{d}_{\mathrm{LOC}}\big((G, o), (G', o')\big) \quad := \quad \sum_{r=1}^{\infty} 2^{-r} \mathbf{1}_{\mathcal{B}_r(G,o) \ncong \mathcal{B}_r(G',o')}. \tag{8}$$

In words, a sequence converges in $\mathcal{G}_\star$ if the local structure around the root ultimately stabilizes. The metric space $(\mathcal{G}_\star, \mathrm{d}_{\mathrm{LOC}})$ is easily seen to be Polish, and will constitute our basic workspace.

Uniform rooting. The graph examples mentioned above are *not* rooted: there is no favored vertex, and we would like to *simultaneously* capture the local neighborhoods of *all* vertices. To do so, the beautifully simple idea of Benjamini & Schramm is to consider the *empirical distribution of all possible rootings*, just as one uses the empirical spectral measures to study the eigenvalues of large matrices. Given a finite graph G, we form a probability measure $\mathcal{L}_G$ on $\mathcal{G}_\star$ by setting

$$\mathcal{L}_G \quad := \quad \frac{1}{|V_G|} \sum_{o \in V_G} \delta_{[G,o]}, \tag{9}$$

where $[G, o]$ denotes the element of $\mathcal{G}_\star$ obtained from G by declaring o as the root, restricting to the corresponding connected component, and forgetting the labels. In words, $\mathcal{L}_G$ is the law of the graph G *seen from a uniformly chosen random vertex*, see Figure 2.

Local weak convergence. Since $\mathcal{G}_\star$ is Polish, so is the space $\mathcal{P}(\mathcal{G}_\star)$ of Borel probability measures on $\mathcal{G}_\star$ equipped with the usual notion of weak convergence (see, e.g. [14]). If $(G_n)_{n\geq 1}$ is a sequence of finite graphs such that $(\mathcal{L}_{G_n})_{n\geq 1}$ admits a limit $\mathcal{L}$ in $\mathcal{P}(\mathcal{G}_\star)$, we call $\mathcal{L}$ the *local weak limit* of $(G_n)_{n\geq 1}$ and write $G_n \to \mathcal{L}$. Thus, $\mathcal{L}$ is the law of some $\mathcal{G}_\star-$valued random variable $(\mathbf{G}, \mathbf{o})$ which describes what G_n asymptotically looks like when seen from a uniformly chosen vertex. We shall

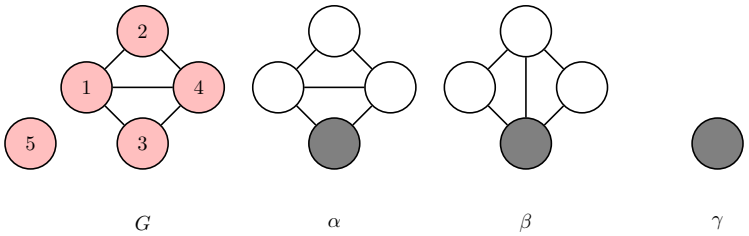


Figure 2: A graph G and its possible rootings $\alpha, \beta, \gamma \in \mathcal{G}_\star$. Here, $\mathcal{L}_G = \frac{2}{5}\delta_\alpha + \frac{2}{5}\delta_\beta + \frac{1}{5}\delta_\gamma$.

often abusively identify $(\mathbf{G}, \mathbf{o})$ with its law $\mathcal{L}$. The convergence $G_n \to \mathcal{L}$ then means that

$$\frac{1}{|V_{G_n}|} \sum_{o \in V_{G_n}} f(G_n, o) \quad \xrightarrow[n\to\infty]{} \quad \mathbb{E}[f(\mathbf{G}, \mathbf{o})], \tag{10}$$

for every continuous, bounded observable $f \colon \mathcal{G}_\star \to \mathbb{R}$. We emphasize that the graphs $(G_n)_{n\geq 1}$ are here deterministic: the randomness of $(\mathbf{G}, \mathbf{o})$ only reflects their spatial heterogeneity. Note that the continuity of f here means that f is *essentially local*: its value at point can be approximated to arbitrary precision by looking at a succificiently large finite ball around the root. Thus, the left-hand side of (10) appears as a spatial average of *local contributions* from the vertices. By standard measure-theoretic arguments, it is actually enough to test this on observables of the form $f(G, o) = \mathbf{1}_{(\mathcal{B}_r(G,o) \cong g)}$, where $r \geq 0$ is an integer and g a finite rooted graph. This reduces local weak convergence to convergence of *subgraph counts*. In the other direction, (10) extends to more general observables: uniform integrability and continuity $\mathcal{L}-$almost everywhere suffice.

Sparsity. Local weak convergence implies convergence of the degree frequencies: for any fixed $k \in \mathbb{Z}_+$, the simple choice $f(G, o) := \mathbf{1}_{(\deg(G,o)=k)}$ in the definition (10) yields

$$\frac{1}{|V_{G_n}|} \sum_{o \in V_{G_n}} \mathbf{1}_{(\deg(G_n,o)=k)} \quad \xrightarrow[n\to\infty]{} \quad \mathbb{P}\left[\deg(\mathbf{G}, \mathbf{o}) = k\right]. \tag{11}$$

Recalling that $\mathbb{P}\left(\deg(\mathbf{G}, \mathbf{o}) < \infty\right) = 1$ by definition of $\mathcal{G}_\star$, we see that most degrees are bounded:

$$\sup_{n\geq 1} \left\{ \frac{1}{|V_{G_n}|} \sum_{o \in V_{G_n}} \mathbf{1}_{(\deg(G_n,o)>\Delta)} \right\} \quad \xrightarrow[\Delta\to\infty]{} \quad 0. \tag{12}$$

Thus, local weak convergence is inherently restricted to sparse graphs. Perhaps surprisingly, this is close to being the only requirement : as we shall see next, *practically every sparse graph sequence admits a local weak limit*. We emphasize that there are various notions of convergence for dense graphs, as well as non-local notions of convergence for sparse graphs, see e.g., the monograph [40].

2.2 Classical examples

Discrete torus. Consider the two-dimensional discrete torus $G_n = (\mathbb{Z}/n\mathbb{Z})^2$ of Figure 3. This graph is transitive (all its rootings are isomorphic), so $\mathcal{L}_{G_n} = \delta_{(G_n,0)}$ where $0 = (0,0)$ is the origin. Since $\mathcal{B}_r(G_n, 0) \cong \mathcal{B}_r(\mathbb{Z}^2, 0)$ for all $n > 2r$, we have $(G_n, 0) \to (\mathbb{Z}^2, 0)$ in $\mathcal{G}_\star$ and hence $\delta_{(G_n,0)} \to \delta_{(\mathbb{Z}^2,0)}$ in $\mathcal{P}(\mathcal{G}_\star)$. In other words, the local weak limit of $(G_n)_{n\geq 1}$ is $\mathcal{L} = \delta_{(\mathbb{Z}^2,0)}$.

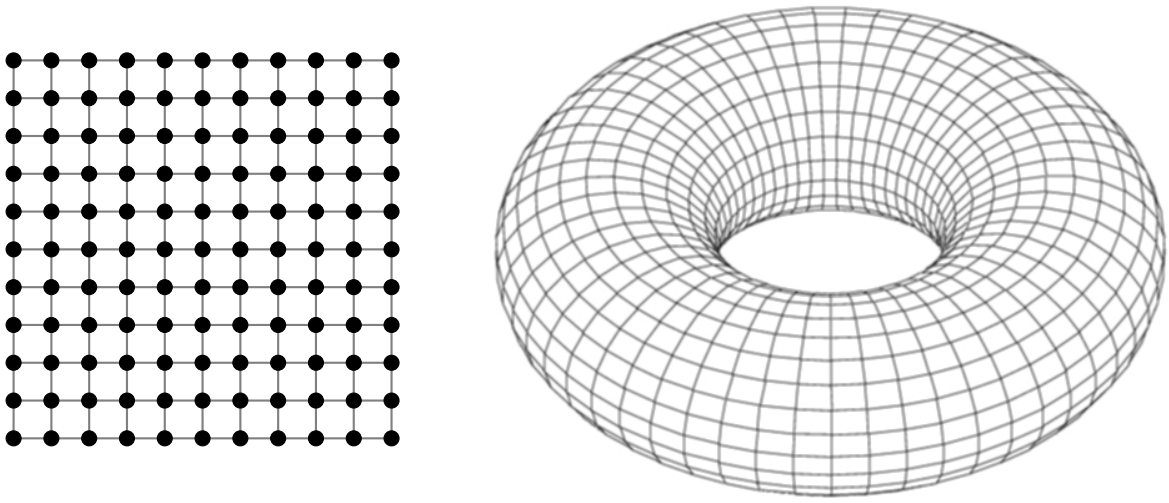


Figure 3: Despite being topologically very different, the square grid $\{1, \dots, n\}^2$ (left) and the torus $(\mathbb{Z}/n\mathbb{Z})^2$ (right) admit the same local weak limit as $n \to \infty$, namely $\mathcal{L} = \delta_{(\mathbb{Z}^2,0)}$.

Square grid. Let us now consider the square grid $G_n = \{0, \dots, n-1\}^2$. The graph is no longer transitive, but the equality $\mathcal{B}_r(G_n, o) \cong \mathcal{B}_r(\mathbb{Z}^2, 0)$ still holds for *most* vertices $o \in V(G_n)$. More precisely, it only fails if o is within distance $r-1$ from the boundary, and the number of such problematic vertices is $o(|V(G_n)|)$ if $r \geq 1$ is kept fixed as $n \to \infty$. Recalling that the root o is chosen uniformly at random in the definition of $\mathcal{L}_{G_n}$, we conclude that the local weak limit of $(G_n)_{n\geq 1}$ is again $\mathcal{L} = \delta_{(\mathbb{Z}^2,0)}$. This extends to higher dimensions, as does the previous example.

Binary tree. Let T_n denote the complete binary tree of height n, see Figure 4. A naive guess might be that the local weak limit $\mathcal{L}$ of $(T_n)_{n\geq 1}$ is just the dirac mass at the infinite binary rooted tree, since the latter is the limit of the sequence of rooted trees (T_n, o_n) where o_n denotes the unique degree-two vertex. This is wrong,however, because the view from o_n is far from being typical. In fact, the trees $(T_n)_{n\geq 1}$ are too heterogeneous for the limit $\mathcal{L}$ to be a dirac mass: for example, roughly half the vertices of T_n are leaves, and so by (11), the root should have degree 1 with probability a half under $\mathcal{L}$. A moment's thought suffices to convince oneself that the local weak limit is in

fact the law of the following random rooted tree, known as the *Canopy tree*: start with disjoint binary trees $T_1, T_2, \ldots$ of all possible heights, and connect them into a single infinite tree by adding the edges $\{o_n, o_{n+1}\}, n \geq 1$. Finally, declare the root to be the random vertex o_N, where N is a geometric random variable with parameter $\frac{1}{2}$, i.e. $\mathbb{P}(N = n) = 2^{-n}$ for all $n \geq 1$.

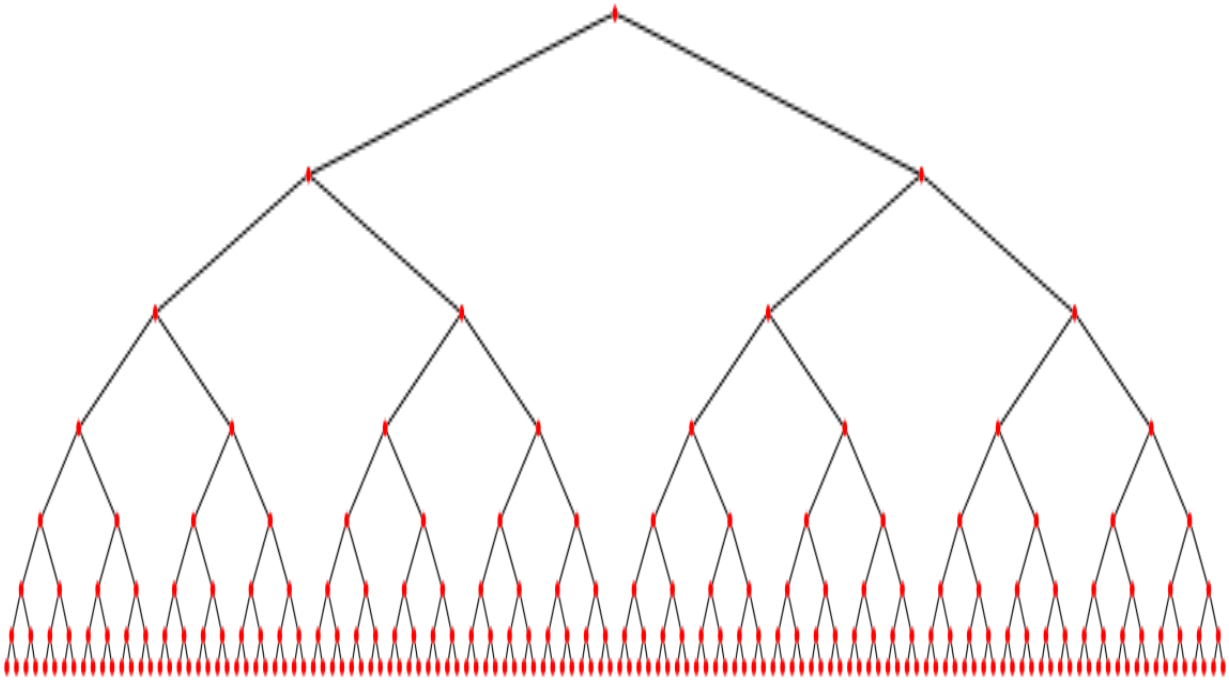


Figure 4: The complete binary tree of height $n = 8$

Regular graphs with high girth. Let $(G_n)_{n\geq 1}$ denote a sequence of $d-$regular graphs whose *girth* (the minimal length of a cycle) tends to infinity as $n \to \infty$. Then for each fixed $r \geq 1$ and all large enough n, we have $\mathcal{B}_r(G_n, o) \cong \mathcal{B}_r(\mathbb{T}_\mathrm{d}, 0)$ for all $o \in V_{G_n}$, where $\mathbb{T}_d$ denotes the $d-$regular infinite tree. Thus, the local weak limit of $(G_n)_{n\geq 1}$ is simply $\mathcal{L} = \delta_{(\mathbb{T}_d,0)}$. The conclusion obviously remains true under the following weaker *locally tree-like* assumption:

$$\text{for each fixed } \ell \geq 3, \text{ the number of cycles of length } \ell \text{ in } G_n \text{ is } o(V(G_n)) \text{ as } n \to \infty. \tag{13}$$

It is not hard to see that this property is satisfied almost-surely when the $d-$regular graphs $(G_n)_{n\geq 1}$ are chosen uniformly at random on vertex sets of diverging size. For this model, the local weak convergence $G_n \to \delta_{(\mathbb{T}_d,0)}$ thus holds almost-surely.

Erdős-Rényi random graphs. For each $n \geq 1$, we contruct a random graph G_n on $\{1, \ldots, n\}$ by letting each of the $\binom{n}{2}$ possible edges be present with probability $0 \leq p_n \leq 1$, independently. In the sparse regime where $np_n \to c \in [0, \infty)$ as $n \to \infty$, the locally tree-like property (13) is easily seen to hold almost-surely. However, the degrees are now random, each having the Binomial$(n-1, p_n)$ distribution. Since the latter converges to Poisson(c) as $n \to \infty$, it is not a surprise that almost-surely, $G_n \to \mathcal{L}$ where $\mathcal{L}$ is the law of a Poisson Galton-Watson tree with mean degree c. We refer

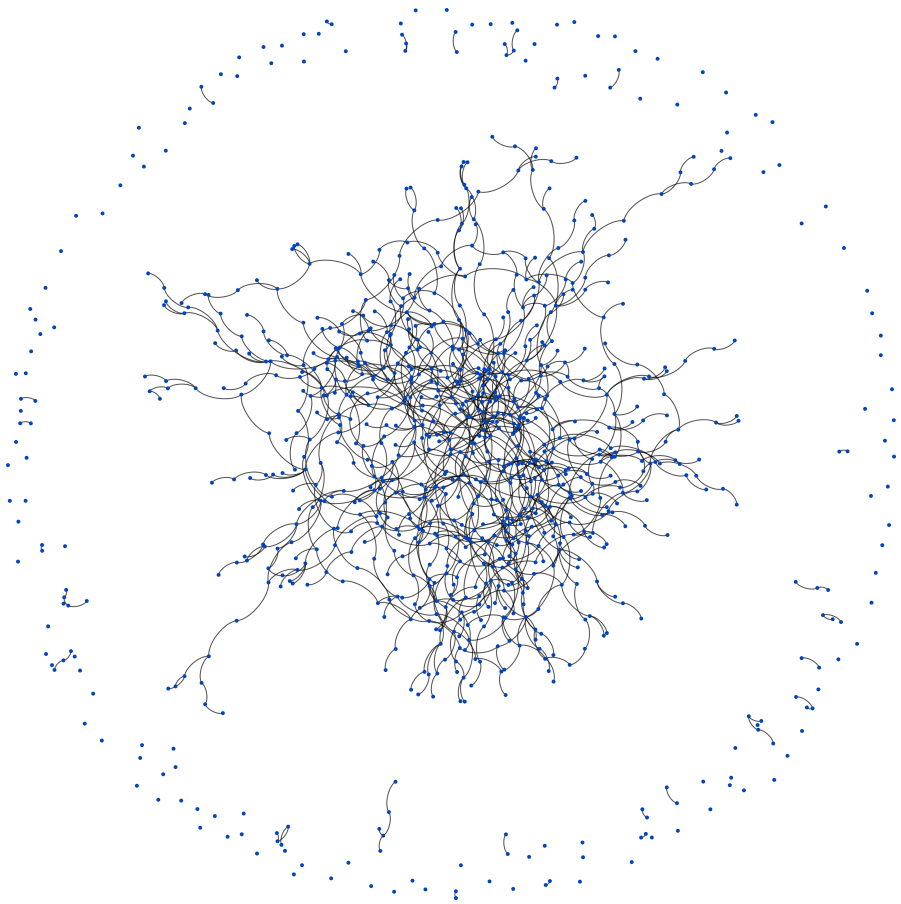


Figure 5: A realization of the Erdős-Rényi random graph with $n = 1000$ and $c = 2$.

to [17] for a detailed proof. It is well-known that $\mathcal{L}$ exhibits a phase transition as c varies: for $c \leq 1$, the tree is almost-surely finite, where as for $c > 1$, the tree is infinite with probability p_c, where $p_c \in (0, 1)$ is the largest root of the equation $1 - p = e^{-cp}$. The convergence $G_n \to \mathcal{L}$ provides a link between this phase transition and an equally famous phase transition for the random graph G_n itself: for $c \leq 1$, most vertices lie in components of bounded size as $n \to \infty$, whereas for $c > 1$, there is an additional *giant* component of size $np_c + o(n)$ visible in the center of Figure 5.

Configuration model. Among the variety of random graph models that have been proposed over the past years, the *configuration model* [15, 43, 44] stands out as one of the most versatile choices, as it offers the possibility to freely specify the degree of each vertex, while keeping as much randomness as possible in the overall structure. Given a sequence $\mathbf{d} = (d_1, \ldots, d_n)$ of non-negative integers with even sum, a random graph $G_{\mathbf{d}}$ on $[n]$ is generated as follows: d_i *half-edges* are attached to each $i \in [n]$, and the $2m = d_1 + \cdots + d_n$ half-edges are paired uniformly at random to form m edges. Loops and multiple edges are then simply removed (a few variants exist [36], but they are all equivalent for our purpose). Now, let a sequence $\mathbf{d}^n = (d_1^n, \ldots, d_n^n)$ be given for each $n \geq 1$, and assume that $\mathbf{d}^n$ approaches some probability distribution $\pi = \{\pi_k\}_{k \geq 0}$ on $\mathbb{Z}_+$ in the sense that

$$\frac{1}{n} \# \{i \in [n] \colon d_i^n = k\} \quad \xrightarrow[n \to \infty]{} \quad \pi_k, \tag{14}$$

for each $k \in \mathbb{Z}_+$. Under the additional sparsity assumption

$$\sup_{n \geq 1} \left\{ \frac{1}{n} \sum_{i=1}^{n} (d_i^n)^2 \right\} \quad < \quad \infty, \tag{15}$$

it was shown in [17] thate $G_{\mathbf{d}^n} \to \mathcal{L}$ almost-surely, where $\mathcal{L}$ is the law of the *Unimodular Galton-Watson tree* with degree distribution π. This is the random rooted tree obtained by a Galton-Watson branching process where the root has a given offspring distribution $\pi = (\pi_k)_{k \geq 0}$ and all descendants have the size-biased offspring distribution $\widehat{\pi} = (\widehat{\pi}_k)_{k \geq 0}$ given by

$$\widehat{\pi}_k \quad = \quad \frac{(k+1)\pi_{k+1}}{\sum_i i \pi_i}. \tag{16}$$

Random degree sequence. Under the assumption (15), a result of Janson [37, 38] asserts that

$$\liminf_{n \to \infty} \mathbb{P}(G_{\mathbf{d}^n} \text{ is simple}) > 0.$$

Moreover, conditionally on being simple, $G_{\mathbf{d}^n}$ is a uniform simple graph with degrees $(d_1^n, \ldots, d_n^n)$. Consequently, the above convergence transfers to the uniform simple graph model. Many classical random graphs, including the popular Erdős-Rényi random graph or the generalized random graph model are actually mixtures of this uniform model: the vertex degrees $d_1, \ldots, d_n$ themselves are random, but two simple graphs with the same degrees occur with the same probability. Thus, the local weak convergence result extends to these models as well. For example, in the Erdős-Rényi case, the vertex degrees satisfy (15) and (14) almost-surely, with π being Poisson. Since $\widehat{\pi} = \pi$ in that case, we recover the fact that the local weak limit is the Poisson-Galton-Watson tree.

Uniform random trees. Pick a tree uniformly at random among the n^{n-2} possible trees that can be built on the vertex set $[n]$. Any sequence of jointly defined such random trees admits almost

surely a special local weak limit $\mathcal{L}$ as $n \to \infty$: the (law of the) so-called *Infinite Skeleton Tree*, see Figure 2.2. To describe this object, consider an infinite sequence of independent random rooted trees (T_1, o_1), $(T_2, o_2), \ldots$ with the Poisson-Galton-Watson distribution with mean 1. Recall that these trees are finite almost-surely, even though their expected size is infinite. The Infinite Skeleton Tree is then constructed by placing an edge between o_k and o_{k+1} for each $k \geq 1$, and declaring o_1 as the root. The convergence was first established by Grimmett [34]. Alternative proofs and generalizations can be found in Devroye [28] and Aldous [3].

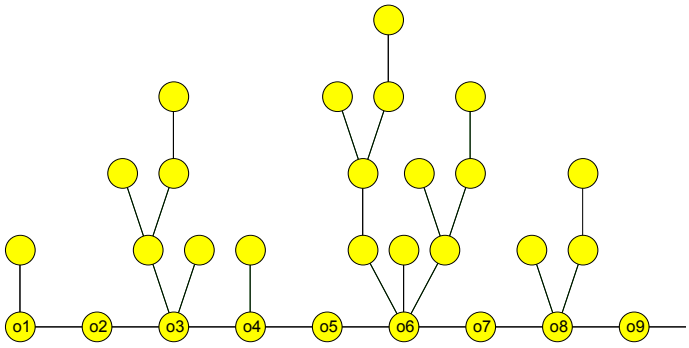


Figure 6: A realization of (a portion of) the Infinite Skeleton Tree

Ubiquity. Local weak convergence is robust under various natural graph-theoretical transformations, deterministic or random. For example, performing a Bernoulli bond percolation with fixed parameter $p \in (0, 1)$ along any convergent graph sequence with diverging size is easily seen to produce an almost-surely convergent random graph sequence. Beyond the above list, many sophisticated random graph models are known to admit a local weak limit. Let us mention, among others, preferential attachment graphs [12], random planar maps [7, 23], and random geometric graphs. We conclude this section with a tightness criterion due to Benjamini, Lyons & Schramm which gives an honest mathematical content to our earlier vague claim that *practically every sparse graph sequence admits a local weak limit.* Of course, passing to subsequences is unavoidable. Note also that the requirement of bounded average degrees does not suffice: if G_n is the star-graph consisting a central vertex connected to n neighbours, then the average degree is less than 2, but the sequence $(G_n)_{n\geq 1}$ does not converge (balls of radius 2 contain all $n + 1$ vertices).

Theorem 1 ([10])**.** *Let $(G_n)_{n\geq 1}$ be any sequence of graphs satisfying the sparsity assumption*

$$\sup_{n\geq 1} \left\{ \frac{1}{|V(G_n)|} \sum_{o \in V(G_n)} \deg(G_n, o) \mathbf{1}_{(\deg(G_n,o)>\Delta)} \right\} \xrightarrow[\Delta\to\infty]{} 0. \tag{17}$$

Then, there exists a subsequence $(G_{n_k})_{k\geq 1}$ which converges in the local weak sense.

2.3 Unimodularity

Not all Borel probability measures on $\mathcal{G}_\star$ arise as local weak limits of finite graphs: the fact that

$$\text{“the root is equally likely to be any vertex”} \tag{18}$$

is somehow *retained* in the limit. Of course, this statement only makes sense for finite graphs, but there exists a rigorous formulation in the infinite case as well: this is called *unimodularity*.

Walk invariance. A naive way to formalize the idea that a random rooted graph $(\mathbf{G}, \mathbf{o})$ is uniformly rooted is to require its law to be walk-invariant, i.e. preserved under the action of re-rooting the graph at a uniformly chosen neighbour of the root. Note however, that this intuition is wrong: the stationary measure π_G for simple random walk on a finite connected graph $G = (V_G, E_G)$ is not uniform on V_G, but proportional to the degrees. In light of this, one should rather expect the degree-biased version of $(\mathbf{G}, \mathbf{o})$ to be walk-invariant: for all Borel observables $f \colon \mathcal{G}_\star \to [0, \infty]$,

$$\mathbb{E}\left[\sum_{x \sim \mathbf{o}} f(\mathbf{G}, x)\right] \;=\; \mathbb{E}\left[\deg(\mathbf{G}, \mathbf{o}) f(\mathbf{G}, \mathbf{o})\right]. \tag{19}$$

In fact, much more is true: the measure π_G is not just *stationary* but *reversible* under the random walk, and a similar refinement can be formulated at the level of local weak limits, as we now see.

Unimodularity. Let $\mathcal{G}_{\star\star}$ denote the analogue of $\mathcal{G}_\star$ for doubly-rooted graphs (G, x, y), with all definitions being adapted in the natural way. A $\mathcal{G}_\star-$valued random variable $(\mathbf{G}, \mathbf{o})$ (or more accurately, its law $\mathcal{L}$) is called *unimodular* if for any Borel function $h \colon \mathcal{G}_{\star\star} \to [0, \infty]$,

$$\mathbb{E}\left[\sum_{x \in V_{\mathbf{G}}} h(\mathbf{G}, x, \mathbf{o})\right] \;=\; \mathbb{E}\left[\sum_{x \in V_{\mathbf{G}}} h(\mathbf{G}, \mathbf{o}, x)\right]. \tag{20}$$

Note that (19) is just the special case where $h(G, x, y) = f(G, x)\mathbf{1}_{x \sim y}$. The identity (20) is called the *Mass Transport Principle* (MTP): if one thinks of $h(G, x, o)$ as an amount of mass sent from x to o, then the expected mass received and sent by the root should coincide. This is clearly true if $\mathcal{L} = \mathcal{L}_G$ with G finite, and can be checked to be preserved under weak convergence. Thus,

Theorem 2. *All local weak limits of finite graphs are unimodular.*

Whether the converse holds is a notorious open problem with deep implications [4, 29, 10].

Applications. The MTP is a powerful tool for studying local weak limits, but choosing the appropriate transport function may seem a bit of an art at first. Let us practice a bit.

Lemma 3 (Everything shows up at the root)**.** *Let* $(\mathbf{G}, \mathbf{o})$ *be a unimodular random rooted graph, and let* $B \subseteq \mathcal{G}_\star$ *be a Borel set such that* $\mathbb{P}((\mathbf{G}, \mathbf{o}) \in B) = 1$*. Then* $\mathbb{P}\left(\forall x \in V_{\mathbf{G}} \colon (\mathbf{G}, x) \in B\right) = 1$*.*

Proof. Define a transport function $h \colon \mathcal{G}_{\star\star} \to [0, \infty]$ by $h(G, x, y) = \mathbf{1}_{(G,x) \notin B}$. Then (20) reads

$$\mathbb{E}\left[\sum_{x \in V_{\mathbf{G}}} \mathbf{1}_{(\mathbf{G},x) \notin B}\right] \quad = \quad \mathbb{E}\left[|V_{\mathbf{G}}| \mathbf{1}_{(\mathbf{G},\mathbf{o}) \notin B}\right]. \tag{21}$$

The integrand on the right-hand side vanishes almost-surely, so the proof is complete. □

A finite connected graph $G = (V_G, E_G)$ obviously always satisfies $|E_G| \geq |V_G| - 1$, or equivalently,

$$\frac{1}{|V_G|} \sum_{o \in V_G} \deg(G, o) \quad \geq \quad 2 - \frac{2}{|V_G|}, \tag{22}$$

with equality if and only if G is a tree. Let us use the MTP to formulate an infinite version of this.

Theorem 4 (Minimal expected degree)**.** *If a unimodular random graph* $(\mathbf{G}, \mathbf{o})$ *is a.-s. infinite, then* $\mathbb{E}\left[\deg(\mathbf{G}, \mathbf{o})\right] \geq 2$*, with equality if and only if* $\mathbf{G}$ *is a.-s. a tree with at most two ends.*

Proof. Define $h \colon \mathcal{G}_{\star\star} \to \{0, 1\}$ by $h(G, x, y) = 1$ iff $\{x, y\}$ is an edge, whose removal leaves x in an infinite connected component. If $G = (V_G, E_G)$ is an infinite connected graph, then

$$\sum_{x \in V_G} f(G, o, x) \quad \leq \quad \deg(G, o) - 2 + \sum_{x \in V_G} f(G, x, o), \tag{23}$$

for every $o \in V_G$. Indeed, the sum on the right-hand side can not be zero; if it equals 1, only one edge connects o to infinity, and (23) is an equality; if it is at least 2, then (23) is trivial. We now apply this to $(\mathbf{G}, \mathbf{o})$, take expectations, and invoke the MTP to conclude that $\mathbb{E}[\deg(\mathbf{G}, \mathbf{o})] \geq 2$. In the equality case, the two sides of (23) have the same expectation and must therefore coincide a.-s.. By the above lemma, this is not only true at the root, but at all vertices, i.e.

$$\forall o \in V_G, \qquad \sum_{x \in V_G} f(G, o, x) \quad = \quad \deg(G, o) - 2 + \sum_{x \in V_G} f(G, x, o). \tag{24}$$

It remains to show that trees with 1 or 2 ends are exactly the infinite connected graphs G satisfying this. If G is not a tree, it contains an infinite path and a cycle intersecting at a single vertex o, and

$$\sum_{x \in V_G} f(G, x, o) \quad \geq \quad 3. \tag{25}$$

This contradicts (24). If G is a tree with at least 3 ends, then one can find a vertex o whose removal creates at least 3 infinite subtrees, and the contradiction (25) arises again. Conversely, it follows directly from the definitions that (24) holds when G is a tree with 1 or 2 ends. □

Corollary 5 (Limits of large trees)**.** *If* $(\mathbf{G}, \mathbf{o})$ *is the local weak limit of a sequence of trees with diverging sizes, then* $\mathbb{E}[\deg(\mathbf{G}, \mathbf{o})] = 2$ *and* $(\mathbf{G}, \mathbf{o})$ *is almost-surely a tree with at most* 2 *ends.*

The skeleton tree is of course a beautiful example that illustrates this result.

3 Application to spectral graph theory

An important feature of local weak convergence is its intrinsic *myopia*: it only captures the geometry of balls of constant radius around typical vertices. A positive consequence is that it is extremely *easy* to converge: as we have seen, any reasonable sequence of sparse graphs admits a local weak limit. One might be tempted to conclude from this that only very basic graph parameters – such as the degree frequencies considered above – will be effectively captured by local weak limits. However, this is beautifully false: over the past decade, a number of *generic continuity theorems* of the form

$$(G_n \to \mathcal{L}) \implies \Big(\lim_{n\to\infty} \phi(G_n) \text{ exists and can be computed directly on } \mathcal{L} \Big) \tag{26}$$

have been established for various fundamental graph parameters ϕ. Examples include: the number of spanning trees [41], the size of a maximum matching [30], the matching polynomial [21], the rank of the adjacency matrix [20], the maximal subgraph density [6], or the free energy of the Ising ferromagnet [27]. As we will now see, spectral statistics also belong to this list.

3.1 Beyond the semi-circle law

Let $G = (V_G, E_G)$ be a finite graph and let $A = A_G \in \mathbb{R}^{V_G \times V_G}$ denote its adjacency matrix:

$$A(x,y) \quad := \quad \begin{cases} 1 & \text{if } \{x,y\} \in E_G \\ 0 & \text{else.} \end{cases} \tag{27}$$

The eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ ($n = |V_G|$) of this symmetric matrix happen to capture a wealth of structural information about the graph G, and we refer the reader to the textbook [25] for a detailed account on this fascinating subject. We shall here be interested in understanding their *typical* asymptotics when G is large. For this purpose, it is convenient to encode the eigenvalues into a probability measure on $\mathbb{R}$, called the empirical spectral distribution of G:

$$\mu_G \quad := \quad \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}. \tag{28}$$

A natural way to give a meaning to the word *typical* consists in sampling G uniformly at random from the $2^{\binom{n}{2}}$ possible graphs on $\{1,\ldots,n\}$. The upper-triangular entries of A are then independent Bernoulli random variables with parameter $p = \frac{1}{2}$, and a celebrated result of Wigner [53] asserts that μ_G will concentrate around (a suitable rescaling of) the *semi-circle law*:

$$\mu_{\cap}(\mathrm{d}\lambda) \quad := \quad \frac{1}{2\pi} \sqrt{(4-\lambda^2)_+}\, \mathrm{d}\lambda, \tag{29}$$

as illustrated on Figure 7. One of the remarkable features of this limiting shape is its *universality*: the specific details of the models are irrelevant. In particular, the conclusion remains valid if we

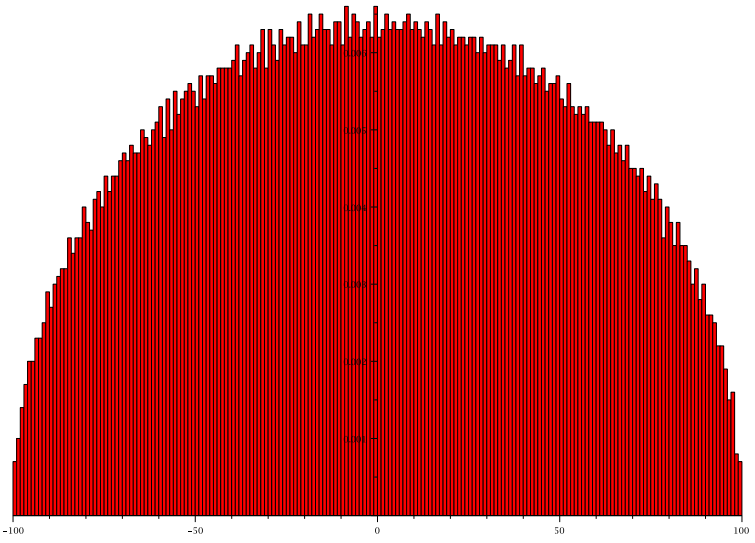


Figure 7: Histogram of the eigenvalues of a uniform random graph on $n = 10^4$ vertices

change the Bernoulli parameter to any value $p = p(n)$ such that $\min\{np(n), n(1-p(n))\} \to \infty$. The independence assumption can also be relaxed: for example, the semi-circle law still arises in the limit if we condition the graph to be $d(n)-$regular with $\min(d(n), n - d(n)) \to \infty$ [51]. Note that in both cases, the graph is required to be dense: $|E_G| \gg |V_G|$. This raises the following question, to which the present lecture notes are devoted:

Question 1. *What does μ_G typically look like when G is* sparse *?*

A quick look at Figure 8 will convince the reader that Wigner's universality breaks down in the sparse regime: when $|E_G| \asymp |V_G|$, the spectral statistics of A depend heavily on the details of the underlying model. From a practical point-of-view, this is actually good news: the spectral measure is then relevant for distinguishing and classifying real-worlds networks [33, 52]! From a more theoretical point-of-view, understanding the asymptotic relation between the geometry and the spectrum of sparse random graphs constitutes a fascinating research program. For the sake of simplicity, we will here focus on the convergence of the empirical spectral distribution and will not discuss refined questions such as local statistics, extremal eigenvalues, eigenvectors, etc. We also note that various other matrices can be naturally associated with graphs: the arguments developed here extend effortlessly as long as these matrices are *local* in the sense that $(G, o) \mapsto A_G^k(o, o)$ is continuous for each $k \geq 0$. This is the case, in particular, for the laplacian matrix, the transition matrix for simple random walk, and their weighted versions.

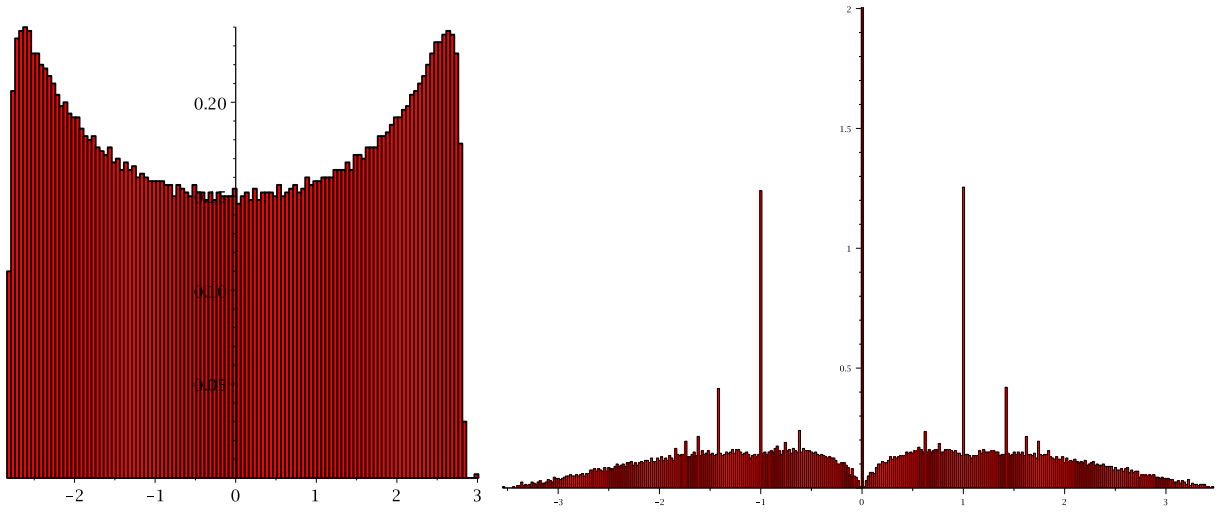


Figure 8: Histogram of eigenvalues of two random graphs with 10000 vertices and 15000 edges: a random regular graph (left) and a Erdős-Rényi (right).

3.2 The spectral continuity theorem

We are now in position to state the main result of this chapter. In the case of bounded degrees, the weak convergence is easy and was noted in [19]. The Kolmogorov-Smirnov refinement is due to [2]. A full proof encompassing the case of unbounded degrees can be found in [18].

Theorem 6 (Spectral continuity)**.** *If $(G_n)_{n\geq 1}$ admits a local weak limit $\mathcal{L}$, then $(\mu_{G_n})_{n\geq 1}$ admits a weak limit $\mu_{\mathcal{L}} \in \mathcal{P}(\mathbb{R})$. Moreover, the convergence holds in the Kolmogorov-Smirnov metric:*

$$\sup_{\lambda\in\mathbb{R}} |\mu_{G_n}(]-\infty,\lambda]) - \mu_{\mathcal{L}}(]-\infty,\lambda])| \quad \xrightarrow[n\to\infty]{} \quad 0.$$

The sole *existence* of a limiting spectral distribution already constitutes a remarkable unification of several results obtained (even recently) for specific models, including random regular graphs [42], the Erdős-Rényi model [54, 39], uniform random trees [13], or the preferential attachment model [45]. Moreover, the Kolmogorov-Smirnov convergence guarantees the convergence of atomic masses:

$$\forall \lambda \in \mathbb{R}, \qquad \mu_{G_n}(\{\lambda\}) \quad \xrightarrow[n\to\infty]{} \quad \mu_{\mathcal{L}}(\{\lambda\}). \tag{30}$$

In particular, the case $\lambda = 0$ yields the convergence of the relative rank of the adjacency matrices. But there is more: as we will now see, the limiting measure $\mu_{\mathcal{L}}$ can be explicitly described in terms of a certain random operator defined on $\mathcal{L}$. Thus, the spectral analysis of sparse graphs can, in principle, be performed directly at the level of their local weak limits. The construction of $\mu_{\mathcal{L}}$ relies on the spectral theorem for self-adjoint operators on Hilbert spaces, which we now recall.

Infinite-dimensional spectral theory. Let $G = (V, E)$ be a countable, locally finite graph. Consider the Hilbert space $\mathcal{H} = \ell^2_{\mathbb{C}}(V)$ and its canonical orthonormal basis $(\mathfrak{e}_o)_{o \in V}$, where

$$\mathfrak{e}_o \colon x \quad \longmapsto \quad \begin{cases} 1 & \text{if } x = o \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

By definition, the adjacency operator $A = A_G$ is the linear operator on $\mathcal{H}$ whose domain is the (dense) subspace of finitely-supported functions, and whose action on the above basis is given by

$$\langle \mathfrak{e}_x | A \mathfrak{e}_y \rangle \quad = \quad \begin{cases} 1 & \text{if } \{x, y\} \in E \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

A is symmetric, and this already ensures that $A - z$ is injective for each $z \in \mathbb{C} \setminus \mathbb{R}$. If in addition the range of $A - z$ is dense, then A (or G itself) is said to be (essentially) *self-adjoint*. In that case, the resolvent $(A - z)^{-1}$ extends to a unique bounded linear operator on $\mathcal{H}$ and for each $o \in V$, the spectral theorem for self-adjoint operators (see, e.g. [47, Chapter VII]) yields the representation

$$\langle \mathfrak{e}_o | (A - z)^{-1} \mathfrak{e}_o \rangle \quad = \quad \int_{\mathbb{R}} \frac{1}{\lambda - z} \, \mu_{(G,o)}(d\lambda), \qquad (z \in \mathbb{C} \setminus \mathbb{R}) \tag{33}$$

for a unique Borel probability measure $\mu_{(G,o)}$ on $\mathbb{R}$ known as the spectral measure of the pair $(A, \mathfrak{e}_o)$.

The spectrum of a unimodular measure. To gain some intuition, let us first consider the case where G is finite: then there is an orthonormal basis of $n = |V_G|$ eigenfunctions $\phi_1, \ldots, \phi_n$ of A with respective eigenvalues $\lambda_1, \ldots, \lambda_n$, and we easily compute

$$\mu_{(G,o)} \quad = \quad \sum_{k=1}^{n} |\phi_k(o)|^2 \delta_{\lambda_k}. \tag{34}$$

In words, $\mu_{(G,o)}$ is a mixture of atoms at the eigenvalues of A, their masses being the squared norm of the orthogonal projection of $\mathfrak{e}_o$ onto the corresponding eigenspaces. In particular, we see that

$$\mu_G \quad = \quad \frac{1}{|V_G|} \sum_{o \in V_G} \mu_{(G,o)}. \tag{35}$$

Since the right-hand side is nothing but an expectation under uniform rooting, it is natural to extend the definition of the spectral distribution (28) to any unimodular measure $\mathcal{L}$ by setting

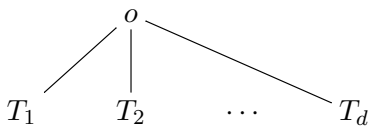
$$\mu_{\mathcal{L}}(\cdot) \quad := \quad \mathbb{E}\left[\mu_{(G,o)}(\cdot)\right] \qquad \text{where} \qquad (G, o) \sim \mathcal{L}. \tag{36}$$

This is the limiting measure appearing in Theorem 6. We emphasize that there are infinite graphs whose adjacency operator is not self-adjoint, see, e.g., [46]. However, such pathological graphs have zero measure under any unimodular law (see [18, Proposition 2.2]), so that the definition (36) makes perfect sense. Regarding measurability issues, let us simply note that the map $(G, o) \mapsto \mu_{(G,o)}$ is continuous when restricted to self-adjoint elements of $\mathcal{G}_\star$ (see e.g., [48][Lemma 2.2]).

3.3 The case of trees

As many graph-theoretical quantities, spectral measures admit a recursive structure on trees.

A recursive formula. Computing the crucial quantity $\mathfrak{s}_{(G,o)}(z) := (A_G - z)^{-1}_{oo}$ appearing in (33) happens to be easy when (G, o) is a finite rooted tree T: let us remove the root o to create $d = \deg(T, o)$ smaller rooted trees $T_1, \dots, T_d$, as depicted in the following diagram:



The adjacency matrix of $T \setminus o$ is block-diagonal, and an easy application of the Schur complement formula yields the following recursive identity:

$$\mathfrak{s}_T(z) \quad = \quad \frac{-1}{z + \sum_{i=1}^{d} \mathfrak{s}_{T_i}(z)}. \tag{37}$$

With some effort, this recursion can be extended to infinite trees (see [48] for details). This is of course a crucial tool for analysing the spectrum of unimodular random trees.

Fixed-point equations. Consider the case where T is a random Galton-Watson tree. Conditionally on the root-degree $d = \deg(T, o)$, the subtrees $T_1, \dots, T_d$ are independent with the same law as T. Consequently, (37) yields a distributional fixed-point equation for the law of the random variable $\mathfrak{s}_T(z)$. The solution can be shown to be unique, and plugging it into (33) gives access to the Cauchy-Stieltjes transform of the spectral measure. Since this transform is injective, we have, in principle, reduced the spectral analysis of unimodular Galton-Watson trees to the study of the distributional fixed point equation (37). In the degenerate case where the degree distribution is a Dirac mass $\pi = \delta_d$, the resolution is trivial and we recover the Kesten-McKay distribution (**??**). For non-degenerate degrees however, such as the case $\pi = \text{Poisson}(c)$ describing the $n \to \infty$ limit of the Erdős-Rényi model with n vertices and edge probability $\frac{c}{n}$, extracting information from (37) turns out to be challenging, and very little has been said about $\mu_{\text{UGWT}}(\pi)$. Notice how the situation trivializes in the large-degree limit $c \to \infty$: since the sum in (37) consists of a large number of i.i.d. terms, it concentrates around its mean and we obtain that $\sqrt{c}\,\mathfrak{s}_{PGWT(c)}(\sqrt{c}z) \to \mathfrak{s}(z)$, where

$$\mathfrak{s}(z) \quad = \quad \frac{-1}{z + \mathfrak{s}(z)}. \tag{38}$$

This is the Stieltjes transform of the semi-circle law. For a second-order refinement, see [31].

4 State of the art and open problems

Atomic mass at zero. If $\mathfrak{s}$ is the Stieltjes transform of a measure μ, then for any fixed $\lambda \in \mathbb{R}$,

$$\varepsilon\, \mathfrak{s}(\lambda + i\varepsilon) \quad \xrightarrow[\varepsilon \to 0]{} \quad \mu(\{\lambda\}). \tag{39}$$

The main result of [20] is the explicit resolution of the equation (37) on a general unimodular Galton-Watson tree in the limit where $z \to 0$ along the imaginary axis, with the following consequence:

Theorem 7 (Mass at zero)**.** *Consider an arbitrary distribution $\pi \in \mathcal{P}(\mathbb{Z}_+)$ with finite mean, and let $\phi(x) = \sum_k \pi_k x^k$ denote its generating series. Then,*

$$\mu_{\mathrm{UGWT}(\pi)}(\{0\}) \quad = \quad \max_{x \in [0,1]} \left\{ \phi(x) + (1-x)\phi'(x) + \phi\left(1 - \frac{\phi'(x)}{\phi'(1)}\right) - 1 \right\}.$$

In particular, Theorem 7 describes the asymptotic dimension of the kernel of the adjacency matrix of random graphs with asymptotic degree distribution π. In the special case $\pi = \mathrm{Poisson}(c)$ corresponding to the Erdős-Rényi model, the formula evaluates to $x_* + e^{-cx_*} + cx_* e^{-cx_*} - 1$, where $x_* \in (0,1)$ denotes the smallest root of $x = e^{-ce^{-cx}}$: this settles a conjecture of Bauer and Golinelli [9] and answers a question of Costello and Vu [26].

Dense atomic support. More generally, one may wonder about the value of $\mu_{\mathrm{UGWT}(\pi)}(\{\lambda\})$ at other locations $\lambda \in \mathbb{R}$. During the 2010 AIM Workshop on Random Matrices, Ben Arous [1, Problem 14] raised the question of determining the set of spectral atoms

$$\Sigma_{p.p.}(\mathcal{L}) \quad := \quad \{\lambda \in \mathbb{R} \colon \mu_{\mathcal{L}}(\{\lambda\}) > 0\}$$

of the local weak limit $\mathcal{L} = \textsc{ugwt}(\mathrm{Poisson}(c))$ corresponding to the Erdős-Rényi model. A trivial outer-bound is obtained by observing that any eigenvalue of a finite graph must, by definition, belong to the ring $\mathbb{A}$ of *totally real algebraic integers*, i.e. roots of real-rooted monic polynomials with integer coefficients: in view of Theorem 6, any local weak limit immediately inherits from the same property. On the other side, a crude inner-bound is given by the fact that any finite tree has positive probability under the Poisson-Galton-Watson law. This leaves us with the inclusions

$$\bigcup_{T \colon \text{finite tree}} \Sigma_{p.p.}(T) \;\subseteq\; \Sigma_{p.p.}(\textsc{ugwt}(\mathrm{Poisson}(c))) \;\subseteq\; \mathbb{A}.$$

Perhaps surprisingly, the inner and outer-bound turn out to coincide, as I established in [49].

Theorem 8 ([49])**.** *Every totally real algebraic integer is an eigenvalue of some finite tree.*

The weaker statement obtained by replacing *tree* by *graph* was conjectured forty years ago by Hoffman [35], and established seventeen years later by Estes [32], see also [8]. The recursion (37) plays a crucial role in the proof. As a by-product, Theorem 8 solves Ben Arous' question:

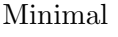
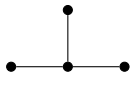
Algebraic integer λ	0	± 1	$\pm\sqrt{2}$	$\pm\sqrt{3}$	$\pm\frac{1+\sqrt{5}}{2}$	$\pm\frac{1-\sqrt{5}}{2}$	$\cdots$
Minimal tree							$\cdots$
Complexity $\tau(\lambda)$	1	2	3	4			$\cdots$

Table 1: The totally real algebraic integers λ with tree-complexity $\tau(\lambda) \leq 4$.

Corollary 9 (Atomic support, [49])**.** *Assume that $\pi_k > 0$ for each $k \geq 1$. Then, the atoms of $\mu_{\mathrm{UGWT}(\pi)}$ are precisely the totally real algebraic integers, i.e., $\Sigma_{p.p.}(\mathrm{UGWT}(\pi)) = \mathbb{A}$.*

Finite atomic support. The work [50] develops a general understanding of the mass $\mu_{\mathcal{L}}(\{\lambda\})$ assigned to an arbitrary $\lambda \in \mathbb{R}$. Let us define the $\lambda-$*support* of a self-adjoint graph $G = (V, E)$ as

$$\mathbb{S}_\lambda \quad := \quad \left\{o \in V : \mu_{(G,o)}(\{\lambda\}) > 0\right\}. \tag{40}$$

When G is finite, it follows from (34) that $o \in \mathbb{S}_\lambda$ if and only if there is an eigenfunction for λ which does not vanish at o. Let us say that a finite graph $G = (V, E)$ is $\lambda-$*prime* if λ is an eigenvalue of the adjacency matrix of G but not of $G \setminus \{o\}$, for any $o \in V$. Note that the eigenvalue λ is then necessarily simple, with a nowhere vanishing associated eigenfunction. We use the standard notation $\deg_S(o)$ for the number of neighbors of $o \in V$ in the subset $S \subseteq V$, and ∂S for the (external) *boundary* of S, consisting of those vertices outside S having at least one neighbor in S. We will use the short-hand ∂o instead of $\partial\{o\}$ to denote the set of neighbors of o.

Theorem 10. *Let $\mathcal{L}$ be a unimodular measure supported on trees, and let $\lambda \in \mathbb{R}$. Then almost-surely under $\mathcal{L}$, the connected components induced by $\mathbb{S}_\lambda$ are finite $\lambda-$prime trees, and*

$$\mu_{\mathcal{L}}(\{\lambda\}) \quad = \quad \mathbb{P}\left(o \in \mathbb{S}_\lambda\right) - \frac{1}{2}\mathbb{E}\left[\deg_{\mathbb{S}_\lambda}(o)\mathbf{1}_{(o\in\mathbb{S}_\lambda)}\right] - \mathbb{P}\left(o \in \partial\mathbb{S}_\lambda\right). \tag{41}$$

Using the Mass Transport Principle, one may easily rewrite (41) into the *component form*

$$\mu_{\mathcal{L}}(\{\lambda\}) \quad = \quad \mathbb{E}\left[\frac{\mathbf{1}_{(o\in\mathbb{S}_\lambda)}}{|\mathcal{C}(\mathbb{S}_\lambda, o)|}\left(1 - \sum_{x\in\partial\mathcal{C}(\mathbb{S}_\lambda,o)} \frac{1}{\deg_{\mathbb{S}_\lambda}(x)}\right)\right]. \tag{42}$$

Now, assume that all degrees lie in $\{\delta, \ldots, \Delta\}$ for some fixed integers $\Delta \geq \delta \geq 3$. Since $|\partial S| \geq |S|(\delta - 2) + 2$ for any finite subset S of vertices, we deduce the L^∞ bound

$$\mu_{\mathcal{L}}(\{\lambda\}) \quad \leq \quad \frac{1}{\tau(\lambda)}\left(1 - \frac{(\delta-2)\tau(\lambda)+2}{\Delta}\right)_+,$$

where $\tau(\lambda)$ is the *tree-complexity* of the totally real algebraic integer λ, defined as the minimum possible size of a tree with eigenvalue λ. This crude inequality has a surprising consequence:

Corollary 11. *Fix two integers $\Delta \geq \delta \geq 3$. If $\mathcal{L}$ is a unimodular network concentrated on trees with degrees in $\{\delta, \dots, \Delta\}$, then $\mu_{\mathcal{L}}$ has only finitely many atoms. More precisely,*

$$\Sigma_{p.p.}(\mathcal{L}) \quad \subseteq \quad \left\{ \lambda \in \mathbb{A} \colon \tau(\lambda) < \frac{\Delta - 2}{\delta - 2} \right\}.$$

For example, if $\frac{\Delta-2}{\delta-2} \leq 2$, then $\Sigma_{p.p.}(\mathcal{L}) \subseteq \{0\}$, while if $\frac{\Delta-2}{\delta-2} \leq 3$ then $\Sigma_{p.p.}(\mathcal{L}) \subseteq \{-1, 0, +1\}$ (see Table 1). The constraint $\delta > 2$ may be relaxed to a control on the *anchored isoperimetric constant*,

$$\mathfrak{i}^\star(G, o) \quad := \quad \lim_{n \to \infty} \inf \left\{ \frac{|\partial S|}{|S|} \colon o \in S \subseteq V(G),\, G_{|S} \text{ is connected},\, n \leq |S| < \infty \right\}. \tag{43}$$

Theorem 12. *Fix $\Delta < \infty$, $\mathfrak{i}^\star > 0$. If $\mathcal{L}$ is a unimodular measure supported on trees with degrees in $\{2, \dots, \Delta\}$ and anchored isoperimetric constant at least $\mathfrak{i}^\star$, then*

$$\Sigma_{p.p.}(\mathcal{L}) \quad \subseteq \quad \left\{ \lambda \in \mathbb{A} \colon \tau(\lambda) < \frac{3\Delta^2}{\mathfrak{i}^\star} \right\}.$$

The anchored isoperimetric constant of $\textsc{ugwt}(\pi)$ is deterministic and strictly positive whenever $\pi_0 = \pi_1 = 0$ and $\pi_2 < 1$ [24, Corollary 1.3]. Thus, we answer [22, Question 1.8] in the affirmative:

Corollary 13. *If π has finite support and $\pi_1 = 0$, then $\Sigma_{p.p.}(\textsc{ugwt}(\pi))$ is finite.*

Emergence of a continuous part. Spectral measures are of course not limited to their pure-point parts. Actually, many fascinating questions concern the continuous part of the spectrum. The main result in [22] is a very general criterion ensuring the existence of a non-trivial continuous part in the spectrum of a unimodular random tree:

Theorem 14. *Let $\mathcal{L}$ be a unimodular measure concentrated on trees. Suppose that the tree contains a bi-infinite path with positive probability under $\mathcal{L}$. Then*

$$\sum_{\lambda \in \mathbb{A}} \mu_{\mathcal{L}}(\{\lambda\}) \quad < \quad 1. \tag{44}$$

In other words, $\mu_{\mathcal{L}}$ contains a non-trivial continuous part.

Note that this is almost optimal: when $\mathcal{L}$ is supported on finite trees, we trivially have $\mu_{\mathcal{L}}(\mathbb{A}) = 1$, i.e. there is no continuous part. The critical case of one-ended trees is still open. In particular, it is unclear whether the spectrum of the Infinite Skeleton Tree contains a non-trivial continuous part. Beyond trees, it was shown in [16] that the spectral measure of the supercritical Bernoulli bond percolation on $\mathbb{Z}^2$ admits a non-trivial continuous part. Extending this to higher dimensions is an important open problem, see [22, Question 1.9].

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