TD1. Brownian motion. Martingales. Stopping times.

Unless stated otherwise, we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion starting form the origin on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. We also denote by $\mathcal{F}_t := \sigma(B_s,\ s \leq t)$ the filtration generated by the Brownian motion.

Exercise 1 Prove that (a.s.) Brownian motion is not monotone on any interval.

Exercise 2 We define the *Brownian bridge* as the process $Z_t = B_t - tB_1$ $(0 \leq t \leq 1)$.

1. Show that Z is a Gaussian process independent of B_1
2. Compute the mean and covariance function of Z.
3. Is $(Z_t)_{0\leq t\leq 1}$ a martingale w.r.t $(\mathcal{F}_t)_{t\geq 0}$?
4. Prove that Z has the same law as the process Y defined by

$$Y_t = \begin{cases} (1-t)B_{\frac{t}{(1-t)}} & (0 \leq t < 1), \\ 0 & (t = 1). \end{cases}$$

Exercise 3 In this exercise, $\mathbf{B}$ is a d-dimensional ($d \in \mathbb{N}$) Brownian motion, that is $\mathbf{B}_t = (B_t^{(1)}, \dots, B_t^{(d)})$, where the $B^{(i)}$'s are independent standard Brownian motions. Let U be a $d \times d$ orthogonal matrix. Prove that the process $(\mathbf{W}_t)_{t\geq 0} = (U\mathbf{B}_t)_{t\geq 0}$ is a d-dimensional Brownian motion.

Exercise 4 (Lévy's characterization of Brownian motion) On a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$, let $M = (M_t)_{t\geq 0}$ be a continuous square-integrable martingale with $M_0 = 0$ and $\langle M_t \rangle = t$ for all $t \geq 0$. Let F be any twice-differentiable function $F \colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ with bounded first and second-order derivatives. Define :

$$Z_t = F(t, M_t) - \frac{1}{2}\int_0^t \Big(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\Big)(u, M_u)\mathrm{d}u, \qquad t \geq 0.$$

1. Prove that Z is a martingale.
2. Applying the previous question to $F(t,x) = \exp(i\theta x + \frac{1}{2}\theta^2 t)$, where $\theta \in \mathbb{R}$, prove that M is a Brownian motion.

Exercise 5 The goal of the exercise is to compute (by hand) the bracket of the square-integrable martingale $(M_t)_{t\geq 0} = (B_t^2 - t)_{t\geq 0}$.

1. Compute $\mathrm{E}(B_t^4 | \mathcal{F}_s)$ for all $0 \leq s \leq t$.
2. Let $X_t = \int_0^t B_s^2 \mathrm{d}s$. Compute $\mathrm{E}(X_t | \mathcal{F}_s)$ for all $0 \leq s \leq t$.
3. Deduce thereof that $\langle M \rangle_t = 4X_t$.

Exercise 6 For $t \geq 0$, define $A_t = \int_0^t B_s \mathrm{d}s$.

1. Justify why A is a well-defined $(\mathcal{F}_t)_{t\geq 0}$-adapted process.
2. Determine the law of A_t for all $t \geq 0$.
3. Is $(A_t)_{t\geq 0}$ a martingale w.r.t $(\mathcal{F}_t)_{t\geq 0}$?

Exercise 7 For $a > 0$, define $T_a = \inf\{t > 0 \colon B_t \geq a\}$.

1. Show that T_a is a $(\mathcal{F}_t)$-stopping time.
2. Prove that for all $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$\mathrm{E}\Big[\exp\Big(\lambda B_{T_a \wedge n} - \frac{1}{2}\lambda^2 (T_a \wedge n)\Big)\Big] = 1.$$

3. Deduce thereof the Laplace transform of T_a :

$$\mathrm{E}[\exp(-uT_a)] = \exp(-a\sqrt{2u}), \qquad (u > 0),$$

and show that $\mathrm{P}(T_a < +\infty) = 1$.

Université Paris-Dauphine
M2 MATH/MASEF

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Stochastic calculus

TD2. Local martingales. Wiener and Itô integrals.

Unless stated otherwise, we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion starting form the origin on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. We also denote by $\mathcal{F}_t := \sigma(B_s,\, s \leq t)$ the filtration generated by the Brownian motion.

Exercise 1 Fix $T \geq 0$. Show that a continuous local martingale $M = (M_t)_{t\in[0,T]}$ is a square-integrable martingale if and only if $M_0 \in L^2$ and $\langle M\rangle_T \in L^1$.

Exercise 2 Let M be a continuous local martingale such that a.s., $\langle M\rangle_\infty = +\infty$. Prove that a.s.,

$$\limsup_{t\to\infty} M_t = +\infty \quad \text{and} \quad \liminf_{t\to\infty} M_t = -\infty.$$

Exercise 3 Let

$$Z_t := (1-t)\int_0^t \frac{\mathrm{d}B_s}{(1-s)}, \qquad 0 \leq t < 1.$$

1. Show that Z is a Gaussian process. Determine its mean and covariance function. How is this process called ?
2. Prove that $Z_t \to 0$ in $L^2(\Omega)$ as $t \to 1$.
3. Prove that $Z_t \to 0$ a.-s. as $t \to 1$. *Hint* : prove that for $n \geq 1$ and $\varepsilon > 0$,

$$\mathrm{P}\Big(\sup_{2^{-(n+1)}\leq t\leq 2^{-n}} |Z_{1-t}| > \varepsilon\Big) \leq \frac{1}{(2^{n-1}\varepsilon^2)}.$$

Exercise 4 Determine the law of the process

$$X_t = e^{-t}\Big(X_0 + \int_0^t e^u \mathrm{d}B_u\Big), \qquad t \geq 0,$$

where X_0 is a $\mathcal{N}(0, 1/2)$ random variable independent of $(B_t)_{t\geq 0}$. Show that the process is stationary. How is it called ?

Exercise 5 Determine the law of the process

$$X_t = \int_0^{\sqrt{t}} \sqrt{2u}\ \mathrm{d}B_u, \qquad t \geq 0.$$

Exercise 6 Let $t \geq 0$ and for $n \in \mathbb{N}$, $0 \leq k \leq n$, define $t_k = tk/n$.

1. Show that

$$B_t^2 = \sum_{k=1}^n 2B_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}) + \sum_{k=1}^n (B_{t_k} - B_{t_{k-1}})^2.$$

2. Deduce thereof that for all $t \geq 0$,

$$B_t^2 - t = \int_0^t 2B_s\ \mathrm{d}B_s\ , \qquad \text{a.s.}$$

3. Retrieve the final result of **Exercise 5** in **TD1**.

Exercise 7 Using the same notation as in the previous exercise :

1. Prove that for all $p > 2$, $\sum_{k=1}^n (B_{t_k} - B_{t_{k-1}})^p \to 0$ in probability as $n \to \infty$.
2. Prove that for all continuous function $f\colon \mathbb{R}_+ \to \mathbb{R}$, $\sum_{k=1}^n f(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2 \to \int_0^t f(B_s)\mathrm{d}s$ in probability as $n \to \infty$.
3. With a similar method to **Exercise 6**, show that $\sin(B_t) = -\frac{1}{2}\int_0^t \sin(B_s)\mathrm{d}s + \int_0^t \cos(B_s)\mathrm{d}B_s$ a.-s.

Université Paris-Dauphine — Academic year 2022-2023

M2 MATH/MASEF — Stochastic calculus

TD3. (Generalized) Itô integrals. Itô processes. Itô's formula.

Unless stated otherwise, we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion starting form the origin on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. We also denote by $\mathcal{F}_t := \sigma(B_s,\, s \leq t)$ the filtration generated by the Brownian motion.

Exercise 1 Check that a process ϕ is progressive if and only if the map $(t, \omega) \mapsto \phi_t(\omega)$ is measurable w.r.t. the progressive σ-field $\mathcal{P}$ defined in the Lecture Notes (Remark 2.2).

Exercise 2 Do the following continuous adapted processes belong to $\mathbb{M}^2(\mathbb{R}^+)$? Or $\mathbb{M}^2$? Or $\mathbb{M}^2_{\mathrm{LOC}}$? (same notation as in the Lecture Notes).

1. $t \in \mathbb{R}^+ \mapsto B_t$;
2. $t \in \mathbb{R}^+ \mapsto \frac{B_t}{1+t^2}$;
3. $t \in \mathbb{R}^+ \mapsto \exp(B_t^2)$.

Exercise 3 Let X be an Itô process started at the origin, that is

$$X_t = \int_0^t \phi_u \mathrm{d}B_u + \int_0^t \psi_u \mathrm{d}u, \qquad t \geq 0,$$

where $\phi \in \mathbb{M}^2_{\mathrm{LOC}}$ and $\psi \in \mathbb{M}^1_{\mathrm{LOC}}$. Prove that X is a square-integrable martingale if and only if $\psi \equiv 0$ and $\phi \in \mathbb{M}^2$.

Exercise 4 Let $\alpha\colon \mathbb{R}^+ \to \mathbb{R}$ be a continuous function. With the help of the stochastic integration by parts formula, check that the process

$$X_t := \int_0^t \exp\Big(\int_s^t \alpha(u)\mathrm{d}u\Big)\mathrm{d}B_s, \qquad t \geq 0,$$

satisfies the stochastic differential equation $\mathrm{d}X_t = \alpha(t)X_t\mathrm{d}t + \mathrm{d}B_t$.

Exercise 5 With the help of Itô's formula, answer to Question (3) in **Exercise 7**, **TD2**.

Exercise 6 Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be continuously twice-differentiable. Show that the process

$$X_t := f(B_t) - \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s, \qquad t \geq 0,$$

is a continuous local martingale. Give a sufficient condition for X to be a martingale.

Exercise 7 Let $n \geq 1$ and $x_1, \dots, x_n$ be distinct points in $\mathbb{R}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f \in C^1(\mathbb{R})$ and $f \in C^2(\mathbb{R} \setminus \{x_1, \dots x_n\})$. We assume that f'' remains bounded in a neighborhood of x_i for all $1 \leq i \leq n$. The goal of the exercise is to prove that Itô's formula is still valid for such a function.

1. Prove that for all $t \geq 0$, the set $\{(\omega, s) \in \Omega \times [0, t] \colon B_s(\omega) \in \{x_1, \dots x_n\}\}$ has zero measure w.r.t. $\mathrm{P} \otimes \mathrm{d}s$, where $\mathrm{d}s$ stands for Lebesgue measure.
2. Prove that Itô's formula is valid when we make the additional assumption that f has compact support. *Hint : approximate f by a sequence of continuously twice-differentiable functions.*
3. Conclude in the general case.

Exercise 8 (Tanaka's formula and local time of Brownian motion) For $\varepsilon > 0$, define

$$\phi_\varepsilon(x) = \begin{cases} |x| & (|x| \geq \varepsilon) \\ \frac{1}{2}(\varepsilon + \frac{x^2}{\varepsilon}) & (|x| < \varepsilon). \end{cases}$$

1. With the help of **Exercise 7**, prove that P-a.-s., for all $t \geq 0$,

$$\phi_\varepsilon(B_t) = \int_0^t \phi'_\varepsilon(B_s)\mathrm{d}B_s + \frac{1}{2\varepsilon}\mathrm{Leb}(\{0 \leq s \leq t \colon |B_s| < \varepsilon\}),$$

   where « Leb » stands for Lebesgue measure.
2. Check that

$$\int_0^t \phi'_\varepsilon(B_s)1_{\{|B_s|<\varepsilon\}}\mathrm{d}B_s \overset{\varepsilon\to 0}{\longrightarrow} 0, \qquad \text{in } L^2(\Omega).$$

3. Deduce thereof **Tanaka's formula** : P-a.-s., for all $t \geq 0$,

$$|B_t| = \int_0^t \mathrm{sgn}(B_s)\mathrm{d}B_s + L_t,$$

   where L_t is the limit in $L^2(\Omega)$ of $\frac{1}{2\varepsilon}\mathrm{Leb}(\{0 \leq s \leq t \colon |B_s| < \varepsilon\})$, as $\varepsilon \to 0$, and

$$\mathrm{sgn}(x) = 1 \text{ if } x > 0, \qquad \mathrm{sgn}(x) = -1 \text{ if } x \leq 0.$$

4. Prove that the process (L_t) is a.-s. continuous and non-decreasing. This process is called **local time** of the Brownian motion (at the origin).

TD4. Multivariate Itô's formula. Exponential martingales and Girsanov's theorem.

Unless stated otherwise, we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion starting from the origin on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. We denote by $\mathcal{F}_t := \sigma(B_s,\, s \leq t)$ the filtration generated by the Brownian motion and write $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$.

Exercise 1 (Transience and strict local martingales) In this exercise, $B = (B^{(1)}, B^{(2)}, B^{(3)})$ is a three-dimensional Brownian motion starting from $x_0 \in \mathbb{R}^3 \setminus \{0\}$ (recall the definition in **Exercise 3, TD1**). We define $G(x) := 1/\|x\|$ for all $x \in \mathbb{R}^3 \setminus \{0\}$, where $\|\cdot\|$ is the Euclidian norm and

$$T_a = \inf\{t \geq 0\colon \|B_t\| = a\}, \qquad a \geq 0.$$

1. Check that $\langle B^{(i)}, B^{(j)}\rangle \equiv 0$ whenever $i \neq j$, where $i, j \in \{1,2,3\}$.
2. Check that $\Delta G(x) = 0$ for all $x \neq 0$.
3. Using Itô's formula, prove that for all $r > 0$, $\{G(B_{t\wedge T_r})\}_{t\geq 0}$ is a martingale.
4. Assume $0 < r < \|x_0\| < R < \infty$. Show that $T_R < \infty$ a.s. (you may use **Exercise 7** in **TD 1**) and
$$\mathrm{P}(T_r < T_R) = \frac{R^{-1} - \|x_0\|^{-1}}{R^{-1} - r^{-1}}.$$
5. Deduce from the previous formula that a.s. $B_t \neq 0$ for all $t \geq 0$.
6. Prove that $\{G(B_t)\}_{t\geq 0}$ is a continuous local martingale.
7. Check that $\{G(B_t)\}_{t\geq 0}$ is a non-negative super-martingale and deduce thereof that $\|B_t\|$ a.s. converges to $+\infty$, as $t \to \infty$.
8. Prove that $\{G(B_t)\}_{t\geq 0}$ is bounded in L^2 but is not a martingale.

Exercise 2 (Exponential martingales : a simple case) Let $\phi\colon \mathbb{R}^+ \to \mathbb{R}$ be a (deterministic) function in $L^2_{\mathrm{loc}}(\mathbb{R}^+)$ and $Z^\phi = (Z^\phi_t)_{t\geq 0}$ the associated Doléans-Dade exponential process. Check that Z^ϕ is a martingale.

Exercise 3 (Exponential martingales : an example) Find a progressive process $X = (X_t)_{t\geq 0}$ such that the process $Z = (Z_t)_{t\geq 0}$ defined by $Z_t = \exp(X_t - B_t^2)$ is a martingale.

Exercise 4 (Change of drift) Let $X = (X_t)_{t\geq 0}$ be an Itô process such that $X_0 = 0$ a.s. and

$$\mathrm{d}X_t = b_1(t)\mathrm{d}t + \sigma(t)\mathrm{d}B_t,$$

where $b_1\colon \mathbb{R}_+ \to \mathbb{R}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+^*$ is in $L^2_{\mathrm{loc}}(\mathbb{R}^+)$ (b_1 and σ are both deterministic). Prove that (under suitable assumptions to be specified) there exists a probability measure Q on $(\Omega, \mathcal{F}_\infty)$ under which X satisfies

$$\mathrm{d}X_t = b_2(t)\mathrm{d}t + \sigma(t)\mathrm{d}\tilde{B}_t,$$

where $b_2\colon \mathbb{R}_+ \to \mathbb{R}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and $\tilde{B}$ is a Brownian motion under Q.

Exercise 5 (Hitting times for Brownian motion with drift) For every $a > 0$ and $b \in \mathbb{R}$, define the following stopping time :

$$T_{a,b} := \inf\{t \geq 0\colon B_t + bt = a\}.$$

The goal of the exercise is to determine $\mathrm{P}(T_{a,b} < \infty)$ using the result of **Exercise 7** in **TD1** (case $b = 0$) via a change of measure argument.

1. Using an appropriate martingale and Girsanov's theorem, prove that for all $t \geq 0$,
$$\mathrm{P}(T_{a,b} > t) = \mathrm{E}\Big[\mathbb{1}_{\{T_a > t\}} \exp\Big(bB_t - \frac{1}{2}b^2 t\Big)\Big].$$

2. Deduce thereof
$$\mathrm{P}(T_{a,b} \le t) = \mathrm{E}\Big[1_{\{T_a \le t\}} \exp\Big(bB_{t\wedge T_a} - \frac{1}{2}b^2(t \wedge T_a)\Big)\Big].$$

3. Conclude that
$$\mathrm{P}(T_{a,b} < \infty) = \begin{cases} 1 & (b \ge 0) \\ \exp(2ab) & (b < 0). \end{cases}$$

Exercise 6 (Onsager-Machlup function) Let $h\colon [0,1] \to \mathbb{R}$ be a twice continuously differentiable (and deterministic) function, with $h(0) = 0$. We equip the space of continuous functions $C_0([0,1], \mathbb{R})$ with the norm $\|f\|_\infty := \max_{x\in[0,1]} |f(x)|$. Using an appropriate martingale and change-of-measure argument, prove that
$$\frac{\mathrm{P}(\|B - h\|_\infty \le \varepsilon)}{\mathrm{P}(\|B\|_\infty \le \varepsilon)} \xrightarrow{\varepsilon\to 0} \exp\Big(-\frac{1}{2}\int_0^1 h'(s)^2 \mathrm{d}s\Big).$$
We admit that the denominator in the left-hand side is positive for all $\varepsilon > 0$.

Exercise 7 (Solutions of S.D.E. via Girsanov's theorem) Let $b\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Assume that there exists $g \in L^2(\mathbb{R}_+)$ such that $|b(t,x)| \le g(t)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Using Girsanov's theorem, find a probability measure Q on $\mathcal{F}_\infty$ under which the process defined by
$$B_t - \int_0^t b(s, B_s)\mathrm{d}s, \qquad t \ge 0,$$
is a Brownian motion.

TD5. Stochastic differential equations. Feynman-Kac formula.

Unless stated otherwise, $(\Omega, \mathcal{A}, \mathrm{P})$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $B = (B_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion.

Exercise 1 (Invariant distribution for the solution of a s.d.e) Let $V\colon \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function with bounded second derivative. We consider the following s.d.e :

$$\mathrm{d}X_t = \mathrm{d}B_t - V'(X_t)\mathrm{d}t.$$

1. Prove that for any given initial condition $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathrm{P})$, this equation has a unique solution (up to indistinguishability).
2. What is the generator associated to this solution ?
3. Define $\phi(x) = \exp(-V(x))$, which we assume to be in $L^2(\mathbb{R})$. Show that without any loss in generality, we may as well assume that $\nu(\mathrm{d}x) := \phi^2(x)\mathrm{d}x$ is a probability distribution on $\mathbb{R}$.
4. Prove that for any $f \in \mathcal{C}^2_c(\mathbb{R})$, $\int (Lf)(x)\nu(\mathrm{d}x) = 0$.
5. Assume that X_0 is distributed as ν and prove that for all $t \geq 0$, X_t is distributed as ν too.
6. Give an example for an explicit function V.

Exercise 2 (Feynman-Kac formula in an interval) Let k and f be two continuous functions from $[0,1]$ to $\mathbb{R}$. We assume that $f(0) = f(1) = 0$. Let $u\colon (t,x) \in [0,\infty) \times [0,1] \to \mathbb{R}$ be a continuous function that is also in $\mathcal{C}^{1,2}((0,\infty)\times(0,1),\mathbb{R})$ and solves the following p.d.e :

$$\begin{array}{lll} & \partial_t u = \frac{1}{2}\partial_x^2 u - ku & (t>0,\ 0<x<1); \\ \text{(initial condition)} & u(0,x) = f(x) & (0 \leq x \leq 1); \\ \text{(boundary condition)} & u(t,1) = u(t,0) = 0 & (t \geq 0). \end{array}$$

For all $n \geq 3$, let $I_n := (\frac{1}{n}, 1 - \frac{1}{n})$ and ϕ_n be a $\mathcal{C}^\infty$ compactly supported function such that

$$\phi_n(x) = 0 \text{ if } x \notin (0,1), \qquad \phi_n(x) = 1 \text{ if } x \in I_n.$$

1. Fix $t > 0$ and define for $s \in [0,t]$, $n \geq 3$,
$$X_n(s) = u(t-s, B_s)\phi_n(B_s)\exp\Big(-\int_0^s k(B_v)\phi_n(B_v)\mathrm{d}v\Big),$$
$$T_n = \inf\{t \geq 0\colon B_t \notin I_n\}.$$
Using Itô's formula, prove that $(X_n(s\wedge T_n))_{s\in[0,t]}$ is a square integrable martingale.
2. Let $T = \inf\{t \geq 0\colon B_t \notin (0,1)\}$. We recall that $\mathrm{E}_x(T) = x(1-x)$ for $x \in [0,1]$, where the subscript stands for the starting point of Brownian motion (see Lecture Notes, *Exit time from an interval*). Deduce from the previous question that
$$u(t,x) = \mathrm{E}_x\Big[f(B_t)\exp\Big(-\int_0^t k(B_s)\mathrm{d}s\Big)1_{\{t<T\}}\Big], \qquad \forall x \in (0,1),$$
and, in the case $k \geq 0$, show that $u(t,x)$ converges to zero uniformly in $x \in (0,1)$ as $t \to \infty$.

Exercise 3 (Brownian motion on a circle) Prove that $X_t = (\cos B_t, \sin B_t)$, $t \geq 0$, is the unique solution of the s.d.e :

$$\mathrm{d}X_t = -\frac{1}{2}X_t\mathrm{d}t + RX_t\mathrm{d}B_t, \qquad X_0 = (1,0),$$

where R is (the matrix of) the rotation with angle $\pi/2$.

Exercise 4 (Killed Brownian motion) We consider $u \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ a bounded solution to the following p.d.e :

$$\begin{aligned} & \partial_t u = \tfrac{1}{2}\partial_x^2 u - ku && (t > 0, x \in \mathbb{R}); \\ \text{(initial condition)} \quad & u(0,x) = f(x) && (x \in \mathbb{R}); \end{aligned}$$

where $f, k \colon \mathbb{R} \to \mathbb{R}$ are measurable, with f bounded and k non-negative. In this exercise, we provide a representation of this solution in terms of a killed process. To this purpose, we introduce $\mathcal{E}$ an exponential random variable with parameter one that is independent of Brownian motion and set the *killing time*

$$\kappa := \inf\left\{t \geq 0 \colon \int_0^t k(B_s)\mathrm{d}s \geq \mathcal{E}\right\}.$$

We now define

$$X_t = \begin{cases} B_t & (t < \kappa) \\ \star & (t \geq \kappa). \end{cases}$$

and extend f to $\mathbb{R} \cup \{\star\}$ by setting $f(\star) = 0$. Finally, $\mathcal{G}_t := \sigma(\mathcal{E}, B_s, s \leq t)$.

1. Using the Feynman-Kac formula, show that $u(t,x) = \mathrm{E}_x(f(X_t))$.
2. Show that $X = (X_t)_{t \geq 0}$ is a Markov process.