Exercises for Stochastic Calculus

September 10, 2020

Throughout the manuscript, we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion starting form 0 on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Also denote by $\mathcal{F}_t := \sigma(B_s :\ s \leq t)$ the filtration generated by the Brownian motion.

Exercise 1 Show that

$$\tilde{B}_t := \begin{cases} tB_{\frac{1}{t}} & t > 0 \\ 0 & t = 0 \end{cases}$$

is a Brownian motion, and that

$$\lim_{t\to+\infty} \frac{B_t}{t} = 0, \quad \text{a.s.}$$

Exercise 2

(a) Determine, for all $n \in \mathbb{N}$, the law of the random variable

$$V_n := \frac{1}{n^\alpha} \int_0^n B_s ds, \qquad \text{where } \alpha \in \mathbb{R} \text{ is given.}$$

(b) Let $\alpha > \frac{3}{2}$. Show that for all $\varepsilon > 0$ we have

$$\sum_n \mathbb{P}\Big[|V_n| > \varepsilon\Big] < \infty.$$

Further prove that

$$\lim_{n\to\infty} V_n = 0, \qquad \text{a.s.}$$

What do you find for the case $\alpha \leq \frac{3}{2}$?

Exercise 3 Define a set $A := \{\limsup_{n\to\infty} \frac{B_n}{\sqrt{n}} \geq x\}$ where $x > 0$ is given.

(a) Show that, for all $m \in \mathbb{N}$, the event A is independent from the σ-field $\mathcal{F}_m = \sigma\big(B_s :\ s \leq m\big)$, and that $\mathbb{P}[A] \in \{0, 1\}$.

(b) Prove that $\mathbb{P}[A] = 1$ and that

$$\mathbb{P}\left[\limsup_{t\to\infty} \frac{B_t}{\sqrt{t}} = +\infty\right] = 1 \quad \text{as well as} \quad \mathbb{P}\left[\liminf_{t\to\infty} \frac{B_t}{\sqrt{t}} = -\infty\right] = 1$$

(c) Let $t \in \mathbb{R}^+$. Show that there exists a null set $\mathcal{N}_t$ such that for all $\omega \notin \mathcal{N}_t$, we can find three strictly decreasing sequences converging to 0, $(h_n(\omega))_n$, $(h'_n(\omega))_n$, $(h''_n(\omega))_n$, verifying

$$B_{t+h_n(\omega)}(\omega) - B_t(\omega) > \sqrt{h_n(\omega)}, \quad B_{t+h'_n(\omega)}(\omega) - B_t(\omega) < -\sqrt{h'_n(\omega)}, \quad B_{t+h''_n(\omega)}(\omega) = B_t(\omega)$$

for all $n \in \mathbb{N}$. Deduce that $\mathbb{P}$-a.s. the function $t \mapsto B_t(\omega)$ is not monotone on any non-trivial interval.

Exercise 4 For all $x \in \mathbb{R}$, define $T_x := \inf\{t \geq 0 : \; B_t = x\}$ under the convention that $\inf(\emptyset) = +\infty$. Show that $T_x < \infty$ a.s. and that

$$\lim_{x \to +\infty} T_x = \lim_{x \to -\infty} T_x = +\infty \quad \text{and} \quad \lim_{x \to 0} T_x = 0, \quad \text{a.s.}$$

Exercise 5 Let us admit the law of the iterated logarithm for Brownian motion, namely,

$$\mathbb{P}\left[\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{2t\ln(\ln(\frac{1}{t}))}}\right] = 1.$$

Show that

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t\ln(\ln(t))}} = 1 \quad \text{and} \quad \liminf_{t \to \infty} \frac{B_t}{\sqrt{2t\ln(\ln(t))}} = -1, \quad \text{a.s.}$$

Exercise 6 Let $0 \leq t_0 < t_1 < \cdots < t_n = t$ and $s \geq t$. Compute the value of the conditional expectations:

- $\mathbb{E}[B_s | \; B_{t_0}, B_{t_1}, \cdots, B_{t_n}]$
- $\mathbb{E}[(B_s^2 - s) | \; B_{t_0}, B_{t_1}, \cdots, B_{t_n}]$
- $\mathbb{E}\left[\exp(\lambda B_s - \frac{\lambda^2}{2}s) \middle| \; B_{t_0}, B_{t_1}, \cdots, B_{t_n}\right]$ for all $\lambda \in \mathbb{C}$
- $\mathbb{E}[f(B_s) | \; B_{t_0}, B_{t_1}, \cdots, B_{t_n}]$ where $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Exercise 7 Prove that $(B_t)_{t \geq 0}$, $(B_t^2 - t)_{t \geq 0}$, $\big(\exp(\lambda B_t - \frac{\lambda^2}{2}t)\big)_{t \geq 0}$ are $(\mathcal{F}_t)$-martingale, and study the long time behaviour, as $t \to \infty$, of the martingales.

Exercise 8 Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function and define $X_t = \int_0^t f(s) dB_s$ for all $t \geq 0$.

(a) Show that the process $X := (X_t)_{t \geq 0}$ is a centred Gaussian process and determine its variance.

(b) Show that X and $\big(\exp(X_t - \frac{1}{2}\int_0^t f^2(s)ds)\big)_{t \geq 0}$ are $(\mathcal{F}_t)$-martingales.

(c) Suppose in addition that for all $s \geq 0$, $f(s) \neq 0$. Define $c = \int_0^{+\infty} f^2(s)ds \in (0, +\infty]$. Denote by $\tau : [0, c) \to \mathbb{R}^+$ the continuous function such that $\tau\left(\int_0^t f^2(s)ds\right) = t$ for all $t \geq 0$. Denote $Z_t = X_{\tau(t)}$. Show that the process $Z = (Z_t)_{t \in [0,c)}$ is a Brownian motion on $[0, c)$.

(d) Further prove that, if $\int_0^{+\infty} f^2(s)ds < \infty$, the process X converge a.s. as well as in L^2 to a random variable X_∞, and determine the law of X_∞.

(e) Suppose that $\int_0^{+\infty} f^2(s)ds = \infty$. Define $Y_t := \frac{X_t}{\sqrt{2g(t)\ln\big(\ln(g(t))\big)}}$ with $g(t) := \int_0^t f^2(s)ds$. Compute

$$\limsup_{t \to \infty} Y_t, \quad \liminf_{t \to \infty} Y_t, \quad \limsup_{t \to \infty} X_t, \quad \liminf_{t \to \infty} X_t, \quad \lim_{t \to \infty} \exp\big(X_t - \frac{1}{2}g(t)\big).$$

Exercise 9

(a) Fix $t > 0$ and denote $B^t_s := B_{t+s} - B_t$. Show that the process $(B^t_s)_{s\geq 0}$ is a Brownian motion independent from $\mathcal{F}_t$.

(b) Denote by $\mathcal{C}_b(\mathbb{R})$ the set of continuous bounded functions $f : \mathbb{R} \to \mathbb{R}$. Given $V \in \mathcal{C}_b(\mathbb{R})$, define for all functions $f \in \mathcal{C}_b(\mathbb{R})$ and for all $x \in \mathbb{R}, t \geq 0$

$$\begin{cases} P_t f(x) := \mathbb{E}[f(B_t + x)] \\ P^v_t f(x) := \mathbb{E}\left[f(B_t + x) \exp\left(\int_0^t V(B_s + x) ds\right)\right] \end{cases}.$$

Show that the functions $x \mapsto P_t f(x)$ and $x \mapsto P^v_t f(x)$ are continuous and bounded.

(c) Show that for all $t, s \geq 0$ we have

$$P_{t+s} = P_t \circ P_s, \qquad P^v_{t+s} = P^v_t \circ P^v_s.$$

(d) Deine $U(t, x) := P_t f(X)$ with $f \in \mathcal{C}_b(\mathbb{R})$ for all $t \geq 0, x \in \mathbb{R}$. Show that the function $U \in \mathcal{C}^\infty$ on $(0, +\infty) \times \mathbb{R}$ and

$$\begin{cases} \frac{\partial}{\partial t} U(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} U(t, x), & (t, x) \in \mathbb{R}^+ \times \mathbb{R} \\ \lim_{t \downarrow 0} U(t, x) = f(x), & x \in \mathbb{R} \end{cases}$$

(e) Let $f, V \in \mathcal{C}_b(\mathbb{R})$. Define $\psi(t, x) := P^v_t f(x)$ for all $t \geq 0, x \in \mathbb{R}$. show that $\psi \in \mathcal{C}^2$ on $\mathbb{R}^+ \times \mathbb{R}$ and

$$\begin{cases} \frac{\partial}{\partial t} \psi(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \psi(t, x) + V(x)\psi(t, x) & (t, x) \in \mathbb{R}^+ \times \mathbb{R} \\ \lim_{t \downarrow 0} \psi(t, x) = f(x) & x \in \mathbb{R} \end{cases}.$$

Exercise 10 Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$ and $x \in (\alpha, \beta)$. Consider the random variable $T_x := \inf\{t \geq 0 : \ x + B_t \in [\alpha, \beta]\}$.

(a) Show that T_x is a.s. finite.

(b) Using the fact that the processes B and $(B^2_t - t)_{t\geq 0}$ are martingales, compute $\mathbb{P}[x + B_{T_x} = \alpha]$, $\mathbb{P}[x + B_{T_x} = \beta]$ and $\mathbb{E}[T_x]$.

(c) Using the fact that $\left(e^{\lambda B_t - \frac{\lambda^2}{2} t}\right)_{t\geq 0}$ is a martingale, compute $\mathbb{E}[e^{-\frac{\lambda^2}{2} T_x}]$ for all $\lambda \in \mathbb{R}$.

(d) Define

$$T^\alpha_x := \inf\{t \geq 0 : \ x + B_t \in [\alpha, +\infty)\} \quad T^\beta_x := \inf\{t \geq 0 : \ x + B_t \in (-\infty, \beta]\}.$$

Compute $\mathbb{E}[e^{-\frac{\lambda^2}{2} T^\alpha_x}]$, $\mathbb{E}[e^{-\frac{\lambda^2}{2} T^\beta_x}]$, $\mathbb{E}[T^\alpha_x]$ and $\mathbb{E}[T^\beta_x]$.

Exercise 11 Define $M_t = \sup_{s \le t} B_s$ and $T^x = \inf\{s \ge 0 : \ B_s \ge x\}$.

(a) Show that for all $x > 0$, we have

$$\mathbb{P}[M_t < x] = \mathbb{P}[B_t < x] - \mathbb{P}[B_t < x, T^x \le t].$$

Using the strong Markov property, deduce that

$$\mathbb{P}[M_t < x] = \mathbb{P}[B_t < x] - \mathbb{P}[B_t > x], \qquad \text{for } x > 0.$$

Determine the law of M_t and that of T^x.

(b) More generally, let $x > 0$ and $y \le x$. Show that

$$\begin{aligned} \mathbb{P}[M_t < x, B_t < y] &= \mathbb{P}[B_t < y] - \mathbb{P}[B_t < y, T^x \le t] \\ &= \mathbb{P}[B_t < y] - \mathbb{P}[B_t > 2x - y]. \end{aligned}$$

Determine the joint law of (M_t, B_t).

Exercise 12 Consider the process $B_t^\theta = B_t + \theta t$ where $\theta \neq 0$ is fixed. Define $M_t^\theta = \sup_{s \le t} B_s^\theta$, $T^x = \inf\{s \ge 0 : \ B_s^\theta \ge x\}$ for some $x > 0$, and $\tau = \inf\{s \ge 0 : \ B_s^\theta \notin [\alpha, \beta]\}$ for some $\alpha < 0 < \beta$.

(a) Determine the joint law of (M_t^θ, B_t^θ).

(b) Compute $\mathbb{E}[e^{-\mu T^x}]$ and $\mathbb{E}[e^{-\mu \tau}]$ for all $\mu > 0$. Then calculate the values of $\mathbb{P}[T^x < \infty]$ and $\mathbb{E}[T^x 1_{\{T^x < \infty\}}]$ (discuss separately the cases $\theta > 0$ and $\theta < 0$).

Exercise 13 Denote by μ the law of the Brownian motion B on the canonical space $\big(C([0,T],\mathbb{R}), \mathcal{B}\big)$ where $\mathcal{B}$ is the Borel field. Show that for all $f \in C\big([0,T],\mathbb{R}\big)$ so that $f(0) = 0$ and for all $\varepsilon > 0$ we have $\mathbb{P}[\sup_{t \in [0,T]} |B_t - f_t| \le \varepsilon] = \mu\{\omega : \ \sup_{t \in [0,T]} |\omega_t - f_t| \le \varepsilon\} > 0$. (Hint: first consider the case $f \in C^1$ and use the Caméron-Martin formula)

Exercise 14 Let $f \in C^1\big([0,T],\mathbb{R}\big)$ and $f(0) = 0$. Using the Caméron-Martin formula, show that for all $\varepsilon \neq 0$, $\delta > 0$ and $M > 0$, we have the inequality:

$$\mathbb{P}\left[\sup_{t \in [0,T]} |\varepsilon B_t - f_t| < \delta\right] \ge e^{-\frac{1}{2\varepsilon^2}\int_0^T |\dot{f}_s|^2 ds - \frac{M}{\varepsilon}} \mathbb{P}\left[\sup_{t \in [0,T]} |\varepsilon B_t| < \delta, \int_0^T \dot{f}_s dB_s \le M\right].$$

Deduce that

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \ln \mathbb{P}\left[\sup_{t \in [0,T]} |\varepsilon B_t - f_t| < \delta\right] \ge -\frac{1}{2}\int_0^T |\dot{f}_s|^2 ds.$$

Exercise 15 For all $\varepsilon \neq 0$, denote by μ_ε the law of the process εB on the canonical space $\big(C([0,T],\mathbb{R}), \mathcal{B}\big)$. Show that for $\varepsilon \neq \varepsilon'$ the laws μ_ε and $\mu_{\varepsilon'}$ are singular.

Exercise 16 Consider the solution of the stochastic differential equation

$$X_t = X_0 + \sigma B_t - b\int_0^t X_s ds \tag{0.1}$$

where $\sigma \neq 0$ and $b > 0$ are fixed, and X_0 is a random variable independent from B.

(a) Define $Y_t = X_t - \sigma B_t$. Show that Y is a solution to an ordinary differential equation, and deduce the explicit expression of X_t.

(b) Show that X_t converges in law to a random variable X_∞ as $t \to +\infty$, and determine the law μ of X_∞.

(c) Show that if X_0 follows the law μ, then X_t follows the same law for all $t \geq 0$.

(d) For all $x \in \mathbb{R}$, denote by X_t^x the solution to the equation (0.1) such that $X_0 = x$. For $f \in C_b(\mathbb{R})$, define $P_t f(x) = \mathbb{E}[f(X_t^x)]$. Show that

$$\mathbb{E}[f(X_t^{X_0})] = \int \mathbb{P}_{X_0}(dx) P_t f(x)$$

and that the function $x \mapsto P_t f(x)$ is bounded continuous. Deduce that for all $s, t \geq 0$ we have

$$P_t \circ P_s = P_{t+s}.$$

(e) Let $U(t,x) = P_t f(x)$ where $t \geq 0, x \in \mathbb{R}, f \in C_b(\mathbb{R})$. Show that the function $(t,x) \mapsto U(t,x)$ belongs to $C^\infty(\mathbb{R}^+ \times \mathbb{R})$ and that U is the solution of the Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial t} U(t,x) = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} U(t,x) - bx\frac{\partial}{\partial x} U(t,x) \\ \lim_{t\to 0} U(t,x) = f(x) \end{cases}$$

Exercise 17 Let $\Phi = (\Phi_s(\omega))_{s\geq 0}$ be a progressively measurable process taking values of $d \times d$ orthogonal matrice. Define $X_t = \int_0^t \Phi_s dB_s$ where B is a d-dimensional Brownian motion. Show that for all $\lambda \in \mathbb{R}^d$, $M_t = e^{i\lambda\cdot X_t + \frac{1}{2}|\lambda|^2 t}$ is an $(\mathcal{F}_t)$-martingale where $\mathcal{F}_t = \sigma\{B_s : s \leq t\}$. Deduce that X is also a d-dimensional Brownian motion.

Exercise 18 Let $\Phi = (\Phi_s(\omega))_{s\geq 0}$ be a progressively measurable process taking values in $\mathbb{R}$, and assume that

$$0 < \int_s^t \Phi_u^2(\omega) du < +\infty \quad \text{a.s. for all } 0 \leq s < t < \infty$$

and

$$\int_0^\infty \Phi_u^2(\omega) du = +\infty \quad \text{a.s.}$$

Let $X_t = \int_0^t \Phi_s dB_s$. Define $\tau_t = \inf\{s \geq 0 : \int_0^s \Phi_u^2 du \geq t\}$ and $\tilde{B}_t(\omega) = X_{\tau_t(\omega)}(\omega)$. Show that for all $\lambda \in \mathbb{R}$ the process $M_t = e^{i\lambda \tilde{B}_t + \frac{1}{2}\lambda^2 t}$ is an $(\mathcal{F}_{\tau_t})$-martingale. Deduce that $\tilde{B}$ is also a Brownian motion. Finally generalize the results to the case with the dimension $d > 1$.

Exercise 19 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and of compact support.

(a) Show that for all $t \geq 0$ that

$$\int_0^t \left(\int_{\mathbb{R}} f(a) 1_{(0,+\infty)}(B_s) da \right) dB_s = \int_{\mathbb{R}} f(a) \left(\int_0^t 1_{(0,+\infty)}(B_s) dB_s \right) da, \quad \text{a.s.}$$

(b) Define $F(x) = \int_{\mathbb{R}} f(a)(x-a)^+ da$. Verify that $F \in C^2(\mathbb{R})$ and

$$\dot{F}(x) = \int_{\mathbb{R}} f(a) 1_{(a,+\infty)}(x) dx, \qquad \ddot{F}(x) = f(x).$$

Then applying the Itô formula to $F(B_t)$ and show that

$$\frac{1}{2} \int_0^t f(B_s) ds = \int_{\mathbb{R}} f(a) \Phi(t,a) da, \quad \text{a.s.} \tag{0.2}$$

for all $f \in C_b(\mathbb{R})$ and $t \geq 0$, where

$$\Phi(t,a) = (B_t - a)^+ - (B_0 - a)^+ - \int_0^t 1_{(a,+\infty)}(B_s) dB_s.$$

Exercise 20 Following the Exercise 19, we assume that the functions $(t,a) \mapsto \Phi(t,a,\omega)$ are continuous for a.s. ω.

(a) Show that for all $(t,a) \in \mathbb{R}^+ \times \mathbb{R}$ we have

$$\Phi(t,a) = \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \int_0^t 1_{[a-\varepsilon, a+\varepsilon]}(B_s) ds$$

(b) Deduce from (0.2) that for all $a \in \mathbb{R}$ we have

$$|B_t - a| = |B_0 - a| + \int_0^t \operatorname{sgn}(B_s - a) dB_s + L(t,a), \quad \text{a.s.}$$

where

$$L(t,a) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[a-\varepsilon, a+\varepsilon]}(B_s) ds \quad \text{and} \quad \operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

Show that the process $\tilde{B}_t = \int_0^t \operatorname{sgn}(B_s - a) dB_s$ is a Brownian motion. What properties the process $(L(t,a))_{t \geq 0}$ satisfies?

Exercise 21 We say that a process B taking values in $\mathbb{C}$ is a complex Brownian motion starting from the origin, if it is in the form:

$$B_t = B_t^1 + i B_t^2,$$

where B^1, B^2 are two independent reel Brownian motion starting from the origin. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Define

$$M_t = \operatorname{Re}[f(B_t)], \quad N_t = \operatorname{Im}[f(B_t)].$$

(a) Show that the processes M, N, MN can be expressed as a stochastic integral w.r.t. the two-dimensional Brownian motion (B^1, B^2).

(b) Show that there exists another two-dimensional Brownian motion $\tilde{B} = (\tilde{B}^1, \tilde{B}^2)$ s.t.

$$\begin{cases} M_t = \mathrm{Re}[f(0)] + \int_0^t |f'(B_s)| d\tilde{B}_s^1 \\ N_t = \mathrm{Re}[f(0)] + \int_0^t |f'(B_s)| d\tilde{B}_s^2 \end{cases}, \qquad \text{a.s.}$$

(Hint: using Exercise 17). Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. Use Itô's formula to develop the expression:

$$\exp\left[i(\lambda_1 M_t + \lambda_2 N_t) + \frac{1}{2}|\lambda|^2 \int_0^t |f'(B_s)|^2 ds\right]$$

(c) Define $f(z) = \alpha e^z$ for all $z \in \mathbb{C}$ where $\alpha \in \mathbb{C}\backslash\{0\}$ is fixed.

- Show that

$$\int_0^\infty |f'(B_s)|^2 ds = |\alpha|^2 \int_0^\infty e^{2B_s^1} ds = +\infty, \quad \text{a.s.}$$

- Show that we can find a complex Brownian motion $(C_t)_{t\geq 0}$ such that

$$\alpha e^{B_t} = \alpha + C_{\rho_t}, \quad \text{a.s.} \tag{0.3}$$

where $\rho_t = \int_0^t |f'(B_s)|^2 ds = |\alpha|^2 \int_0^t e^{2B_s^1} ds$.

- Deduce from (0.3) that for all $x \in \mathbb{C}\backslash\{0\}$

$$\mathbb{P}[\exists t > 0,\ B_T = x] = 0 \quad \text{and} \quad \mathbb{P}[\liminf_{t\to\infty} |B_t - x| = 0] = 1.$$

(The property above implies that the paths of B is a.s. dense in the complex plane, that is, the two-dimensional Brownian motion is recurrent.)

Exercise 22 Let $B = (B^1, \cdots, B^d)$ be a d-dimensional Brownian motion. Let $0 < \alpha < \frac{1}{2}$ and $\gamma > 0$. Show that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ we have

$$\mathbb{P}[|B_n| \leq \gamma n^\alpha] \leq \frac{C}{n^{d(\frac{1}{2}-\alpha)}}$$