TD1. Brownian motion. Martingales. Stopping times.

Unless stated otherwise, we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion starting form the origin on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. We also denote by $\mathcal{F}_t := \sigma(B_s,\ s \leq t)$ the filtration generated by the Brownian motion.

Exercise 1 We define the *Brownian bridge* as the process $Z_t = B_t - tB_1$ $(0 \leq t \leq 1)$.

1. Show that Z is a Gaussian process independent of B_1
2. Compute the mean and covariance function of Z.
3. Is $(Z_t)_{0\leq t\leq 1}$ a martingale w.r.t $(\mathcal{F}_t)_{t\geq 0}$?
4. Prove that Z has the same law as the process Y defined by

$$Y_t = \begin{cases} (1-t)B_{\frac{t}{(1-t)}} & (0 \leq t < 1), \\ 0 & (t = 1). \end{cases}$$

Exercise 2 For $t \geq 0$, define $A_t = \int_0^t B_s \mathrm{d}s$.

1. Justify why A is a well-defined $(\mathcal{F}_t)_{t\geq 0}$-adapted process.
2. Determine the law of A_t for all $t \geq 0$.
3. Is $(A_t)_{t\geq 0}$ a martingale w.r.t $(\mathcal{F}_t)_{t\geq 0}$?

Exercise 3 The goal of the exercise is to compute (by hand) the bracket of the square-integrable martingale $(M_t)_{t\geq 0} = (B_t^2 - t)_{t\geq 0}$.

1. Compute $\mathrm{E}(B_t^4 | \mathcal{F}_s)$ for all $0 \leq s \leq t$.
2. Let $X_t = \int_0^t B_u^2 \mathrm{d}u$. Compute $\mathrm{E}(X_t|\mathcal{F}_s)$ for all $0 \leq s \leq t$.
3. Deduce thereof that $\langle M \rangle_t = 4X_t$.

Exercise 4 In this exercise, $\mathbf{B}$ is a d-dimensional $(d \in \mathbb{N})$ standard Brownian motion, that is $\mathbf{B}_t = (B_t^{(1)}, \dots, B_t^{(d)})$, where the $B^{(i)}$'s are independent standard Brownian motions. Let U be a $d \times d$ (deterministic) orthogonal matrix. Prove that the process $(\mathbf{W}_t)_{t\geq 0} = (U\mathbf{B}_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion. By default, vectors are here considered to be column vectors in all matrix operations.

Exercise 5 For $a > 0$, define $T_a = \inf\{t > 0 \colon B_t \geq a\}$.

1. Show that T_a is a $(\mathcal{F}_t)$-stopping time.
2. Prove that for all $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$\mathrm{E}\Big[\exp\Big(\lambda B_{T_a \wedge n} - \frac{1}{2}\lambda^2 (T_a \wedge n)\Big)\Big] = 1.$$

3. Deduce thereof the Laplace transform of T_a :

$$\mathrm{E}[\exp(-uT_a)] = \exp(-a\sqrt{2u}), \qquad (u > 0),$$

   and show that $\mathrm{P}(T_a < +\infty) = 1$.

Exercise 6 (Lévy's characterization of Brownian motion) On a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$, let $M = (M_t)_{t\geq 0}$ be a continuous square-integrable martingale with $M_0 = 0$ and $\langle M_t \rangle = t$ for all $t \geq 0$. Let F be any twice-differentiable function $F \colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ with bounded first and second-order derivatives. Define :

$$Z_t = F(t, M_t) - \int_0^t \Big(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\Big)(u, M_u)\mathrm{d}u, \qquad t \geq 0.$$

1. Prove that Z is a martingale.
2. Applying the previous question to $F(t, x) = \exp(i\theta x + \frac{1}{2}\theta^2 t)$, where $\theta \in \mathbb{R}$, prove that M is a Brownian motion.

Exercise 7 Prove that Brownian motion is not monotone on any interval (a.s.).

Hints.

It is highly recommended to allow you some time to think about the problems first, before looking at the hints.

Exercise 1

1. First show that the collection of random variables $(B_1, Z_t, 0 \leq t \leq 1)$ as a whole is Gaussian or, equivalently, that the constant process $t \in [0,1] \mapsto B_1$ and the process Z are jointly Gaussian.
2. Computation.
3. Is $(Z_t)_{0 \leq t \leq 1}$ adapted to $(\mathcal{F}_t)_{t \geq 0}$?
4. Check the finite-dimensional marginals.

Exercise 2

1. Approximate A_t by a Riemann sum.
2. Follow the previous hint.
3. Compute $\mathrm{E}(A_t - A_s|\mathcal{F}_s)$ when $0 \leq s \leq t$.

Exercise 3

1. Expand $(B_s + [B_t - B_s])^4$
2. Try exchanging the integral sign and the conditional expectation.
3. Recall the definition of the bracket and use the previous questions.

Exercise 4

1. If X is a random vector with square-integrable components, show that its covariance matrix equals $\mathrm{E}(XX^\intercal)$ where the row vector $X^\intercal$ is the transpose of the column vector X.
2. Compute $\mathrm{E}(\mathbf{W}_s \mathbf{W}_t^\intercal)$.

Exercise 5

1. $T_a = \inf\{t > 0 \colon B_t \in [a, +\infty[\}$.
2. Use Doob's optional stopping theorem.
3. Let $n \to \infty$. Justify the limit.

Exercise 6 Do a Taylor-expansion of $F(t', x') - F(t, x)$ when $|x - x'| + |t - t'|$ is small, then choose "$(x, x') = (M_t, M_{t'})$" and take conditional expectation w.r.t. $\mathcal{F}_t$ $(t \leq t')$.

Exercise 7

1. Check that Brownian motion has zero probability of being nondecreasing on the unit time interval.
2. What is the probability that $B_{\frac{k+1}{n}} - B_{\frac{k}{n}} \geq 0$ for all $k \in \{0, \ldots, n-1\}$?

TD2. Local martingales. Wiener and Itô integrals.

Unless stated otherwise, we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion starting form the origin on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. We also denote by $\mathcal{F}_t := \sigma(B_s,\, s \leq t)$ the filtration generated by the Brownian motion.

Exercise 1 Fix $T \geq 0$. Show that a continuous local martingale $M = (M_t)_{t\in[0,T]}$ is a square-integrable martingale if and only if $M_0 \in L^2$ and $\langle M\rangle_T \in L^1$.

Exercise 2 Let M be a continuous local martingale such that a.s., $\langle M\rangle_\infty = +\infty$. Prove that a.s.,

$$\limsup_{t\to\infty} M_t = +\infty \quad \text{and} \quad \liminf_{t\to\infty} M_t = -\infty.$$

Exercise 3 Let

$$Z_t := (1-t)\int_0^t \frac{\mathrm{d}B_s}{(1-s)}, \qquad 0 \leq t < 1.$$

1. Show that Z is a Gaussian process. Determine its mean and covariance function. How is this process called ?
2. Prove that $Z_t \to 0$ in $L^2(\Omega)$ as $t \to 1$.
3. Prove that $Z_t \to 0$ a.-s. as $t \to 1$.

Exercise 4 Determine the law of the process

$$X_t = e^{-t}\Big(X_0 + \int_0^t e^u \mathrm{d}B_u\Big), \qquad t \geq 0,$$

where X_0 is a $\mathcal{N}(0, 1/2)$ random variable independent of $(B_t)_{t\geq 0}$. Show that the process is stationary. How is it called ?

Exercise 5 Determine the law of the process

$$X_t = \int_0^{\sqrt{t}} \sqrt{2u}\ \mathrm{d}B_u, \qquad t \geq 0.$$

Exercise 6 Let $t \geq 0$ and for $n \in \mathbb{N}$, $0 \leq k \leq n$, define $t_k = tk/n$.

1. Show that
$$B_t^2 = \sum_{k=1}^n 2B_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}) + \sum_{k=1}^n (B_{t_k} - B_{t_{k-1}})^2.$$
2. Deduce thereof that for all $t \geq 0$,
$$B_t^2 - t = \int_0^t 2B_s\ \mathrm{d}B_s\ , \qquad \text{a.s.}$$
3. Retrieve the final result of **Exercise 5** in **TD1**.

Exercise 7 Using the same notation as in the previous exercise :

1. Prove that for all $p > 2$, $\sum_{k=1}^n (B_{t_k} - B_{t_{k-1}})^p \to 0$ in probability as $n \to \infty$.
2. Prove that for all continuous function $f\colon \mathbb{R}_+ \to \mathbb{R}$, $\sum_{k=1}^n f(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2 \to \int_0^t f(B_s)\mathrm{d}s$ in probability as $n \to \infty$.
3. With a similar method to **Exercise 6**, show that $\sin(B_t) = -\frac{1}{2}\int_0^t \sin(B_s)\mathrm{d}s + \int_0^t \cos(B_s)\mathrm{d}B_s$ a.-s.

Hints.

Exercise 1 (Second implication) Assume $M_0 \in L^2$ and $\langle M\rangle_T \in L^1$. Consider the localizing sequence :

$$T_n := \inf\{t \geq 0 \colon |M_t| \geq n\}.$$

Exercise 2 Assume for simplicity that $M_0 = 0$ a.-s. Let $a, b > 0$ and consider

$$\sigma_a^- = \inf\{t \geq 0 \colon M_t \leq -a\}, \qquad \sigma_b^+ = \inf\{t \geq 0 \colon M_t \geq b\}.$$

Using Doob's theorem, show that

$$\mathrm{P}(\sigma_a^- < \sigma_b^+) = \frac{b}{a+b}, \qquad \mathrm{P}(\sigma_b^+ < \sigma_a^-) = \frac{a}{a+b},$$

and let $b \to \infty$.

Exercise 3 Let

$$Z_t := (1-t)\int_0^t \frac{\mathrm{d}B_s}{(1-s)}, \qquad 0 \leq t < 1.$$

1. Use that $t \in [0,1) \mapsto Y_t := \int_0^t \frac{\mathrm{d}B_s}{(1-s)}$ is a Wiener process.
2. $\mathrm{E}(Z_t^2) = \ldots$.
3. You can use a time-reversal argument. Alternatively, prove that for $n \geq 1$ and $\varepsilon > 0$,

$$\mathrm{P}\Big(\sup_{2^{-(n+1)} \leq t \leq 2^{-n}} |Z_{1-t}| > \varepsilon\Big) \leq \frac{1}{(2^{n-1}\varepsilon^2)}.$$

   To this end, use Doob's maximal inequality on the martingale Y and conclude via Borel-Cantelli's lemma.

Exercise 6 1. Computation

2. Recall the definition of quadratic variation. Recall that convergence of a sequence of random variables in probability implies a.s. convergence on asubsequence.

Exercise 7 1. Use

$$\left|\sum_{k=1}^n (B_{t_k} - B_{t_{k-1}})^p\right| \leq \max_k |B_{t_k} - B_{t_{k-1}}|^{p-2} \times \sum_{k=1}^n (B_{t_k} - B_{t_{k-1}})^2.$$

2. Prove it for step functions and then use a density argument.

TD3. (Generalized) Itô integrals. Itô processes. Itô's formula.

Unless stated otherwise, we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion starting form the origin on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. We also denote by $\mathcal{F}_t := \sigma(B_s,\, s \leq t)$ the filtration generated by the Brownian motion.

Exercise 1 Check that a process ϕ is progressive if and only if the map $(t, \omega) \mapsto \phi_t(\omega)$ is measurable w.r.t. the progressive σ-field $\mathcal{P}$ defined in the Lecture Notes (Remark 2.2).

Exercise 2 Do the following continuous adapted processes belong to $\mathbb{M}^2(\mathbb{R}^+)$? Or $\mathbb{M}^2$? Or $\mathbb{M}^2_{\mathrm{LOC}}$? (same notation as in the Lecture Notes).

1. $t \in \mathbb{R}^+ \mapsto B_t$;
2. $t \in \mathbb{R}^+ \mapsto \frac{B_t}{1+t^2}$;
3. $t \in \mathbb{R}^+ \mapsto \exp(B_t^2)$.

Exercise 3 Let X be an Itô process started at the origin, that is

$$X_t = \int_0^t \phi_u \mathrm{d}B_u + \int_0^t \psi_u \mathrm{d}u, \qquad t \geq 0,$$

where $\phi \in \mathbb{M}^2_{\mathrm{LOC}}$ and $\psi \in \mathbb{M}^1_{\mathrm{LOC}}$. Prove that X is a square-integrable martingale if and only if $\psi \equiv 0$ and $\phi \in \mathbb{M}^2$.

Exercise 4 Let $\alpha\colon \mathbb{R}^+ \to \mathbb{R}$ be a continuous (deterministic) function. With the help of the stochastic integration by parts formula, check that the process

$$X_t := \int_0^t \exp\Big(\int_s^t \alpha(u)\mathrm{d}u\Big)\mathrm{d}B_s, \qquad t \geq 0,$$

satisfies the stochastic differential equation $\mathrm{d}X_t = \alpha(t)X_t\mathrm{d}t + \mathrm{d}B_t$.

Exercise 5 With the help of Itô's formula, answer to Question (3) in **Exercise 7**, **TD2**.

Exercise 6 Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be continuously twice-differentiable. Show that the process

$$X_t := f(B_t) - \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s, \qquad t \geq 0,$$

is a continuous local martingale. Give a sufficient condition for X to be a martingale.

Exercise 7 Let $n \geq 1$ and $x_1, \ldots, x_n$ be distinct points in $\mathbb{R}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f \in C^1(\mathbb{R})$ and $f \in C^2(\mathbb{R} \setminus \{x_1, \ldots x_n\})$. We assume that f'' remains bounded in a neighborhood of x_i for all $1 \leq i \leq n$. The goal of the exercise is to prove that Itô's formula is still valid for such a function.

1. Prove that for all $t \geq 0$, the set $\{(\omega, s) \in \Omega \times [0, t] \colon B_s(\omega) \in \{x_1, \ldots x_n\}\}$ has zero measure w.r.t. $\mathrm{P} \otimes \mathrm{d}s$, where $\mathrm{d}s$ stands for Lebesgue measure.
2. Prove that Itô's formula is valid when we make the additional assumption that f has compact support. *Hint : approximate f by a sequence of continuously twice-differentiable functions.*
3. Conclude in the general case.

Exercise 8 (Tanaka's formula and local time of Brownian motion) For $\varepsilon > 0$, define

$$\phi_\varepsilon(x) = \begin{cases} |x| & (|x| \geq \varepsilon) \\ \frac{1}{2}(\varepsilon + \frac{x^2}{\varepsilon}) & (|x| < \varepsilon). \end{cases}$$

1. With the help of **Exercise 7**, prove that P-a.-s., for all $t \geq 0$,

$$\phi_\varepsilon(B_t) - \phi_\varepsilon(0) = \int_0^t \phi'_\varepsilon(B_s)\mathrm{d}B_s + \frac{1}{2\varepsilon}\mathrm{Leb}(\{0 \leq s \leq t \colon |B_s| < \varepsilon\}),$$

   where « Leb » stands for Lebesgue measure.
2. Check that
$$\int_0^t \phi'_\varepsilon(B_s)1_{\{|B_s|<\varepsilon\}}\mathrm{d}B_s \overset{\varepsilon\to 0}{\longrightarrow} 0, \qquad \text{in } L^2(\Omega).$$
3. Deduce thereof **Tanaka's formula** : P-a.-s., for all $t \geq 0$,

$$|B_t| = \int_0^t \mathrm{sgn}(B_s)\mathrm{d}B_s + L_t,$$

   where L_t is the limit in $L^2(\Omega)$ of $\frac{1}{2\varepsilon}\mathrm{Leb}(\{0 \leq s \leq t \colon |B_s| < \varepsilon\})$, as $\varepsilon \to 0$, and

$$\mathrm{sgn}(x) = 1 \text{ if } x > 0, \qquad \mathrm{sgn}(x) = -1 \text{ if } x \leq 0.$$

4. Prove that the process (L_t) is a.-s. continuous and non-decreasing. This process is called **local time** of the Brownian motion (at the origin).
5. Show that $(|B_t| - L_t)_{t\geq 0}$ is a Brownian motion.

Hints.

Exercise 3 Use Remark 3.2 in lectures notes and Exercice 1 in TD2.

Exercise 4 Write X as a product of Itô processes and apply integration by parts.

Exercise 6 Use Exercise 3.

Exercise 7 Item (2) : use convolution with C^∞ functions and stochastic dominated convergence.

Exercise 8 Item (5). Use Lévy's characterization of Brownian motion.

TD4. Multivariate Itô's formula. Exponential martingales and Girsanov's theorem.

Unless stated otherwise, we denote by $B = (B_t)_{t\geq 0}$ a Brownian motion starting from the origin on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. We denote by $\mathcal{F}_t := \sigma(B_s,\, s \leq t)$ the filtration generated by the Brownian motion and write $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$.

Exercise 1 (Transience and strict local martingales) In this exercise, $B = (B^{(1)}, B^{(2)}, B^{(3)})$ is a three-dimensional Brownian motion starting from $x_0 \in \mathbb{R}^3 \setminus \{0\}$ (recall the definition in **Exercise 3, TD1**). We define $G(x) := 1/\|x\|$ for all $x \in \mathbb{R}^3 \setminus \{0\}$, where $\|\cdot\|$ is the Euclidian norm and

$$T_a = \inf\{t \geq 0\colon \|B_t\| = a\}, \qquad a \geq 0.$$

1. Check that $\langle B^{(i)}, B^{(j)}\rangle \equiv 0$ whenever $i \neq j$, where $i, j \in \{1, 2, 3\}$.
2. Check that $\Delta G(x) = 0$ for all $x \neq 0$.
3. Using Itô's formula, prove that for all $r > 0$, $\{G(B_{t\wedge T_r})\}_{t\geq 0}$ is a martingale.
4. Assume $0 < r < \|x_0\| < R < \infty$. Show that $T_R < \infty$ a.s. (you may use **Exercise 7** in **TD 1**) and
$$\mathrm{P}(T_r < T_R) = \frac{R^{-1} - \|x_0\|^{-1}}{R^{-1} - r^{-1}}.$$
5. Deduce from the previous formula that a.s. $B_t \neq 0$ for all $t \geq 0$.
6. Prove that $\{G(B_t)\}_{t\geq 0}$ is a continuous local martingale.
7. Check that $\{G(B_t)\}_{t\geq 0}$ is a non-negative super-martingale and deduce thereof that $\|B_t\|$ a.s. converges to $+\infty$, as $t \to \infty$.
8. Prove that $\{G(B_t)\}_{t\geq 0}$ is bounded in L^2 but is not a martingale.

Exercise 2 (Exponential martingales : a simple case) Let $\phi\colon \mathbb{R}^+ \to \mathbb{R}$ be a (deterministic) function in $L^2_{\mathrm{loc}}(\mathbb{R}^+)$ and $Z^\phi = (Z^\phi_t)_{t\geq 0}$ the associated Doléans-Dade exponential process. Check that Z^ϕ is a martingale.

Exercise 3 (Exponential martingales : an example) Find a progressive process $X = (X_t)_{t\geq 0}$ such that the process $Z = (Z_t)_{t\geq 0}$ defined by $Z_t = \exp(X_t - B_t^2)$ is a martingale.

Exercise 4 (Change of drift) Let $X = (X_t)_{t\geq 0}$ be an Itô process such that $X_0 = 0$ a.s. and

$$\mathrm{d}X_t = b_1(t)\mathrm{d}t + \sigma(t)\mathrm{d}B_t,$$

where $b_1\colon \mathbb{R}_+ \to \mathbb{R}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+^*$ is in $L^2_{\mathrm{loc}}(\mathbb{R}^+)$ (b_1 and σ are both deterministic). Prove that (under suitable assumptions to be specified) there exists a probability measure Q on $(\Omega, \mathcal{F}_\infty)$ under which X satisfies

$$\mathrm{d}X_t = b_2(t)\mathrm{d}t + \sigma(t)\mathrm{d}\tilde{B}_t,$$

where $b_2\colon \mathbb{R}_+ \to \mathbb{R}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and $\tilde{B}$ is a Brownian motion under Q.

Exercise 5 (Hitting times for Brownian motion with drift) For every $a > 0$ and $b \in \mathbb{R}$, define the following stopping time :

$$T_{a,b} := \inf\{t \geq 0\colon B_t + bt = a\}.$$

The goal of the exercise is to determine $\mathrm{P}(T_{a,b} < \infty)$ using the result of **Exercise 7** in **TD1** (case $b = 0$) via a change of measure argument.

1. Using an appropriate martingale and Girsanov's theorem, prove that for all $t \geq 0$,
$$\mathrm{P}(T_{a,b} > t) = \mathrm{E}\Big[1_{\{T_a > t\}} \exp\Big(bB_t - \frac{1}{2}b^2 t\Big)\Big].$$

2. Deduce thereof
$$\mathrm{P}(T_{a,b} \le t) = \mathrm{E}\Big[1_{\{T_a \le t\}} \exp\Big(bB_{t\wedge T_a} - \frac{1}{2}b^2(t \wedge T_a)\Big)\Big].$$

3. Conclude that
$$\mathrm{P}(T_{a,b} < \infty) = \begin{cases} 1 & (b \ge 0) \\ \exp(2ab) & (b < 0). \end{cases}$$

Exercise 6 (Onsager-Machlup function) Let $h\colon [0,1] \to \mathbb{R}$ be a twice continuously differentiable (and deterministic) function, with $h(0) = 0$. We equip the space of continuous functions $C_0([0,1], \mathbb{R})$ with the norm $\|f\|_\infty := \max_{x\in[0,1]} |f(x)|$. Using an appropriate martingale and change-of-measure argument, prove that
$$\frac{\mathrm{P}(\|B - h\|_\infty \le \varepsilon)}{\mathrm{P}(\|B\|_\infty \le \varepsilon)} \xrightarrow{\varepsilon\to 0} \exp\Big(-\frac{1}{2}\int_0^1 h'(s)^2 \mathrm{d}s\Big).$$
We admit that the denominator in the left-hand side is positive for all $\varepsilon > 0$.

Exercise 7 (Solutions of S.D.E. via Girsanov's theorem) Let $b\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Assume that there exists $g \in L^2(\mathbb{R}_+)$ such that $|b(t,x)| \le g(t)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Using Girsanov's theorem, find a probability measure Q on $\mathcal{F}_\infty$ under which the process defined by
$$B_t - \int_0^t b(s, B_s)\mathrm{d}s, \qquad t \ge 0,$$
is a Brownian motion.

Université Paris-Dauphine
M2 MATH/MASEF

Academic year 2023-2024
Stochastic calculus

TD5. Stochastic differential equations. Feynman-Kac formula.

Unless stated otherwise, $(\Omega, \mathcal{A}, \mathrm{P})$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $B = (B_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion.

Exercise 1 (Invariant distribution for the solution of a s.d.e) Let $V\colon \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function with bounded second derivative. We consider the following s.d.e :

$$\mathrm{d}X_t = \mathrm{d}B_t - V'(X_t)\mathrm{d}t.$$

1. Prove that for any given initial condition $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathrm{P})$, this equation has a unique solution (up to indistinguishability).
2. What is the generator associated to this solution ?
3. Define $\phi(x) = \exp(-V(x))$, which we assume to be in $L^2(\mathbb{R})$. Show that without any loss in generality, we may as well assume that $\nu(\mathrm{d}x) := \phi^2(x)\mathrm{d}x$ is a probability distribution on $\mathbb{R}$.
4. Prove that for any $f \in \mathcal{C}^2_c(\mathbb{R})$, $\int (Lf)(x)\nu(\mathrm{d}x) = 0$.
5. Assume that X_0 is distributed as ν and prove that for all $t \geq 0$, X_t is distributed as ν too.
6. Give an example for an explicit function V.

Exercise 2 (Feynman-Kac formula in an interval) Let k and f be two continuous functions from $[0, 1]$ to $\mathbb{R}$. We assume that $f(0) = f(1) = 0$. Let $u\colon (t, x) \in [0, \infty) \times [0, 1] \to \mathbb{R}$ be a continuous function that is also in $\mathcal{C}^{1,2}((0, \infty) \times (0, 1), \mathbb{R})$ and solves the following p.d.e :

$$\begin{array}{lll} & \partial_t u = \frac{1}{2}\partial_x^2 u - ku & (t > 0,\ 0 < x < 1); \\ \text{(initial condition)} & u(0, x) = f(x) & (0 \leq x \leq 1); \\ \text{(boundary condition)} & u(t, 1) = u(t, 0) = 0 & (t \geq 0). \end{array}$$

For all $n \geq 3$, let $I_n := (\frac{1}{n}, 1 - \frac{1}{n})$ and ϕ_n be a $\mathcal{C}^\infty$ compactly supported function such that

$$\phi_n(x) = 0 \text{ if } x \notin (0, 1), \qquad \phi_n(x) = 1 \text{ if } x \in I_n.$$

1. Fix $t > 0$ and define for $s \in [0, t]$, $n \geq 3$,

$$X_n(s) = u(t - s, B_s)\phi_n(B_s) \exp\Big(- \int_0^s k(B_v)\phi_n(B_v)\mathrm{d}v\Big),$$
$$T_n = \inf\{t \geq 0\colon B_t \notin I_n\}.$$

   Using Itô's formula, prove that $(X_n(s \wedge T_n))_{s\in[0,t]}$ is a square integrable martingale.
2. Let $T = \inf\{t \geq 0\colon B_t \notin (0, 1)\}$. We recall that $\mathrm{E}_x(T) = x(1-x)$ for $x \in [0, 1]$, where the subscript stands for the starting point of Brownian motion (see Lecture Notes, *Exit time from an interval*). Deduce from the previous question that

$$u(t, x) = \mathrm{E}_x\Big[f(B_t) \exp\Big(- \int_0^t k(B_s)\mathrm{d}s\Big)1_{\{t<T\}}\Big], \qquad \forall x \in (0, 1),$$

   and, in the case $k \geq 0$, show that $u(t, x)$ converges to zero uniformly in $x \in (0, 1)$ as $t \to \infty$.

Exercise 3 (Brownian motion on a circle) Prove that $X_t = (\cos B_t, \sin B_t)$, $t \geq 0$, is the unique solution of the s.d.e :

$$\mathrm{d}X_t = -\frac{1}{2}X_t\mathrm{d}t + RX_t\mathrm{d}B_t, \qquad X_0 = (1, 0),$$

where R is (the matrix of) the rotation with angle $\pi/2$.

Exercise 4 (Killed Brownian motion) We consider $u \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ a bounded solution to the following p.d.e :

$$\begin{array}{rll} & \partial_t u = \frac{1}{2}\partial_x^2 u - ku & (t > 0, x \in \mathbb{R}); \\ \text{(initial condition)} & u(0,x) = f(x) & (x \in \mathbb{R}); \end{array}$$

where $f, k\colon \mathbb{R} \to \mathbb{R}$ are measurable, with f bounded and k non-negative. In this exercise, we provide a representation of this solution in terms of a killed process. To this purpose, we introduce $\mathcal{E}$ an exponential random variable with parameter one that is independent of Brownian motion and set the *killing time*

$$\kappa := \inf\left\{t \geq 0\colon \int_0^t k(B_s)\mathrm{d}s \geq \mathcal{E}\right\}.$$

We now define

$$X_t = \begin{cases} B_t & (t < \kappa) \\ \star & (t \geq \kappa). \end{cases}$$

and extend f to $\mathbb{R} \cup \{\star\}$ by setting $f(\star) = 0$. Finally, $\mathcal{G}_t := \sigma(\mathcal{E}, B_s, s \leq t)$.

1. Using the Feynman-Kac formula, show that $u(t,x) = \mathrm{E}_x(f(X_t))$.
2. Show that $X = (X_t)_{t\geq 0}$ is a Markov process.