Exam 2018/2019

January 9, 2019, from 09:00 to 12:00
Documents allowed, Internet not allowed
Do what you can, and do not worry

$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space, with complete and right continuous filtration.
$B = (B_t)_{t\geq 0}$ is a d-dimensional Brownian motion issued from the origin, $d \geq 1$.
If Z is a semi-martingale, we denote by $\langle Z\rangle$ the increasing process of its local martingale part.
If $Z = Z_0 + M + V$, do not confuse $\langle Z\rangle = \langle M\rangle$ with the finite variation part V of Z.

Exercise 1 (Nature of an integral)**.** Set $d = 1$. Let us consider the following integral, for $t \geq 0$,

$$I_t = \int_0^t B_s \mathrm{d}s.$$

1. Is it a Lebesgue–Stieltjes integral? A Wiener integral? An Itô integral? Justify your answer
2. Show that $\mathrm{d}(tB_t) = B_t \mathrm{d}t + t\mathrm{d}B_t$
3. Deduce from the preceding question that $I_t = \int_0^t (t-s)\mathrm{d}B_s$ for all $t \geq 0$
4. Deduce from the preceding question that $I_t \sim \mathscr{N}(0, \frac{1}{3}t^3)$ for all $t \geq 0$
5. For all $t \geq 0$, $n \geq 1$, $0 \leq k \leq n$, let us define $t_k = \frac{k}{n}t$. Show that
$$\sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n}\sum_{j=0}^{n-2}(n-j-1)(B_{t_{j+1}} - B_{t_j}).$$
6. Deduce from the preceding question another proof that $I_t \sim \mathscr{N}(0, \frac{1}{3}t^3)$ for all $t \geq 0$
7. Is the process $(I_t)_{t\geq 0}$ a martingale?

Exercise 2 (Study of a special process)**.** Set $d = 2$. For all $t \geq 0$, we write $B_t = (X_t, Y_t)$ and

$$A_t = \int_0^t X_s \mathrm{d}Y_s - \int_0^t Y_s \mathrm{d}X_s.$$

1. Show that $\langle A\rangle = \int_0^\bullet (X_s^2 + Y_s^2)\mathrm{d}s$ and that the process A is a square integrable martingale
2. From now on let $\lambda > 0$. Show that for all $t \geq 0$,
$$\mathbb{E}\mathrm{e}^{\mathrm{i}\lambda A_t} = \mathbb{E}\cos(\lambda A_t).$$
3. From now on, let $f : \mathbb{R}_+ \to \mathbb{R}$ be $\mathscr{C}^2$, and let us define the continuous semi-martingales
$$(Z_t)_{t\geq 0} = (\cos(\lambda A_t))_{t\geq 0} \quad\text{and}\quad (W_t)_{t\geq 0} = \Big(-\frac{f'(t)}{2}(X_t^2 + Y_t^2) + f(t)\Big)_{t\geq 0}.$$
Show that for all $t \geq 0$,
$$Z_t = 1 - \lambda\int_0^t \sin(\lambda A_s)\mathrm{d}A_s - \frac{\lambda^2}{2}\int_0^t (X_s^2 + Y_s^2)Z_s \mathrm{d}s.$$
and
$$W_t = f(0) - \int_0^t f'(s)X_s \mathrm{d}X_s - \int_0^t f'(s)Y_s \mathrm{d}Y_s - \frac{1}{2}\int_0^t f''(s)(X_s^2 + Y_s^2)\mathrm{d}s,$$
and deduce that
$$\langle Z, W\rangle = 0.$$

4. Show that if f solves $f'' = f'^2 - \lambda^2$ then Ze^W is a continuous local martingale and

$$Z_t e^{W_t} = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s dY_s.$$

5. Let $r > 0$. By using $f(t) = -\log\cosh(\lambda(r-t))$ deduce from the previous question that

$$\mathbb{E} e^{i\lambda A_r} = \frac{1}{\cosh(\lambda r)}.$$

Exercise 3 (Criterion for a stochastic differential equation)**.** Set $d = 1$. Let σ, b be two functions $\mathbb{R} \to \mathbb{R}$ such that for some finite constant $C < \infty$ and for all $x, y \in \mathbb{R}$,

$$|\sigma(x) - \sigma(y)| \le C\sqrt{x-y} \quad \text{and} \quad |b(x) - b(y)| \le C|x-y|$$

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt. \qquad \text{(SDE)}$$

A solution X is a continuous semi-martingale with canonical decomposition $X = X_0 + M + V$ with $X_0 \in L^2$, local martingale part $M = \int_0^\bullet \sigma(X_s) dB_s$, and finite variation part $V = \int_0^\bullet b(X_s) ds$. Note that the continuity of σ, X, b gives that almost surely, for all $t \ge 0$, $s \mapsto \sigma(X_s) + b(X_s)$ is locally bounded.

1. Let Z be a continuous semi-martingale such that $\langle Z \rangle = \int_0^\bullet \varphi_s ds$ for a progressive process φ such that $0 \le \varphi \le C|Z|$ for some constant $C < \infty$. Prove that for all $t \ge 0$ and all $a > 0$,

$$\mathbb{E} \int_0^t \frac{\mathbf{1}_{0<|Z_s|\le a}}{|Z_s|} d\langle Z \rangle_s \le Ct.$$

2. Deduce from the preceding question that for all $t \ge 0$,

$$\lim_{n\to\infty} n\mathbb{E} \int_0^t \mathbf{1}_{\{0<|Z_s|\le \frac{1}{n}\}} d\langle Z \rangle_s = 0.$$

3. For all $n \ge 1$, $x \in \mathbb{R}$, let us define $g_n(x) = 2n(1+nx)\mathbf{1}_{x\in[-\frac{1}{n},0)} + 2n\mathbf{1}_{x=0} + 2n(1-nx)\mathbf{1}_{x\in(0,\frac{1}{n}]}$.
   Let $f_n : \mathbb{R} \to \mathbb{R}$ be the twice differentiable function such that $f_n'' = g_n$ and $f_n(0) = f_n'(0) = 0$.
   Show that for all $x \in \mathbb{R}$, the following properties hold true:

   (a) $f_n'(x) \in [-1, 1]$ and $\lim_{n\to\infty} f_n'(x) = \operatorname{sign}(x) = \mathbf{1}_{x>0} - \mathbf{1}_{x<0}$

   (b) $|f_n(x)| \le |x|$ and $\lim_{n\to\infty} f_n(x) = |x|$.

4. By using Itô formula, prove that for all continuous semi-martingale $Z = (Z_t)_{t\ge 0}$, all $t \ge 0$,

$$\int_0^t \mathbf{1}_{Z_s=0} d\langle Z \rangle_s = 0.$$

5. From now on, let X and X' be two solutions of (SDE) on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge0}, \mathbb{P})$ and with respect to the Brownian motion B. Show that for all $t \ge 0$,

$$\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X_s'))^2 ds.$$

6. By using the assumption on σ, deduce from the preceding questions that for all $t \ge 0$,

$$\lim_{n\to\infty} \mathbb{E} \int_0^t g_n(X_s - X_s') d\langle X - X' \rangle_s = 0.$$

7. Set $Z = X - X'$. From now on, let T be a stopping time such that the semi-martingale $(Z_{t\wedge T})_{t\geq 0}$ is bounded. By using notably the assumption on σ, prove that for all $t \geq 0$, $n \geq 1$,

$$\mathbb{E}(f_n(Z_{t\wedge T})) = \mathbb{E}(f_n(Z_0)) + \mathbb{E}\int_0^{t\wedge T} f_n'(Z_s)(b(X_s) - b(X_s'))\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^{t\wedge T} f_n''(Z_s)\mathrm{d}\langle Z\rangle_s.$$

8. Deduce from the preceding questions and the assumption on b that for all $t \geq 0$,

$$\mathbb{E}(|X_{t\wedge T} - X'_{t\wedge T}|) = \mathbb{E}(|X_0 - X_0'|) + \mathbb{E}\int_0^{t\wedge T} (b(X_s) - b(X_s'))\mathrm{sign}(X_s - X_s')\mathrm{d}s.$$

9. By using the Grönwall lemma, deduce that if $X_0 = X_0'$ then $X_t = X_t'$ for all $t \geq 0$.