Exam 2020/2021

October 28, 2020, from 13:45 to 16:45
Documents allowed, Internet not allowed
Do what you can, and do not worry

$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space, with complete and right continuous filtration. $B = (B_t)_{t\geq 0}$ is a d-dimensional Brownian motion issued from the origin, $d \geq 1$.

Exercise 1 (Representation of a process)**.** Take $d = 1$ and $x \in \mathbb{R}$.

1. Recall the computations and reasoning showing that the process $(Z_t)_{t\geq 0}$ defined by

$$Z_t = x\mathrm{e}^{-t} + \mathrm{e}^{-t} M_t \quad \text{where} \quad M_t = \sqrt{2}\int_0^t \mathrm{e}^s \mathrm{d}B_s$$

   is the unique solution of the stochastic differential equation $Z_0 = x$, $\mathrm{d}Z_t = \sqrt{2}\mathrm{d}B_t - Z_t \mathrm{d}t$.
2. Show that for all $t \geq 0$, $Z_t \overset{\text{law}}{=} x\mathrm{e}^{-t} + \mathrm{e}^{-t} B_{\mathrm{e}^{2t}-1}$.
3. Can we have, for all $t \geq 0$, $Z_t = x\mathrm{e}^{-t} + \mathrm{e}^{-t} B_{\mathrm{e}^{2t}-1}$?
4. Show that the process $(M_t)_{t\geq 0}$ is a continuous local martingale with, for all $t \geq 0$, $\langle M\rangle_t = \mathrm{e}^{2t} - 1$.
5. Deduce that there exists a Brownian motion $(W_t)_{t\geq 0}$ such that for all $t \geq 0$, $Z_t = x\mathrm{e}^{-t} + \mathrm{e}^{-t} W_{\mathrm{e}^{2t}-1}$.

Exercise 2 (Study of a special process)**.** Let $d = 1$, $\alpha \geq 0$, $x \geq 0$. Let X be a continuous semi-martingale taking values in $\mathbb{R}_+$ and solving the stochastic differential equation:

$$X_t = x + 2\int_0^t \sqrt{X_s}\mathrm{d}B_s + \alpha t, \quad t \geq 0.$$

Let $f : [0, +\infty) \to [0, +\infty)$ be continuous and $\varphi : [0, +\infty) \to (0, +\infty)$ be positive and $\mathscr{C}^2$, solving the ordinary differential equation $\varphi'' = 2f\varphi$ with boundary conditions $\varphi(0) = 1$ and $\varphi'(1) = 0$. Note that $\varphi > 0$.

1. Could you give an explicit example of process X for special values of α?
2. Show that φ decreases on the interval $[0, 1]$
3. Show that $u = \varphi'/(2\varphi)$ solves the differential equation $u' + 2u^2 = f$
4. Show that for all $t \geq 0$,

$$u(t)X_t - \int_0^t f(s)X_s\mathrm{d}s = u(0)x + \int_0^t u(s)\mathrm{d}X_s - 2\int_0^t u(s)^2 X_s\mathrm{d}s.$$

5. For all $t \geq 0$, let us define $Y_t = u(t)X_t - \int_0^t f(s)X_s\mathrm{d}s$. Show that

$$\varphi(t)^{-\frac{\alpha}{2}}\mathrm{e}^{Y_t} = \mathrm{e}^{N_t - \frac{1}{2}\langle N\rangle_t} \quad \text{where} \quad N_t = u(0)x + 2\int_0^t u(s)\sqrt{X_s}\mathrm{d}B_s$$

6. Show that

$$\mathbb{E}\exp\Big(-\int_0^1 f(s)X_s\mathrm{d}s\Big) = \varphi(1)^{\frac{\alpha}{2}}\mathrm{e}^{\frac{x}{2}\varphi'(0)}$$

7. From now on, let $\lambda > 0$. Prove that

$$\mathbb{E}\exp\Big(-\lambda\int_0^1 X_s\mathrm{d}s\Big) = (\cosh(\sqrt{2\lambda}))^{-\frac{\alpha}{2}}\mathrm{e}^{-\frac{x}{2}\sqrt{2\lambda}\tanh\sqrt{2\lambda}}$$

8. Prove that for all $\lambda > 0$ and $y \in \mathbb{R}$,

$$\mathbb{E}\exp\Big(-\lambda\int_0^1 (y + B_s)^2\mathrm{d}s\Big) = (\cosh(\sqrt{2\lambda}))^{-\frac{1}{2}}\mathrm{e}^{-\frac{y^2}{2}\sqrt{2\lambda}\tanh\sqrt{2\lambda}}$$

Exercise 3 (Strict local martingales). We take $d = 3$, $X = x + B$, $0 < r < |x| < R < \infty$, and, for all $a \geq 0$,

$$T_a = \inf\{t \geq 0 : |X_t| = a\}.$$

1. Show that if $M = (M_t)_{t\geq 0}$ is a continuous local martingale with for all $t \geq 0$, $|M_t| \leq U$ where $U \in \mathrm{L}^1$, then M is a martingale. Does it remain true if the domination condition is replaced by "M is u.i."?
2. Show that if $Z = (Z_t)_{t\geq 0}$ is d-dimensional, adapted, taking values in an open set $D \subset \mathbb{R}^d$, such that its components are continuous local martingales, and for all $1 \leq j,k \leq d$, $\langle Z^j, Z^k\rangle = V\mathbf{1}_{j=k}$ for a finite variation process V, then, for all harmonic $u : D \to \mathbb{R}$, the process $u(Z)$ is a local martingale.
3. Show that $|\bullet|^{-1}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$.
4. Show that $T_R < \infty$ almost surely and
$$\mathbb{P}(T_r < T_R) = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}.$$
5. Deduce from the previous formula that a.s. for all $t \geq 0$, $X_t \neq 0$.
6. Show that a.s. $\lim_{t\to\infty} |B_t| = +\infty$. Hint: show that $|X|^{-1}$ is a non-negative super-martingale.
7. Show that $|X|^{-1}$ is bounded in L^2. Hint: density of B_t in spherical coordinates.
8. Show that $|X|^{-1}$ is a continuous local martingale, but is not a martingale.

Exercise 4 (Strict local martingales and stochastic integrals).

1. Give an example of an Itô stochastic integral which is a local martingale but not a martingale, without using the previous exercise.

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