Stochastic calculus – exam 2021

We always work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ on which is defined a $(\mathcal{F}_t)_{t\geq 0}$–Brownian motion $B = (B_t)_{t\geq 0}$.

Problem 1 (10 points)

The goal of this problem is to determine the law of $X_\star := \sup_{t\geq 0} X_t$, where X solves the SDE

$$\mathrm{d}X_t \;=\; \frac{1}{1+X_t^2}\,\mathrm{d}B_t - \frac{1}{2\left(1+X_t^2\right)^2}\,\mathrm{d}t, \qquad X_0 = 0.$$

1. Justify the existence and uniqueness of a solution $X = (X_t)_{t\geq 0}$.

2. Prove that $M := (e^{X_t})_{t\geq 0}$ is a local martingale, and explicitate its quadratic variation.

3. In this question, we fix $a, b > 0$, and set $T = T_{-a} \wedge T_b$ where $T_r := \inf\{t \geq 0 \colon X_t = r\}$.

   (a) Prove that $(M_{t\wedge T})_{t\geq 0}$ is a square-integrable martingale.

   (b) Justify the following identity:

   $$\forall t \geq 0, \qquad \mathbb{E}\left[M_{t\wedge T}^2\right] \;=\; 1 + \mathbb{E}\left[\int_0^{t\wedge T} \frac{e^{2X_u}}{(1+X_u^2)^2}\,\mathrm{d}u\right].$$

   (c) Deduce from this identity that $\mathbb{E}[T] < \infty$.

   (d) Justify the following formula:

   $$\mathbb{P}(T_b < T_{-a}) \;=\; \frac{1-e^{-a}}{e^b - e^{-a}}.$$

4. Deduce the value of $\mathbb{P}(T_b < \infty)$ for all $b > 0$. Relate this to $X_\star$ and conclude.

5. More generally, determine the law of $X_\star := \sup_{t\geq 0} X_t$ when $X = (X_t)_{t\geq 0}$ solves the SDE

   $$\mathrm{d}X_t \;=\; f(X_t)\,\mathrm{d}B_t - \frac{f^2(X_t)}{2}\,\mathrm{d}t, \qquad X_0 = 0,$$

   with f a strictly positive, bounded, Lipschitz function.

Problem 2 (10 points)

The goal of this problem is to determine all bounded solutions $v\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ to the PDE

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) &= \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2}(t,x) - \dfrac{x^2}{2}v(t,x) \\ v(0,x) &= 1. \end{cases}$$

To this end, we fix a bounded solution v, and $x \in \mathbb{R}$ and we write $X = x + B$.

1. Fix $t \geq 0$, and let $M = (M_s)_{s\in[0,t]}$ be defined by
$$\forall s \in [0,t], \qquad M_s \;:=\; v(t-s, X_s)e^{-\frac{1}{2}\int_0^s X_u^2\,\mathrm{d}u}.$$
Prove that M is a martingale, and deduce the following formula:
$$v(t,x) \;=\; \mathbb{E}\left[e^{-\frac{1}{2}\int_0^t X_u^2\,\mathrm{d}u}\right].$$

2. Establish the following identity:
$$\forall t \geq 0, \qquad \int_0^t X_u\,\mathrm{d}B_u \;=\; \frac{X_t^2 - t - x^2}{2}.$$

3. Show that the process $Z = (Z_t)_{t\geq 0}$ defined below is a martingale:
$$\forall t \geq 0, \qquad Z_t \;:=\; \exp\left\{-\int_0^t X_u\,\mathrm{d}B_u - \frac{1}{2}\int_0^t X_u^2\,\mathrm{d}u\right\}.$$

4. Construct a probability measure $\mathbb{Q}$ under which the process $W = (W_t)_{t\geq 0}$ defined by
$$\forall t \geq 0, \qquad W_t \;:=\; B_t + \int_0^t X_u\,\mathrm{d}u,$$
is a $(\mathcal{F}_t)_{t\geq 0}-$Brownian motion, and express $v(t,x)$ as an expectation under $\mathbb{Q}$.

5. Show that the process X satisfies a Langevin equation driven by the Brownian motion W, and deduce an explicit expression for X, in terms of W.

6. Deduce that for each $t \geq 0$, the distribution of X_t under $\mathbb{Q}$ is $\mathcal{N}\left(xe^{-t}, \frac{1-e^{-2t}}{2}\right)$.

7. For a random variable $Y \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma \in [0,1)$, show that
$$\mathbb{E}\left[e^{\frac{Y^2}{2}}\right] \;=\; \frac{e^{\frac{\mu^2}{2(1-\sigma^2)}}}{\sqrt{1-\sigma^2}}.$$

8. Deduce that for all $t \geq 0$, the function v admits the expression
$$v(t,x) \;=\; \frac{1}{\sqrt{C(t)}}\exp\left\{-\frac{x^2 T(t)}{2}\right\},$$
where $C, T\colon \mathbb{R}_+ \to \mathbb{R}_+$ are classical functions that you should explicitate. Conclude.