Stochastic calculus – exam 2021

We always work on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) on which is defined a \((\mathcal{F}_t)_{t \geq 0}\)-Brownian motion \(B = (B_t)_{t \geq 0}\).

Problem 1 (10 points)

The goal of this problem is to determine the law of \(X_* := \sup_{t \geq 0} X_t\), where \(X\) solves the SDE
\[
dX_t = \frac{1}{1+X_t^2} dB_t - \frac{1}{2(1+X_t^2)^2} dt, \quad X_0 = 0.
\]

1. Justify the existence and uniqueness of a solution \(X = (X_t)_{t \geq 0}\).

This is an homogeneous SDE with coefficients \(\sigma: x \mapsto \frac{1}{1+x^2}\) and \(b: x \mapsto \frac{-1}{2(1+x^2)^2}\). These two functions are Lipshitz, because they are continuously differentiable and their derivatives \(\sigma': x \mapsto \frac{-2x}{(1+x^2)^2}\) and \(b': x \mapsto \frac{2x}{(1+x^2)^3}\) vanish at infinity. Thus, the SDE admits a unique solution starting from any \(X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})\), hence in particular from \(X_0 = 0\).

2. Prove that \(M := (e^{X_t})_{t \geq 0}\) is a local martingale, and explicitate its quadratic variation.

\(M\) is in fact the exponential local martingale associated with the progressive, bounded process \(\phi: t \mapsto \frac{1}{1+X_t^2}\). Specifically, we have \(M_t = \exp(\int_0^t \phi_u du - \frac{1}{2} \int_0^t \phi_u^2 du)\). The general theory ensures that \(M\) is a continuous local martingale, with quadratic variation \(\langle M \rangle_t = \int_0^t M_u^2 \phi_u^2 du\).

3. In this question, we fix \(a, b > 0\), and set \(T = T_a \wedge T_b\) where \(T_r := \inf\{t \geq 0 : X_t = r\}\).

(a) Prove that \((M_{t \wedge T})_{t \geq 0}\) is a square-integrable martingale.

Being the hitting time of the closed set \([-a, b]\) by the continuous and adapted process \(X\), \(T\) is a stopping time. Thus, the stopped process \(M^T := (M_{t \wedge T})_{t \geq 0}\) is a local martingale. But the continuity of \(M\) and the definition of \(T\) ensure that the process \(M^T\) takes values in \([-a, b]\). Thus, it is in fact a true, square-integrable martingale.

(b) Justify the following identity:
\[
\forall t \geq 0, \quad \mathbb{E}[M_{t \wedge T}^2] = 1 + \mathbb{E} \left[ \int_0^{t \wedge T} \frac{\mathbb{e}^{2X_u}}{(1+X_u^2)^2} du \right].
\]

We know that \((M^T)^2 - (M^T)\) is a martingale. In particular, it has constant expectation, i.e. \(\mathbb{E}[M_{t \wedge T}^2 - (M^T)_{t \wedge T}] = \mathbb{E}[M_0^2] = 1\) for all \(t \geq 0\). Rearranging yields the desired identity.
(c) Deduce from this identity that $E[T] < \infty$.

In the above identity, the left-hand side is at most $e^{2b}$, while the right-hand side is at least $1 + E[T \wedge t](e^{-2a}/(1+6^2t))$. This implies that $E[T \wedge t]$ is bounded by a constant $C_{a,b} < \infty$, which does not depend on $t$. Taking $t \to \infty$ (monotone convergence) yields $E[T] \leq C_{a,b}$.

(d) Justify the following formula:

$$\mathbb{P}(T_b < T_a) = \frac{1 - e^{-a}}{e^b - e^{-a}}.$$ 

Since $M_T$ is a martingale, we have $E[M_{T\wedge t}] = E[M_0] = 1$ for all $t \geq 0$. Letting $t \to \infty$ yields $E[M_T] = 1$. Indeed, we have $T < \infty$ a.s. because $E[T] < \infty$, and we have the domination $|M_{T\wedge t}| \leq e^b$. Now, since $M_T$ takes values in $\{e^{-a}, e^b\}$, we have $E[M_T] = pe^b + (1-p)e^{-a}$, where $p = \mathbb{P}(T_b < T_a)$. Thus, $p = (1 - e^{-a})/(e^b - e^{-a})$, as desired.

4. Deduce the value of $\mathbb{P}(T_b < \infty)$ for all $b > 0$. Relate this to $X_*$ and conclude.

The random variables $(T_{-a})_{a>0}$ are clearly increasing with $a$. Moreover, for each $t \geq 0$, we have $\mathbb{P}(\lim_{a \to \infty} T_{-a} \leq t) = \mathbb{P}(\inf_{u \in [0,t]} X_u = -\infty) = 0$. Passing to the limit as $t \to \infty$, we obtain $\mathbb{P}(\lim_{a \to \infty} T_{-a} < \infty) = 0$. In other words, $T_{-a} \to +\infty$ a.s. as $a \to \infty$. We may thus send $a \to \infty$ in the formula obtained in the previous question to obtain (by monotone convergence) that $\mathbb{P}(T_b < \infty) = e^{-b}$. But the continuity of $X$ implies that $\mathbb{P}(X_* \geq b) = \mathbb{P}(T_b < \infty)$, so we conclude that $X_*$ is an Exponential random variable with mean 1.

5. More generally, determine the law of $X_* := \sup_{t \geq 0} X_t$ when $X = (X_t)_{t \geq 0}$ solves the SDE

$$dX_t = f(X_t) dB_t - \frac{f^2(X_t)}{2} dt, \quad X_0 = 0,$$

with $f$ a strictly positive, bounded, Lipschitz function.

The answer is exactly the same. First, the assumptions on $f$ imply that $f^2$ is Lipschitz, because $|f^2(x) - f^2(y)| = |f(x) - f(y)| |f(x) + f(y)| \leq 2\kappa \|f\|_\infty |x - y|$, where $\kappa$ denotes the Lipschitz constant of $f$. Thus, the SDE has a unique solution. Moreover, $M = e^X$ is the exponential local martingale associated with $t \mapsto f(X_t)$. Thus, the stopped process $M_T$ is a local martingale, and it is bounded so it is a square-integrable martingale. As above, we have

$$E \left[ M_{T\wedge t}^2 \right] = E \left[ M_0^2 \right] + E \left[ (M)_{T\wedge t} \right] = 1 + E \left[ \int_0^{t\wedge T} e^{2X_u} f^2(X_u) du \right].$$

The left-hand is at most $e^{2b}$, and the right-hand side is at least $1 + E[t \wedge T]e^{-2a} \min_{a,b} f^2$. This shows that $E[T \wedge t]$ is bounded by a constant $C_{a,b}$. The end of the proof is the same.
Problem 2 (10 points)

The goal of this problem is to determine all bounded solutions $v: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ to the PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t, x) &= \frac{1}{2} \frac{\partial^2 v}{\partial x^2}(t, x) - \frac{x^2}{2} v(t, x) \\ v(0, x) &= 1. \end{cases}$$

To this end, we fix a bounded solution $v$, and $x \in \mathbb{R}$ and we write $X = x + B$.

1. Fix $t \geq 0$, and let $M = (M_s)_{s \in [0, t]}$ be defined by

$$\forall s \in [0, t], \quad M_s := v(t - s, X_s) e^{-\frac{1}{2} \int_0^s X_u^2 du}.$$ 

Prove that $M$ is a martingale, and deduce the following formula:

$$v(t, x) = \mathbb{E}\left[e^{-\frac{1}{2} \int_0^t X_u^2 du}\right].$$

One possibility is to compute the stochastic differential of $M$ and check that the drift term is zero. Since $M$ is bounded, we may then deduce that it is a true martingale. Alternatively, we recognize a special case of the general PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t, x) &= -h(x)v(t, x) + b(x) \frac{\partial v}{\partial x}(t, x) + \frac{1}{2} \sigma^2(x) \frac{\partial^2 v}{\partial x^2}(t, x), \\ v(0, x) &= f. \end{cases}$$

for which Feynman-Kac’s formula gives the representation $v(t, x) = \mathbb{E}[f(X_t^x)e^{-\int_0^t h(X_u^x) du}]$, where $X^x$ solves the homogeneous SDE $dX_t^x = \sigma(X_t^x) dB_t + b(X_t^x) dt$, $X_0^x = x$. In our case, we have $b = 0$, $\sigma = 1$, $h(x) = x^2$ and $f \equiv 1$. Thus, $X^x = x + B$, and the claim follows.

2. Establish the following identity:

$$\forall t \geq 0, \quad \int_0^t X_u dB_u = \frac{X_t^2 - t - x^2}{2}.$$ 

Both sides are Itô processes. They take the same value (zero) at time $t = 0$, and they have the same stochastic differentials (by Itô’s formula), so they must coincide.
3. Show that the process \( Z_t = (Z_t)_{t \geq 0} \) defined below is a martingale:

\[
\forall t \geq 0, \quad Z_t := \exp \left\{ - \int_0^t X_u \, dB_u - \frac{1}{2} \int_0^t X_u^2 \, du \right\}.
\]

The process \( Z \) is the exponential local martingale associated with \( X \). Moreover, the previous question implies that \( 0 \leq Z_t \leq e^{\frac{1 + x^2}{2}} \), so that

\[
\forall t \geq 0, \quad \mathbb{E} \left[ \sup_{s \in [0,t]} |Z_s| \right] < \infty.
\]

This condition suffices to conclude that the local martingale \( Z \) is in fact a martingale.

4. Construct a probability measure \( Q \) under which the process \( W_t = (W_t)_{t \geq 0} \) defined by

\[
\forall t \geq 0, \quad W_t := B_t + \int_0^t X_u \, du,
\]

is a \((\mathcal{F}_t)_{t \geq 0}\)–Brownian motion, and express \( v(t, x) \) as an expectation under \( Q \).

This is Girsanov’s theorem, valid here because \( Z \) is a martingale. For each \( t \geq 0 \), the formula

\[
\forall A \in \mathcal{F}_t, \quad Q_t(A) := \mathbb{E}[Z_t \mathbf{1}_A],
\]

defines a probability measure \( Q_t \) on \((\Omega, \mathcal{F}_t)\), and these measures are consistent as \( t \) increases. Thus, they must all be restrictions of a single probability measure \( Q \) on \( \mathcal{F}_\infty := \sigma(\bigcup_{t \geq 0} \mathcal{F}_t) \), under which \( W \) is a \((\mathcal{F}_t)_{t \geq 0}\)–Brownian motion. In view of Question 1, we have

\[
v(t, x) = \mathbb{E} \left[ Z_t e^{\int_0^t X_u \, dB_u} \right] = \mathbb{E}_Q \left[ e^{\int_0^t X_u \, dB_u} \right].
\]

5. Show that the process \( X \) satisfies a Langevin equation driven by the Brownian motion \( W \), and deduce an explicit expression for \( X \), in terms of \( W \).

By differentiating the very definition of \( W \), we see that the process \( X = x + B \) solves

\[
\frac{dX_t}{dt} = dB_t - X_t \, dt, \quad X_0 = x.
\]

This is the classical Langevin equation on the filtered space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}_\infty, Q)\) equipped with the Brownian motion \( W \). The solution is of course the Ornstein-Uhlenbeck process:

\[
\forall t \geq 0, \quad X_t := xe^{-t} + \int_0^t e^{u-t} \, dW_u,
\]

as shown in class (or re-obtained via the change of variable \( Y_t = e^t X_t \)).
6. Deduce that for each \( t \geq 0 \), the distribution of \( X_t \) under \( Q \) is \( \mathcal{N}(xe^{-t}, \frac{1-e^{-2t}}{2}) \).

Under \( Q \), we have \( \int_0^t e^u \, dW_u \sim \mathcal{N}(0, \frac{e^{2t}-1}{2}) \) (Wiener integral), so the result follows.

7. For a random variable \( Y \sim \mathcal{N}(\mu, \sigma^2) \) with \( \mu \in \mathbb{R} \) and \( \sigma \in [0, 1) \), show that

\[
E \left[ \frac{Y^2}{e^{t/2}} \right] = \frac{e^{\frac{\mu^2}{2(1-\sigma^2)}}}{\sqrt{1-\sigma^2}}.
\]

Writing \( Y = \mu + \sigma Y_0 \) with \( Y_0 \sim \mathcal{N}(0, 1) \), we have

\[
E \left[ \frac{Y^2}{e^{t/2}} \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{\mu^2}{2(1-\sigma^2)} + \frac{\sigma^2 y^2}{2} - \frac{\sigma \mu y}{1-\sigma^2}} \, dy
\]

\[
= \frac{e^{\frac{\mu^2}{2(1-\sigma^2)}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1-\sigma^2}{4} (y - \frac{\mu \sigma}{1-\sigma^2})^2} \, dy,
\]

and the result follows because the last integral is equal to \( \sqrt{\frac{2\pi}{1-\sigma^2}} \).

8. Deduce that for all \( t \geq 0 \), the function \( v \) admits the expression

\[
v(t, x) = \frac{1}{\sqrt{C(t)}} \exp \left\{ -\frac{x^2 T(t)}{2} \right\},
\]

where \( C, T : \mathbb{R}_+ \to \mathbb{R}_+ \) are classical functions that you should explicitate. Conclude.

Combining Questions 2 and 4, we obtain

\[
v(t, x) = e^{\frac{1+x^2}{2}} \mathbb{E}_Q \left[ \frac{x^2}{e^{T/2}} \right].
\]

Now, Questions 6 and 7 allow us to compute the expectation on the right-hand side (take \( Y = X_t, \mu = xe^{-t} \) and \( \sigma^2 = \frac{1-e^{-2t}}{2} \)). Re-arranging yields the desired expression, with

\[
C(t) = \frac{e^t + e^{-t}}{2} = \cosh(t)
\]

\[
T(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \tanh(t).
\]

Conversely, a direct computation shows that the above expression indeed satisfies the desired PDE, because the pair \((T, C)\) satisfies the ODE \((T', C') = (1-T^2, TC)\) with initial condition \((0, 1)\). Moreover, this expression is \([0, 1]-valued\), because \( T \geq 0 \) and \( C \geq 1 \). Thus, the PDE admits a unique bounded solution, and it is given by the above formula.