Stochastic calculus – exam 2022

We always work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ on which is defined a $(\mathcal{F}_t)_{t\geq 0}-$Brownian motion $B = (B_t)_{t\geq 0}$.

Exercise 1 (5 points)

Let $X = (X_t)_{t\geq 0}$ solve the stochastic differential equation

$$\mathrm{d}X_t \quad := \quad \frac{X_t}{2}\,\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = 0.$$

1. Justify that this equation admits a unique solution, and find it explicitly.
2. Set $Y_t := e^{\frac{t}{2}} B_{1-e^{-t}}$. Show that Y has the same law as X. Deduce the $t \to \infty$ behavior of X_t.
3. Find a necessary and sufficient condition on $F \in \mathcal{C}^2(\mathbb{R})$ for $(F(X_t))_{t\geq 0}$ to be a local martingale.
4. Deduce that the process $M = (M_t)_{t\geq 0}$ defined as follows is a martingale:
$$M_t \quad := \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X_t} e^{-\frac{u^2}{2}}\,\mathrm{d}u.$$
5. Find an expression for $\mathbb{P}(T_a < \infty)$ for all $a \geq 0$, where $T_a := \inf\{t \geq 0 \colon X_t \geq a\}$.

Exercise 2 (5 points)

Let $F \in \mathcal{C}^2(\mathbb{R})$ be such that $F(0) = 0$ and F', F'' are bounded. Let $X = (X_t)_{t\geq 0}$ solve

$$\mathrm{d}X_t \quad := \quad \mathrm{d}B_t - F'(X_t)\,\mathrm{d}t, \qquad X_0 = 0.$$

1. Justify that this SDE admits a unique solution.
2. Set $G := (F')^2 - F''$. Compute the stochastic differential of the process $W := (W_t)_{t\geq 0}$, where
$$W_t \quad := \quad F(X_t) + \frac{1}{2}\int_0^t G(X_u)\,\mathrm{d}u.$$
3. Write W in integral form and deduce that e^W is a martingale.
4. Prove that for any measurable functions $f \colon \mathbb{R} \to \mathbb{R}_+$, we have the identity
$$\forall t \geq 0, \qquad \mathbb{E}\left[f(X_t)\right] \quad = \quad \mathbb{E}\left[f(B_t) e^{-F(B_t) - \frac{1}{2}\int_0^t G(B_u)\,\mathrm{d}u}\right].$$

Problem (10 points)

In this problem, we fix two Lipschitz functions $b, \sigma \colon \mathbb{R} \to \mathbb{R}$ and for each $x \in \mathbb{R}$, we let X^x solve

$$\begin{cases} \mathrm{d}X_t^x &=& b(X_t^x)\,\mathrm{d}t + \sigma(X_t^x)\,\mathrm{d}B_t \\ X_0^x &=& x. \end{cases} \tag{1}$$

Given two initial conditions $x, y \in \mathbb{R}$, we define two processes $\psi = (\psi_t)_{t\geq 0}$ and $\phi = (\phi_t)_{t\geq 0}$ by

$$\psi_t \;:=\; \frac{b(X_t^x) - b(X_t^y)}{X_t^x - X_t^y}\mathbf{1}_{(X_t^x \neq X_t^y)} \quad \text{and} \quad \phi_t \;:=\; \frac{\sigma(X_t^x) - \sigma(X_t^y)}{X_t^x - X_t^y}\mathbf{1}_{(X_t^x \neq X_t^y)}.$$

1. Compute the stochastic differential of the process $V = (V_t)_{t\geq 0}$ defined by
$$V_t \;:=\; \exp\left\{-\int_0^t \phi_u\,\mathrm{d}B_u + \int_0^t \left(\frac{\phi_u^2}{2} - \psi_u\right)\mathrm{d}u\right\}.$$

2. Express the stochastic differential of the process $W = X^x - X^y$ in terms of W, ψ, ϕ

3. Compute the stochastic differential of VW and deduce the following identity.
$$\forall t \geq 0, \qquad X_t^x - X_t^y \;=\; (x-y)\exp\left\{\int_0^t \phi_u\,\mathrm{d}B_u + \int_0^t \left(\psi_u - \frac{\phi_u^2}{2}\right)\mathrm{d}u\right\}.$$

4. Deduce that when $x \neq y$, the indicators in the definition of ψ, ϕ can be safely removed.

5. Fix $p \geq 1$. Prove that the process $M = (M_t)_{t\geq 0}$ defined as follows is a martingale:
$$M_t \;:=\; \exp\left\{p\int_0^t \phi_u\,\mathrm{d}B_u - \frac{p^2}{2}\int_0^t \phi_u^2\,\mathrm{d}u\right\}.$$

6. Deduce the existence of a constant $c \in (0,\infty)$, independent of t and p, such that
$$\forall t \geq 0, \quad \forall p \geq 1, \qquad \|X_t^x - X_t^y\|_{L^p} \;\leq\; |x-y|e^{cpt}.$$

7. Deduce that the semi-group $(P_t)_{t\geq 0}$ associated with (1) enjoys the following properties:
   (a) If $f \colon \mathbb{R} \to \mathbb{R}$ is bounded and non-decreasing, then so is $P_t f$ for each $t \geq 0$.
   (b) If $f \colon \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz, then so is $P_t f$ for each $t \geq 0$.
   (c) If $f \colon \mathbb{R} \to \mathbb{R}$ is bounded and continuous, then so is $P_t f$ for each $t \geq 0$.

8. Prove that if f, b, σ are in $\mathcal{C}_b^1(\mathbb{R})$, then so is $P_t f$ for all $t \geq 0$ and
$$\forall x \in \mathbb{R}, \quad (P_t f)'(x) \;=\; \mathbb{E}\left[f'(X_t^x)e^{\int_0^t \sigma'(X_u^x)\,\mathrm{d}B_u + \int_0^t \left(b'(X_u^x) - \frac{(\sigma'(X_u^x))^2}{2}\right)\mathrm{d}u}\right].$$

9. We finally assume that f, b, σ are in $\mathcal{C}_b^2(\mathbb{R})$, and we admit that $P_t f \in \mathcal{C}_b^2(\mathbb{R})$ for each $t \geq 0$. Prove that the function $v \colon (t,x) \mapsto (P_t f)(x)$ solves a PDE that you should explicitate.