Stochastic calculus – exam 2022

We always work on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) on which is defined a \((\mathcal{F}_t)_{t \geq 0}\)-Brownian motion \(B = (B_t)_{t \geq 0}\).

**Exercise 1 (5 points)**

Let \(X = (X_t)_{t \geq 0}\) solve the stochastic differential equation

\[
dX_t := \frac{X_t}{2} \, dt + dB_t, \quad X_0 = 0.
\]

1. Justify that this equation admits a unique solution, and find it explicitly.

2. Set \(Y_t := e^{\frac{t}{2}} B_1 e^{-t}\). Show that \(Y\) has the same law as \(X\). Deduce the \(t \to \infty\) behavior of \(X_t\).

3. Find a necessary and sufficient condition on \(F \in \mathcal{C}^2(\mathbb{R})\) for \((F(X_t))_{t \geq 0}\) to be a local martingale.

4. Deduce that the process \(M = (M_t)_{t \geq 0}\) defined as follows is a martingale:

\[
M_t := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X_t} e^{-\frac{u^2}{2}} \, du.
\]

5. Find an expression for \(\mathbb{P}(T_a < \infty)\) for all \(a \geq 0\), where \(T_a := \inf\{t \geq 0 : X_t \geq a\}\).

**Exercise 2 (5 points)**

Let \(F \in \mathcal{C}^2(\mathbb{R})\) be such that \(F(0) = 0\) and \(F', F''\) are bounded. Let \(X = (X_t)_{t \geq 0}\) solve

\[
dX_t := dB_t - F'(X_t) \, dt, \quad X_0 = 0.
\]

1. Justify that this SDE admits a unique solution.

2. Set \(G := (F')^2 - F''\). Compute the stochastic differential of the process \(W := (W_t)_{t \geq 0}\), where

\[
W_t := F(X_t) + \frac{1}{2} \int_0^t G(X_u) \, du.
\]

3. Write \(W\) in integral form and deduce that \(e^W\) is a martingale.

4. Prove that for any measurable functions \(f : \mathbb{R} \to \mathbb{R}_+\), we have the identity

\[
\forall t \geq 0, \quad \mathbb{E}[f(X_t)] = \mathbb{E}\left[f(B_t) e^{-F(B_t)} - \frac{1}{2} \int_0^t G(B_u) \, du\right].
\]
Problem (10 points)

In this problem, we fix two Lipschitz functions \( b, \sigma : \mathbb{R} \to \mathbb{R} \) and for each \( x \in \mathbb{R} \), we let \( X^x \) solve

\[
\begin{align*}
\frac{dX_t^x}{dt} &= b(X_t^x) dt + \sigma(X_t^x) dB_t \\
X_0^x &= x.
\end{align*}
\]  

(1)

Given two initial conditions \( x, y \in \mathbb{R} \), we define two processes \( \psi = (\psi_t)_{t \geq 0} \) and \( \phi = (\phi_t)_{t \geq 0} \) by

\[
\psi_t := \frac{b(X_t^x) - b(X_t^y)}{X_t^x - X_t^y} 1_{(X_t^x \neq X_t^y)} \quad \text{and} \quad \phi_t := \frac{\sigma(X_t^x) - \sigma(X_t^y)}{X_t^x - X_t^y} 1_{(X_t^x \neq X_t^y)}.
\]

1. Compute the stochastic differential of the process \( V = (V_t)_{t \geq 0} \) defined by

\[
V_t := \exp \left\{ - \int_0^t \phi_u dB_u + \int_0^t \left( \frac{\phi_u^2}{2} - \psi_u \right) du \right\}.
\]

2. Express the stochastic differential of the process \( W = X^x - X^y \) in terms of \( W, \psi, \phi \)

3. Compute the stochastic differential of \( VW \) and deduce the following identity.

\[
\forall t \geq 0, \quad X_t^x - X_t^y = (x - y) \exp \left\{ - \psi_0 \int_0^t \phi_u dB_u + \int_0^t \left( \psi_u - \frac{\phi_u^2}{2} \right) du \right\}.
\]

4. Deduce that when \( x \neq y \), the indicators in the definition of \( \psi, \phi \) can be safely removed.

5. Fix \( p \geq 1 \). Prove that the process \( M = (M_t)_{t \geq 0} \) defined as follows is a martingale:

\[
M_t := \exp \left\{ p \int_0^t \phi_u dB_u - \frac{p^2}{2} \int_0^t \phi_u^2 du \right\}.
\]

6. Deduce the existence of a constant \( c \in (0, \infty) \), independent of \( t \) and \( p \), such that

\[
\forall t \geq 0, \quad \forall p \geq 1, \quad \|X_t^x - X_t^y\|_{L^p} \leq |x - y| e^{ct}.
\]

7. Deduce that the semi-group \( (P_t)_{t \geq 0} \) associated with (1) enjoys the following properties:

   (a) If \( f : \mathbb{R} \to \mathbb{R} \) is bounded and non-decreasing, then so is \( P_t f \) for each \( t \geq 0 \).

   (b) If \( f : \mathbb{R} \to \mathbb{R} \) is bounded and Lipschitz, then so is \( P_t f \) for each \( t \geq 0 \).

   (c) If \( f : \mathbb{R} \to \mathbb{R} \) is bounded and continuous, then so is \( P_t f \) for each \( t \geq 0 \).

8. Prove that if \( f, b, \sigma \) are in \( C^1_b(\mathbb{R}) \), then so is \( P_t f \) for all \( t \geq 0 \) and

\[
\forall x \in \mathbb{R}, \quad (P_t f)'(x) = \mathbb{E} \left[ f'(X_t^x) e^{\int_0^t \sigma'(X_u^x) dB_u + \int_0^t \sigma'(X_u^x) - \frac{(\sigma'(X_u^x))^2}{2} du} \right].
\]

9. We finally assume that \( f, b, \sigma \) are in \( C^2_b(\mathbb{R}) \), and we admit that \( P_t f \in C^2_b(\mathbb{R}) \) for each \( t \geq 0 \).

Prove that the function \( v : (t, x) \mapsto (P_t f)(x) \) solves a PDE that you should explicitate.