Stochastic calculus – exam 2023

Phones and lecture notes are not allowed.

We always work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *on which is defined a* $(\mathcal{F}_t)_{t\geq 0}-$*Brownian motion* $B = (B_t)_{t\geq 0}$.

Exercise 1 (5 points)

Explicitate all bounded functions $v \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ that solve the partial differential equation

$$\partial_t v(t,x) \;\; = \;\; -v(t,x) - x\partial_x v(t,x) + \frac{1}{2}\partial_{xx} v(t,x),$$

with initial condition $v(0,x) = \cos(x)$ for all $x \in \mathbb{R}$.

Exercise 2 (5 points)

Fix two continuous functions $b, \sigma \colon \mathbb{R}_+ \to \mathbb{R}$ and consider the stochastic differential equation

$$\mathrm{d}X_t \;\; := \;\; b(t)\,\mathrm{d}t + \sigma(t) X_t\,\mathrm{d}B_t,$$

with initial condition $X_0 = \zeta \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$.

1. Justify that this equation admits a unique solution $X = (X_t)_{t\geq 0}$.

2. Solve this equation explicitly in the special case where $b \equiv 0$, and express the solution in terms of the process $W = (W_t)_{t\geq 0}$ defined as follows:

$$W_t \;\; := \;\; \int_0^t \sigma(u)\,\mathrm{d}B_u - \frac{1}{2}\int_0^t \sigma^2(u)\,\mathrm{d}u.$$

3. Coming back to the general case, compute the stochastic differential of the process Xe^{-W} and deduce an explicit expression for X, in terms of b, ζ and W.

Problem (10 points)

The goal of this problem is to compute the following Laplace transform:

$$L_t(a,b) \quad := \quad \mathbb{E}\left[\exp\left\{-aB_t^2 - \frac{b^2}{2}\int_0^t B_u^2\,du\right\}\right] \qquad (a,b,t \geq 0).$$

1. Compute $L_t(a,0)$ for all $a,t \geq 0$. We henceforth assume that $b > 0$.
2. Find $\psi \in \mathbb{M}^1_{\text{loc}}$ so that the process Z defined below is a local martingale:
$$Z_t \quad := \quad \exp\left\{-b\int_0^t B_u\,\mathrm{d}B_u - \int_0^t \psi_u\,\mathrm{d}u\right\}.$$
3. Express Z_t in terms of the random variables B_t and $\int_0^t B_u^2\,\mathrm{d}u$ only, and deduce that
$$L_t(a,b) \quad = \quad \mathbb{E}\left[Z_t \exp\left\{\left(\frac{b}{2} - a\right)B_t^2\right\}\right]\exp\left(-\frac{bt}{2}\right).$$
4. Fix $t \geq 0$. Construct a probability measure $\mathbb{Q}_t$ on $(\Omega, \mathcal{F}_t)$ under which the process $W = (W_s)_{s\in[0,t]}$ defined by $W_s := B_s + b\int_0^s B_u\,\mathrm{d}u$ is a Brownian motion.
5. Show that for all $t \geq 0$,
$$B_t \quad = \quad \int_0^t e^{b(u-t)}\,\mathrm{d}W_u.$$
6. Determine the law of B_t under $\mathbb{Q}_t$ and deduce the formula
$$L_t(a,b) \quad = \quad \frac{1}{\sqrt{\cosh(bt) + \frac{2a}{b}\sinh(bt)}}.$$