Stochastic calculus – exam 2021

We always work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ on which is defined a $(\mathcal{F}_t)_{t\geq 0}$–Brownian motion $B = (B_t)_{t\geq 0}$.

Problem 1 (10 points)

The goal of this problem is to determine the law of $X_\star := \sup_{t\geq 0} X_t$, where X solves the SDE

$$\mathrm{d}X_t \quad = \quad \frac{1}{1+X_t^2}\,\mathrm{d}B_t - \frac{1}{2\left(1+X_t^2\right)^2}\,\mathrm{d}t, \qquad X_0 = 0.$$

1. Justify the existence and uniqueness of a solution $X = (X_t)_{t\geq 0}$.
   This is an homogeneous SDE with coefficients $\sigma\colon x \mapsto \frac{1}{1+x^2}$ and $b\colon x \mapsto \frac{-1}{2(1+x^2)^2}$. These two functions are Lipshitz, because they are continuously differentiable and their derivatives $\sigma'\colon x \mapsto \frac{-2x}{(1+x^2)^2}$ and $b'\colon x \mapsto \frac{2x}{(1+x^2)^3}$ vanish at infinity. Thus, the SDE admits a unique solution starting from any $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$, hence in particular from $X_0 = 0$.

2. Prove that $M := (e^{X_t})_{t\geq 0}$ is a local martingale, and explicitate its quadratic variation.
   M is in fact the exponential local martingale associated with the progressive, bounded process $\phi\colon t \mapsto \frac{1}{1+X_t^2}$. Specifically, we have $M_t = \exp(\int_0^t \phi_u\,\mathrm{d}u - \frac{1}{2}\int_0^t \phi_u^2\,\mathrm{d}u)$. The general theory ensures that M is a continuous local martingale, with quadratic variation $\langle M\rangle_t = \int_0^t M_u^2\phi_u^2\,\mathrm{d}u$.

3. In this question, we fix $a, b > 0$, and set $T = T_{-a} \wedge T_b$ where $T_r := \inf\{t \geq 0\colon X_t = r\}$.

   (a) Prove that $(M_{t\wedge T})_{t\geq 0}$ is a square-integrable martingale.
   Being the hitting time of the closed set $\{-a, b\}$ by the continuous and adapted process X, T is a stopping time. Thus, the stopped process $M^T := (M_{t\wedge T})_{t\geq 0}$ is a local martingale. But the continuity of M and the definition of T ensure that the process M^T takes values in $[-a, b]$. Thus, it is in fact a true, square-integrable martingale.

   (b) Justify the following identity:

   $$\forall t \geq 0, \qquad \mathbb{E}\left[M_{t\wedge T}^2\right] \quad = \quad 1 + \mathbb{E}\left[\int_0^{t\wedge T} \frac{e^{2X_u}}{(1+X_u^2)^2}\,\mathrm{d}u\right].$$

   We know that $(M^T)^2 - \langle M^T\rangle$ is a martingale. In particular, it has constant expectation, i.e. $\mathbb{E}[M_{t\wedge T}^2 - \langle M\rangle_{T\wedge t}] = \mathbb{E}[M_0^2] = 1$ for all $t \geq 0$. Rearranging yields the desired identity.

(c) Deduce from this identity that $\mathbb{E}[T] < \infty$.

In the above identity, the left-hand side is at most e^{2b}, while the right-hand side is at least $1 + \mathbb{E}[T \wedge t]\frac{e^{-2a}}{(1+b^2)^2}$. This implies that $\mathbb{E}[T \wedge t]$ is bounded by a constant $C_{a,b} < \infty$, which does not depend on t. Taking $t \to \infty$ (monotone convergence) yields $\mathbb{E}[T] \le C_{a,b}$.

(d) Justify the following formula:

$$\mathbb{P}(T_b < T_{-a}) \quad = \quad \frac{1 - e^{-a}}{e^b - e^{-a}}.$$

Since M^T is a martingale, we have $\mathbb{E}[M_{T \wedge t}] = \mathbb{E}[M_0] = 1$ for all $t \ge 0$. Letting $t \to \infty$ yields $\mathbb{E}[M_T] = 1$. Indeed, we have $T < \infty$ a.-s. because $\mathbb{E}[T] < \infty$, and we have the domination $|M_{T \wedge t}| \le e^b$. Now, since M_T takes values in $\{e^{-a}, e^b\}$, we have $\mathbb{E}[M_T] = pe^b + (1-p)e^{-a}$, where $p = \mathbb{P}(T_b < T_{-a})$. Thus, $p = (1 - e^{-a})/(e^b - e^{-a})$, as desired.

4. Deduce the value of $\mathbb{P}(T_b < \infty)$ for all $b > 0$. Relate this to $X_\star$ and conclude.

The random variables $(T_{-a})_{a>0}$ are clearly increasing with a. Moreover, for each $t \ge 0$, we have $\mathbb{P}(\lim_{a\to\infty} T_{-a} \le t) = \mathbb{P}(\inf_{u\in[0,t]} X_u = -\infty) = 0$. Passing to the limit as $t \to \infty$, we obtain $\mathbb{P}(\lim_{a\to\infty} T_{-a} < \infty) = 0$. In other words, $T_{-a} \to +\infty$ a.-s. as $a \to \infty$. We may thus send $a \to \infty$ in the formula obtained in the previous question to obtain (by monotone convergence) that $\mathbb{P}(T_b < \infty) = e^{-b}$. But the continuity of X implies that $\mathbb{P}(X_\star \ge b) = \mathbb{P}(T_b < \infty)$, so we conclude that $X_\star$ is an Exponential random variable with mean 1.

5. More generally, determine the law of $X_\star := \sup_{t\ge 0} X_t$ when $X = (X_t)_{t\ge 0}$ solves the SDE

$$\mathrm{d}X_t \quad = \quad f(X_t)\,\mathrm{d}B_t - \frac{f^2(X_t)}{2}\,\mathrm{d}t, \qquad X_0 = 0,$$

with f a strictly positive, bounded, Lipschitz function.

The answer is exactly the same. First, the assumptions on f imply that f^2 is Lipschitz, because $|f^2(x) - f^2(y)| = |f(x) - f(y)||f(x) + f(y)| \le 2\kappa\|f\|_\infty|x - y|$, where κ denotes the Lipschitz constant of f. Thus, the SDE has a unique solution. Moreover, $M = e^X$ is the exponential local martingale associated with $t \mapsto f(X_t)$. Thus, the stopped process M^T is a local martingale, and it is bounded so it is a square-integrable martingale. As above, we have

$$\mathbb{E}\left[M^2_{t\wedge T}\right] \quad = \quad \mathbb{E}\left[M_0^2\right] + \mathbb{E}\left[\langle M\rangle_{t\wedge T}\right] \quad = \quad 1 + \mathbb{E}\left[\int_0^{t\wedge T} e^{2X_u} f^2(X_u)\,\mathrm{d}u\right].$$

The left-hand is at most e^{2b}, and the right-hand side is at least $1 + \mathbb{E}[t \wedge T]e^{-2a}\min_{[-a,b]} f^2$. This shows that $\mathbb{E}[T \wedge t]$ is bounded by a constant $C_{a,b}$. The end of the proof is the same.

Problem 2 (10 points)

The goal of this problem is to determine all bounded solutions $v\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ to the PDE

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) & = & \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2}(t,x) - \dfrac{x^2}{2}v(t,x) \\ v(0,x) & = & 1. \end{cases}$$

To this end, we fix a bounded solution v, and $x \in \mathbb{R}$ and we write $X = x + B$.

1. Fix $t \geq 0$, and let $M = (M_s)_{s\in[0,t]}$ be defined by

$$\forall s \in [0,t], \qquad M_s \quad := \quad v(t-s, X_s)e^{-\frac{1}{2}\int_0^s X_u^2 \,\mathrm{d}u}.$$

   Prove that M is a martingale, and deduce the following formula:

$$v(t,x) \quad = \quad \mathbb{E}\left[e^{-\frac{1}{2}\int_0^t X_u^2 \,\mathrm{d}u}\right].$$

   One possibility is to compute the stochastic differential of M and check that the drift term is zero. Since M is bounded, we may then deduce that it is a true martingale. Alternatively, we recognize a special case of the general PDE

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) & = & -h(x)v(t,x) + b(x)\dfrac{\partial v}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2(x)\dfrac{\partial^2 v}{\partial x^2}(t,x) \\ v(0,x) & = & f. \end{cases},$$

   for which Feynman-Kac's formula gives the representation $v(t,x) = \mathbb{E}[f(X_t^x)e^{-\int_0^t h(X_u^x)\,\mathrm{d}u}]$, where X^x solves the homogeneous SDE $\mathrm{d}X_t^x = \sigma(X_t^x)\,\mathrm{d}B_t + b(X_t^x)\,\mathrm{d}t$, $X_0^x = x$. In our case, we have $b = 0$, $\sigma = 1$, $h(x) = x^2$ and $f \equiv 1$. Thus, $X^x = x + B$, and the claim follows.

2. Establish the following identity:

$$\forall t \geq 0, \qquad \int_0^t X_u \,\mathrm{d}B_u \quad = \quad \frac{X_t^2 - t - x^2}{2}.$$

   Both sides are Itô processes. They take the same value (zero) at time $t = 0$, and they have the same stochastic differentials (by Itô's formula), so they must coincide.

3. Show that the process $Z = (Z_t)_{t \geq 0}$ defined below is a martingale:

$$\forall t \geq 0, \qquad Z_t \;:=\; \exp\left\{-\int_0^t X_u \,\mathrm{d}B_u - \frac{1}{2}\int_0^t X_u^2 \,\mathrm{d}u\right\}.$$

The process Z is the exponential local martingale associated with X. Moreover, the previous question implies that $0 \leq Z_t \leq e^{\frac{t+x^2}{2}}$, so that

$$\forall t \geq 0, \qquad \mathbb{E}\left[\sup_{s \in [0,t]} |Z_s|\right] \;<\; \infty.$$

This condition suffices to conclude that the local martingale Z is in fact a martingale.

4. Construct a probability measure $\mathbb{Q}$ under which the process $W = (W_t)_{t \geq 0}$ defined by

$$\forall t \geq 0, \qquad W_t \;:=\; B_t + \int_0^t X_u \,\mathrm{d}u,$$

is a $(\mathcal{F}_t)_{t \geq 0}-$Brownian motion, and express $v(t,x)$ as an expectation under $\mathbb{Q}$.
This is Girsanov's theorem, valid here because Z is a martingale. For each $t \geq 0$, the formula

$$\forall A \in \mathcal{F}_t, \quad \mathbb{Q}_t(A) \;:=\; \mathbb{E}[Z_t \mathbf{1}_A],$$

defines a probability measure $\mathbb{Q}_t$ on $(\Omega, \mathcal{F}_t)$, and these measures are consistent as t increases. Thus, they must all be restrictions of a single probability measure $\mathbb{Q}$ on $\mathcal{F}_\infty := \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$, under which W is a $(\mathcal{F}_t)_{t \geq 0}-$Brownian motion. In view of Question 1, we have

$$v(t,x) \;=\; \mathbb{E}\left[Z_t e^{\int_0^t X_u \,\mathrm{d}B_u}\right] \;=\; \mathbb{E}^Q\left[e^{\int_0^t X_u \,\mathrm{d}B_u}\right].$$

5. Show that the process X satisfies a Langevin equation driven by the Brownian motion W, and deduce an explicit expression for X, in terms of W.
By differentiating the very definition of W, we see that the process $X = x + B$ solves

$$\mathrm{d}X_t \;=\; \mathrm{d}W_t - X_t \,\mathrm{d}t, \qquad X_0 = x.$$

This is the classical Langevin equation on the filtered space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}_\infty, \mathbb{Q})$ equipped with the Brownian motion W. The solution is of course the Ornstein-Uhlenbeck process:

$$\forall t \geq 0, \quad X_t \;:=\; xe^{-t} + \int_0^t e^{u-t} \,\mathrm{d}W_u,$$

as shown in class (or re-obtained via the change of variable $Y_t = e^t X_t$).

6. Deduce that for each $t \geq 0$, the distribution of X_t under $\mathbb{Q}$ is $\mathcal{N}\left(xe^{-t}, \frac{1-e^{-2t}}{2}\right)$.

Under $\mathbb{Q}$, we have $\int_0^t e^u \, \mathrm{d}W_u \sim \mathcal{N}(0, \frac{e^{2t}-1}{2})$ (Wiener integral), so the result follows.

7. For a random variable $Y \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma \in [0, 1)$, show that

$$\mathbb{E}\left[e^{\frac{Y^2}{2}}\right] = \frac{e^{\frac{\mu^2}{2(1-\sigma^2)}}}{\sqrt{1-\sigma^2}}.$$

Writing $Y = \mu + \sigma Y_0$ with $Y_0 \sim \mathcal{N}(0, 1)$, we have

$$\begin{aligned} \mathbb{E}\left[e^{\frac{Y^2}{2}}\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{\mu^2+\sigma^2 y^2 - 2\mu\sigma y - y^2}{2}} \, \mathrm{d}y \\ &= \frac{e^{\frac{\mu^2}{2(1-\sigma)^2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1-\sigma^2}{2}\left(y - \frac{\mu\sigma}{1-\sigma^2}\right)^2} \, \mathrm{d}y, \end{aligned}$$

and the result follows because the last integral is equal to $\sqrt{\frac{2\pi}{1-\sigma^2}}$.

8. Deduce that for all $t \geq 0$, the function v admits the expression

$$v(t, x) = \frac{1}{\sqrt{C(t)}} \exp\left\{-\frac{x^2 T(t)}{2}\right\},$$

where $C, T \colon \mathbb{R}_+ \to \mathbb{R}_+$ are classical functions that you should explicitate. Conclude.

Combining Questions 2 and 4, we obtain

$$v(t, x) = e^{-\frac{t+x^2}{2}} \mathbb{E}^Q\left[e^{\frac{X_t^2}{2}}\right].$$

Now, Questions 6 and 7 allow us to compute the expectation on the right-hand side (take $Y = X_t$, $\mu = xe^{-t}$ and $\sigma^2 = \frac{1-e^{-2t}}{2}$). Re-arranging yields the desired expression, with

$$\begin{aligned} C(t) &= \frac{e^t + e^{-t}}{2} = \cosh(t) \\ T(t) &= \frac{e^t - e^{-t}}{e^t + e^{-t}} = \tanh(t). \end{aligned}$$

Conversely, a direct computation shows that the above expression indeed satisfies the desired PDE, because the pair (T, C) satisfies the ODE $(T', C') = (1 - T^2, TC)$ with initial condition $(0, 1)$. Moreover, this expression is $[0, 1]-$valued, because $T \geq 0$ and $C \geq 1$. Thus, the PDE admits a unique bounded solution, and it is given by the above formula.