Stochastic calculus – exam 2022

We always work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *on which is defined a* $(\mathcal{F}_t)_{t\geq 0}-$*Brownian motion* $B = (B_t)_{t\geq 0}$.

Exercise 1 (5 points)

Let $X = (X_t)_{t\geq 0}$ solve the stochastic differential equation

$$\mathrm{d}X_t \quad := \quad \frac{X_t}{2}\,\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = 0.$$

1. Justify that this equation admits a unique solution, and find it explicitly.

   This is a special case of the Langevin equation: the variable $Z_t := e^{-t/2}X_t$ solves $\mathrm{d}Z_t = e^{-t/2}\,\mathrm{d}B_t$ with initial condition $Z_0 = 0$, i.e. $Z_t = \int_0^t e^{-u/2}\,\mathrm{d}B_u$. Thus, we have

   $$X_t \quad = \quad e^{\frac{t}{2}} \int_0^t e^{-\frac{u}{2}}\,\mathrm{d}B_u.$$

2. Set $Y_t := e^{\frac{t}{2}} B_{1-e^{-t}}$. Show that Y has the same law as X. Deduce the $t \to \infty$ behavior of X_t.

   By the properties of the Wiener integral, X is a centered Gaussian process with covariance

   $$\mathrm{Cov}(X_s, X_t) \quad = \quad e^{\frac{t+s}{2}} \int_0^{t\wedge s} e^{-u}\,\mathrm{d}u \quad = \quad e^{\frac{t+s}{2}}\left(1 - e^{-t\wedge s}\right).$$

   Y is also a centered Gaussian process (its coordinates belong to $\mathrm{Vect}(B)$), with covariance

   $$\mathrm{Cov}(Y_s, Y_t) \quad = \quad e^{\frac{t+s}{2}}\,\mathrm{Cov}(B_{1-e^{-s}}, B_{1-e^{-t}}) \quad = \quad e^{\frac{t+s}{2}}\left(1 - e^{-t\wedge s}\right).$$

   Since the law of a Gaussian process is determined by the mean and covariance, we conclude that $X \overset{d}{=} Y$. Now, it is clear that Y_t tends to $+\infty$ or $-\infty$, each with probability $1/2$ (depending on the sign of B_1) and the same must be true for X (for a continuous function $(x_t)_{t\geq 0}$, the events $\{x_t \to +\infty\}$ and $\{x_t \to -\infty\}$ can be expressed in the product $\sigma-$field).

3. Find a necessary and sufficient condition on $F \in \mathcal{C}^2(\mathbb{R})$ for $(F(X_t))_{t\geq 0}$ to be a local martingale.

   Using Itô's formula and the definition of X, we find

   $$\mathrm{d}F(X_t) \quad = \quad F'(X_t)\,\mathrm{d}B_t + \left(F'(X_t) + \frac{F''(X_t)}{2}\right)\mathrm{d}t$$

It then follows from the general properties of Itô processes that $(F(X_t))_{t\geq 0}$ is a local martingale if and only if $F'(X_t) + \frac{F''(X_t)}{2} = 0$ almost-surely, for all $t \geq 0$. But this is equivalent to $F' + \frac{F''}{2} = 0$, because $F' + \frac{F''}{2}$ is continuous and $X_t \sim \mathcal{N}(0, e^t - 1)$ has full support for $t > 0$.

4. Deduce that the process $M = (M_t)_{t\geq 0}$ defined as follows is a martingale:

$$M_t \quad := \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X_t} e^{-\frac{u^2}{2}} \, \mathrm{d}u.$$

The Gaussian cumulative distribution function $F(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} \, \mathrm{d}u$ clearly satisfies $F' + \frac{F''}{2} = 0$, so $F(X)$ is a local martingale, and in fact a martingale because F is bounded.

5. Find an expression for $\mathbb{P}(T_a < \infty)$ for all $a \geq 0$, where $T_a := \inf\{t \geq 0 \colon X_t \geq a\}$.

M is a continuous martingale, and T_a is a stopping time (hitting time of the closed set $[a, +\infty)$ by the continuous adapted process X). Thus, Doob's optional stopping Theorem ensures that

$$\mathbb{E}\left[M_{t\wedge T_a}\right] \quad = \quad \mathbb{E}[M_0] \quad = \quad \frac{1}{2}.$$

We now send $t \to \infty$. On the event $\{T_a < \infty\}$, we have $M_{t\wedge T_a} \to F(a)$ by the continuity of X. On the event $\{T_a = +\infty\}$, we can not have $X_t \to +\infty$, so we must have $X_t \to -\infty$ by Question 2, and hence $M_{t\wedge T_a} \to 0$. We thus have $M_{t\wedge T_a} \to F(a)\mathbf{1}_{T_a<\infty}$ almost-surely, and also in L^1 because M is bounded. Taking expectations, we conclude that

$$\mathbb{P}(T_a < \infty) \quad = \quad \frac{1}{2F(a)}, \qquad \text{with} \qquad F(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{u^2}{2}} \, \mathrm{d}u.$$

Exercise 2 (5 points)

Let $F \in \mathcal{C}^2(\mathbb{R})$ be such that $F(0) = 0$ and F', F'' are bounded. Let $X = (X_t)_{t\geq 0}$ solve

$$\mathrm{d}X_t \quad := \quad \mathrm{d}B_t - F'(X_t)\, \mathrm{d}t, \qquad X_0 = 0.$$

1. Justify that this SDE admits a unique solution.

This is an homogeneous SDE whose coefficients $\sigma(\cdot) = 1$ and $b(\cdot) = -F'(\cdot)$ are Lipschitz (the second because F'' is continuous and bounded). Thus, the SDE has a unique solution.

2. Set $G := (F')^2 - F''$. Compute the stochastic differential of the process $W := (W_t)_{t\geq 0}$, where

$$W_t \quad := \quad F(X_t) + \frac{1}{2}\int_0^t G(X_u)\,\mathrm{d}u.$$

Using the linearity of the stochastic differential and Itô's formula, we find

$$\begin{aligned} \mathrm{d}W_t &= F'(X_t)\,\mathrm{d}X_t + \frac{1}{2}F''(X_t)\,\mathrm{d}\langle X\rangle_t + \frac{1}{2}G(X_t)\,\mathrm{d}t \\ &= F'(X_t)\,\mathrm{d}B_t - \frac{1}{2}\left(F'(X_t)\right)^2\,\mathrm{d}t, \end{aligned}$$

where the second line uses the definitions of G and X.

3. Write W in integral form and deduce that e^W is a martingale.

Since $W_0 = F(0) = 0$, the previous question gives

$$W_t \quad = \quad \int_0^t F'(X_u)\,\mathrm{d}B_u - \frac{1}{2}\int_0^t \left(F'(X_u)\right)^2\,\mathrm{d}u.$$

Thus, e^W is the exponential local martingale Z^ϕ associated with $\phi_t := F'(X_t)$. Since F' is bounded, Novikov's criterion $\mathbb{E}\left[e^{\frac{1}{2}\int_0^t \phi_u^2\,\mathrm{d}u}\right] < \infty$ holds for all $t \geq 0$. Thus, e^W is a martingale.

4. Prove that for any measurable functions $f\colon \mathbb{R} \to \mathbb{R}_+$, we have the identity

$$\forall t \geq 0, \qquad \mathbb{E}\left[f(X_t)\right] \quad = \quad \mathbb{E}\left[f(B_t)e^{-F(B_t)-\frac{1}{2}\int_0^t G(B_u)\,\mathrm{d}u}\right].$$

We have $X_t = B_t - \int_0^t \phi_u\,\mathrm{d}u$ where $\phi_t := F'(X_t)$. Since $Z^\phi = e^W$ is a martingale, Girsanov's Theorem ensures that X is a Brownian motion under $\mathbb{Q}$, where for every $t \geq 0$,

$$A \in \mathcal{F}_t \quad \Longrightarrow \quad \mathbb{Q}(A) \;=\; \mathbb{E}[e^{W_t}\mathbf{1}_A].$$

By linearity and density, this formula implies $\mathbb{E}^{\mathbb{Q}}\left[Y\right] = \mathbb{E}[e^{W_t}Y]$ for any non-negative, $\mathcal{F}_t$−measurable random variable Y. In particular, we may take $Y = f(X_t)e^{-W_t}$ to obtain

$$\begin{aligned} \mathbb{E}\left[f(X_t)\right] &= \mathbb{E}^{\mathbb{Q}}\left[f(X_t)e^{-W_t}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[f(X_t)e^{-F(X_t)-\frac{1}{2}\int_0^t G(X_u)\,\mathrm{d}u}\right] \\ &= \mathbb{E}\left[f(B_t)e^{-F(B_t)-\frac{1}{2}\int_0^t G(B_u)\,\mathrm{d}u}\right], \end{aligned}$$

where the last identity uses the fact that X is a Brownian motion under $\mathbb{Q}$.

Problem (10 points)

In this problem, we fix two Lipschitz functions $b, \sigma \colon \mathbb{R} \to \mathbb{R}$ and for each $x \in \mathbb{R}$, we let X^x solve

$$\begin{cases} \mathrm{d}X_t^x &= b(X_t^x)\,\mathrm{d}t + \sigma(X_t^x)\,\mathrm{d}B_t \\ X_0^x &= x. \end{cases} \tag{1}$$

Given two initial conditions $x, y \in \mathbb{R}$, we define two processes $\psi = (\psi_t)_{t \geq 0}$ and $\phi = (\phi_t)_{t \geq 0}$ by

$$\psi_t := \frac{b(X_t^x) - b(X_t^y)}{X_t^x - X_t^y}\mathbf{1}_{(X_t^x \neq X_t^y)} \quad \text{and} \quad \phi_t := \frac{\sigma(X_t^x) - \sigma(X_t^y)}{X_t^x - X_t^y}\mathbf{1}_{(X_t^x \neq X_t^y)}.$$

1. Compute the stochastic differential of the process $V = (V_t)_{t \geq 0}$ defined by

$$V_t := \exp\left\{ -\int_0^t \phi_u\,\mathrm{d}B_u + \int_0^t \left(\frac{\phi_u^2}{2} - \psi_u \right) \mathrm{d}u \right\}.$$

Applying Itô's formula to the function exp, we readily find

$$\mathrm{d}V_t = V_t \left(\phi_t^2 - \psi_t \right) \mathrm{d}t - V_t \phi_t \,\mathrm{d}B_t.$$

2. Express the stochastic differential of the process $W = X^x - X^y$ in terms of W, ψ, ϕ

In view of the SDE (1) and the linearity of the stochastic differential, we have

$$\begin{aligned} \mathrm{d}W_t &= (b(X_t^x) - b(X_t^y))\,\mathrm{d}t + (\sigma(X_t^x) - \sigma(X_t^y))\,\mathrm{d}B_t \\ &= W_t(\psi_t\,\mathrm{d}t + \phi_t\,\mathrm{d}B_t), \end{aligned}$$

where the second line uses the identities $W_t\psi_t = b(X_t^x) - b(X_t^y)$ and $W_t\phi_t = \sigma(X_t^x) - \sigma(X_t^y)$.

3. Compute the stochastic differential of VW and deduce the following identity.

$$\forall t \geq 0, \qquad X_t^x - X_t^y = (x - y)\exp\left\{ \int_0^t \phi_u\,\mathrm{d}B_u + \int_0^t \left(\psi_u - \frac{\phi_u^2}{2} \right) \mathrm{d}u \right\}.$$

By the stochastic integration-by-parts formula and the previous questions, we have

$$\begin{aligned} \mathrm{d}(V_t W_t) &= W_t\,\mathrm{d}V_t + V_t\,\mathrm{d}W_t + \mathrm{d}\langle V, W \rangle_t \\ &= V_t W_t \left(\phi_t^2\,\mathrm{d}t - \psi_t\,\mathrm{d}t - \phi_t\,\mathrm{d}B_t \right) + V_t W_t (\psi_t\,\mathrm{d}t + \phi_t\,\mathrm{d}B_t) - V_t W_t \phi_t^2\,\mathrm{d}t \\ &= 0. \end{aligned}$$

We conclude that the process VW is constant equal to $V_0W_0 = x-y$. In other words, we have $W = (x-y)V^{-1}$, which is exactly the claimed identity.

4. Deduce that when $x \neq y$, the indicators in the definition of ψ, ϕ can be safely removed.
   Clearly, the right-hand side of the expression given for $X_t^x - X_t^y$ does not vanish.

5. Fix $p \geq 1$. Prove that the process $M = (M_t)_{t\geq 0}$ defined as follows is a martingale:
$$M_t \quad := \quad \exp\left\{p\int_0^t \phi_u \, \mathrm{d}B_u - \frac{p^2}{2}\int_0^t \phi_u^2 \, \mathrm{d}u\right\}.$$
   This is the exponential local martingale associated with the process $p\phi$. Recalling that b, σ are $K-$Lipschitz for some constant $K < \infty$, we know that ϕ, ψ are bounded by K. In particular, Novikov's criterion $\mathbb{E}\left[e^{\frac{1}{2}\int_0^t (p\phi_u)^2 \, \mathrm{d}u}\right] < \infty$ is satisfied for all $t \geq 0$, so M is a martingale.

6. Deduce the existence of a constant $c \in (0,\infty)$, independent of t and p, such that
$$\forall t \geq 0, \quad \forall p \geq 1, \qquad \|X_t^x - X_t^y\|_{L^p} \quad \leq \quad |x-y|e^{cpt}.$$
   In view of the definition of M, we deduce from the identity in Question 3 that
$$\begin{aligned} |X_t^x - X_t^y|^p \quad &= \quad |x-y|^p M_t \exp\left\{p\int_0^t \psi_u \, \mathrm{d}u + \frac{p^2-p}{2}\int_0^t \phi_u^2 \, \mathrm{d}u\right\} \\ &\leq \quad |x-y|^p M_t \exp\left\{pKt + \frac{p^2-p}{2}K^2 t\right\}. \end{aligned}$$
   Taking expectations yields the result, with $c = \max(\frac{K^2}{2}, 2)$.

7. Deduce that the semi-group $(P_t)_{t\geq 0}$ associated with (1) enjoys the following properties:

   (a) If $f\colon \mathbb{R} \to \mathbb{R}$ is bounded and non-decreasing, then so is $P_t f$ for each $t \geq 0$.
   Let $f\colon \mathbb{R} \to \mathbb{R}$ be bounded and non-decreasing, and let $x \leq y$. The identity proved in Question 3 ensures that $X_t^x \leq X_t^y$ almost-surely for all $t \geq 0$, and hence $f(X_t^x) \leq f(Y_t^x)$. Taking expectations yields $(P_t f)(x) \leq (P_t f)(y)$, which shows that $P_t f$ is non-decreasing. The fact that $P_t f$ is bounded is clear, since $\|P_t f\|_\infty \leq \|f\|_\infty$.

   (b) If $f\colon \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz, then so is $P_t f$ for each $t \geq 0$.
   If $f\colon \mathbb{R} \to \mathbb{R}$ is bounded and $K-$Lipschitz, then for any $x, y \in \mathbb{R}$ we can write
$$\begin{aligned} |P_t f(x) - P_t f(y)| \quad &= \quad |\mathbb{E}\left[f(X_t^x)\right] - \mathbb{E}\left[f(X_t^y)\right]| \\ &\leq \quad \mathbb{E}\left[|f(X_t^x) - f(X_t^y)|\right] \\ &\leq \quad K\mathbb{E}\left[|X_t^x - X_t^y|\right] \\ &\leq \quad Ke^{ct}|x-y|, \end{aligned}$$

where the last line uses Question 6 with $p = 1$. This shows that $P_t f$ is Ke^{ct} Lipschitz.

(c) If $f\colon \mathbb{R} \to \mathbb{R}$ is bounded and continuous, then so is $P_t f$ for each $t \geq 0$.
Fix $t \geq 0$ and a real-valued sequence $(x_n)_{n\geq 1}$ that converges to x. Question 6 implies that $X_t^{x_n} \to X_t^x$ in L^p hence in distribution, which precisely means that $P_t f(x_n) \to (P_t f)(x)$ for every bounded continuous $f\colon \mathbb{R} \to \mathbb{R}$, as desired.

8. Prove that if f, b, σ are in $\mathcal{C}_b^1(\mathbb{R})$, then so is $P_t f$ for all $t \geq 0$ and

$$\forall x \in \mathbb{R}, \quad (P_t f)'(x) \quad = \quad \mathbb{E}\left[f'(X_t^x) e^{\int_0^t \sigma'(X_u^x)\,\mathrm{d}B_u + \int_0^t \left(b'(X_u^x) - \frac{(\sigma'(X_u^x))^2}{2} \right) \mathrm{d}u} \right].$$

For fixed $u \geq 0$, the formula in Question 3 shows that $x \mapsto X_u^x$ is continuous. Since b, σ are differentiable, we deduce that

$$\begin{aligned} \frac{b(X_u^x) - b(X_u^y)}{X_u^x - X_u^y} &\xrightarrow[y\to x, y\neq x]{} b'(X_u^x) \\ \frac{\sigma(X_u^x) - \sigma(X_u^y)}{X_u^x - X_u^y} &\xrightarrow[y\to x, y\neq x]{} \sigma'(X_u^x). \end{aligned}$$

Since moreover b, σ are Lipshitz, the left-hand sides are uniformly bounded, so we may invoke the dominated convergence theorem and its stochastic version to obtain in probability:

$$\begin{aligned} \int_0^t \left(\frac{b(X_u^x) - b(X_u^y)}{X_u^x - X_u^y} \right) \mathrm{d}u &\xrightarrow[y\to x, y\neq x]{} \int_0^t b'(X_u^x)\,\mathrm{d}u \\ \int_0^t \left(\frac{\sigma(X_u^x) - \sigma(X_u^y)}{X_u^x - X_u^y} \right)^2 \mathrm{d}u &\xrightarrow[y\to x, y\neq x]{} \int_0^t (\sigma'(X_u^x))^2\,\mathrm{d}u \\ \int_0^t \left(\frac{\sigma(X_u^x) - \sigma(X_u^y)}{X_u^x - X_u^y} \right) \mathrm{d}B_u &\xrightarrow[y\to x, y\neq x]{} \int_0^t \sigma'(X_u^x)\,\mathrm{d}B_u \end{aligned}$$

In view of Questions 3 and 4, we deduce that in probability,

$$\frac{X_t^x - X_t^y}{x - y} \xrightarrow[y\to x, y\neq x]{} \exp\left\{ \int_0^t \sigma'(X_u^x)\,\mathrm{d}B_u + \int_0^t \left(b'(X_u^x) - \frac{(\sigma'(X_u^x))^2}{2} \right) \mathrm{d}u \right\}.$$

This convergence actually holds in every L^p, $p \geq 1$ because the left-hand side is bounded in every $L^p, p \geq 1$ (Question 6). On the other hand, we have the uniformly bounded convergence

$$\frac{f(X_t^x) - f(X_t^y)}{X_t^x - X_t^y} \xrightarrow[y\to x, y\neq x]{} f'(X_t^x),$$

because $f \in \mathcal{C}_b^1(\mathbb{R})$. Multiplying the last two displays yields the desired conclusion.

9. We finally assume that f, b, σ are in $\mathcal{C}_b^2(\mathbb{R})$, and we admit that $P_t f \in \mathcal{C}_b^2(\mathbb{R})$ for each $t \geq 0$. Prove that the function $v\colon (t,x) \mapsto (P_t f)(x)$ solves a PDE that you should explicitate.

   Write L for the generator associated with the SDE (1), and recall that for $g \in \mathcal{C}_b^2(\mathbb{R})$, we have

$$\forall x \in \mathbb{R}, \qquad (Lg)(x) \;=\; b(x)g'(x) + \frac{\sigma^2(x)}{2} g''(x).$$

   Now for each $t \geq 0$, we have $P_t f \in \mathcal{C}_b^2(\mathbb{R})$, so we may take $g = P_t f$ and combine this with Kolmogorov's equation $\frac{\partial}{\partial t} P_t f = L P_t f$ to conclude that $v(t,x) = (P_t f)(x)$ solves the PDE

$$\frac{\partial v}{\partial t}(t,x) \;=\; b(x)\,\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2(x)}{2}\,\frac{\partial^2 v}{\partial x^2}(t,x). \tag{2}$$

   on $\mathbb{R}_+ \times \mathbb{R}$. Note that v is bounded by $\|f\|_\infty$ and that

$$\forall x \in \mathbb{R}, \quad v(0,x) \;=\; f(x). \tag{3}$$

   Conversely, we have seen in class that v is the only bounded solution to (**??**)-(**??**).