Stochastic calculus – exam 2023

Phones and lecture notes are not allowed.

We always work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *on which is defined a* $(\mathcal{F}_t)_{t\geq 0}-$*Brownian motion* $B = (B_t)_{t\geq 0}$.

Exercise 1 (5 points)

Explicitate all bounded functions $v \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ that solve the partial differential equation

$$\partial_t v(t,x) \;=\; -v(t,x) - x\partial_x v(t,x) + \frac{1}{2}\partial_{xx} v(t,x),$$

with initial condition $v(0,x) = \cos(x)$ for all $x \in \mathbb{R}$.

This is a differential equation of the form

$$\begin{cases} \partial_t v(t,x) &=& -h(x)v(t,x) + b(x)\partial_x v(t,x) + \frac{\sigma^2(x)}{2}\partial_{xx} v(t,x) \\ v(0,x) &=& f(x), \end{cases}$$

with $h(x) = 1$, $b(x) = -x$, $\sigma(x) = 1$, and $f(x) = \cos(x)$ for all $x \in \mathbb{R}$. Feyman-Kac's formula guarantees that the only bounded solution $v \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ (if any) must be given by

$$v(t,x) \;=\; \mathbb{E}_x\left[f(X_t^x)e^{-\int_0^t h(X_u^x)\,\mathrm{d}u}\right] \;=\; e^{-t}\mathbb{E}\left[\cos(X_t^x)\right],$$

where X_t^x denotes the unique solution to the SDE

$$\begin{cases} \mathrm{d}X_t^x &=& b(X_t^x)\,\mathrm{d}t + \sigma(X_t^x)\,\mathrm{d}B_t \\ X_0^x &=& x. \end{cases}$$

With $b(x) = -x$ and $\sigma(x) = 1$, we recognize the Langevin equation, whose solution is

$$X_t^x \;=\; xe^{-t} + e^{-t}\int_0^t e^u\,\mathrm{d}B_u \;\overset{d}{=}\; \mathcal{N}\left(xe^{-t}, \frac{1-e^{-2t}}{2}\right).$$

In particular, its characteristic function is $\mathbb{E}[e^{iX_t^x}] = \exp\left(ixe^{-t} - \frac{1-e^{-2t}}{4}\right)$. Taking real parts, we obtain $\mathbb{E}[\cos(X_t^x)] = \cos(xe^{-t})\exp\left(-\frac{1-e^{-2t}}{4}\right)$, and we conclude that

$$v(t,x) \;=\; \cos(xe^{-t})\exp\left\{-t - \frac{1-e^{-2t}}{4}\right\}.$$

Conversely, it is easy to check that this is indeed a bounded solution to the desired PDE.

Exercise 2 (5 points)

Fix two continuous functions $b, \sigma \colon \mathbb{R}_+ \to \mathbb{R}$ and consider the stochastic differential equation

$$\mathrm{d}X_t \quad := \quad b(t)\,\mathrm{d}t + \sigma(t)X_t\,\mathrm{d}B_t,$$

with initial condition $X_0 = \zeta \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$.

1. Justify that this equation admits a unique solution $X = (X_t)_{t\geq 0}$.

   This is an SDE of the form $\mathrm{d}X_t := b(t,x)\,\mathrm{d}t + \sigma(t,x)\,\mathrm{d}B_t$, with coefficients $b(t,x) = b(t)$ and $\sigma(t,x) = \sigma(t)x$. The first is constant (hence uniformly Lipschitz) in the space variable x. The second is uniformly Lipschitz in x, provided we restrict the time variable t to a compact interval (being continuous, σ is bounded on any compact interval). Finally, the functions $t \mapsto b(t,0)$ and $t \mapsto \sigma(t,0)$ are continuous, hence locally square-integrable. Thus, the stochastic Picard-Lindelhöf Theorem applies.

2. Solve this equation explicitly in the special case where $b \equiv 0$, and express the solution in terms of the process $W = (W_t)_{t\geq 0}$ defined as follows:

$$W_t \quad := \quad \int_0^t \sigma(u)\,\mathrm{d}B_u - \frac{1}{2}\int_0^t \sigma^2(u)\,\mathrm{d}u.$$

   When $b = 0$, we recognize the SDE satisfied by the Black-Scholes process $X = \zeta e^W$.

3. Coming back to the general case, compute the stochastic differential of the process Xe^{-W} and deduce an explicit expression for X, in terms of b, ζ and W.

   By Itô's formula, we know that e^{-W} is an Itô process with stochastic differential

$$\begin{aligned} \mathrm{d}(e^{-W_t}) \quad &= \quad e^{-W_t}\left(-\,\mathrm{d}W_t + \frac{1}{2}\,\mathrm{d}\langle W\rangle_t\right) \\ &= \quad e^{-W_t}\left(\sigma^2(t)\,\mathrm{d}t - \sigma(t)\,\mathrm{d}B_t\right). \end{aligned}$$

   By the stochastic integration-by-parts formula, Xe^{-W} is in turn an Itô process, with

$$\begin{aligned} \mathrm{d}(X_t e^{-W_t}) \quad &= \quad X_t\,\mathrm{d}(e^{-W_t}) + e^{-W_t}\,\mathrm{d}X_t + \,\mathrm{d}\langle X, e^{-W}\rangle_t \\ &= \quad e^{-W_t}\left(\sigma^2(t)X_t\,\mathrm{d}t - \sigma(t)X_t\,\mathrm{d}B_t + \sigma(t)X_t\,\mathrm{d}B_t + b(t)\,\mathrm{d}t - \sigma^2(t)X_t\,\mathrm{d}t\right) \\ &= \quad b(t)e^{-W_t}\,\mathrm{d}t. \end{aligned}$$

Recalling that $X_0 = \zeta$, we obtain $X_t e^{-W_t} = \zeta + \int_0^t b(u) e^{-W_u}\,\mathrm{d}u$, or equivalently,

$$X_t \;=\; \zeta e^{W_t} + \int_0^t b(u) e^{W_t - W_u}\,\mathrm{d}u.$$

Problem (10 points)

The goal of this problem is to compute the following Laplace transform:

$$L_t(a,b) \;:=\; \mathbb{E}\left[\exp\left\{-aB_t^2 - \frac{b^2}{2}\int_0^t B_u^2\,du\right\}\right] \quad (a,b,t \geq 0).$$

1. Compute $L_t(a,0)$ for all $a,t \geq 0$. We henceforth assume that $b > 0$.

   By definition, we have for all $a,t \geq 0$,

$$L_t(a,0) \;=\; \mathbb{E}\left[e^{-aB_t^2}\right] \;=\; \mathbb{E}\left[e^{-atB_1^2}\right] \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{(1+2at)z^2}{2}}\,\mathrm{d}z \;=\; \frac{1}{\sqrt{1+2at}}.$$

2. Find $\psi \in \mathbb{M}^1_{\mathrm{loc}}$ so that the process Z defined below is a local martingale:

$$Z_t \;:=\; \exp\left\{-b\int_0^t B_u\,\mathrm{d}B_u - \int_0^t \psi_u\,\mathrm{d}u\right\}.$$

   We simply choose $\psi = \frac{b^2}{2}B^2$, so that $Z = Z^\phi$ is the exponential local martingale associated with the process $\phi = -bB$.

3. Express Z_t in terms of the random variables B_t and $\int_0^t B_u^2\,\mathrm{d}u$ only, and deduce that

$$L_t(a,b) \;=\; \mathbb{E}\left[Z_t \exp\left\{\left(\frac{b}{2} - a\right)B_t^2\right\}\right]\exp\left(-\frac{bt}{2}\right).$$

   Using the identity $2\int_0^t B_u\,\mathrm{d}B_u = B_t^2 - t$, we readily find

$$Z_t \;=\; \exp\left\{-\frac{b}{2}(B_t^2 - t) - \frac{b^2}{2}\int_0^t B_t^2\,\mathrm{d}u\right\}.$$

It follows that

$$Z_t \exp\left\{\left(\frac{b}{2}-a\right)B_t^2 - \frac{bt}{2}\right\} \;=\; \exp\left\{-aB_t^2 - \frac{b^2}{2}\int_0^t B_u^2\,du\right\},$$

and taking expectations gives the desired expression for $L_t(a,b)$.

4. Fix $t \geq 0$. Construct a probability measure $\mathbb{Q}_t$ on $(\Omega, \mathcal{F}_t)$ under which the process $W = (W_s)_{s\in[0,t]}$ defined by $W_s := B_s + b\int_0^s B_u\,du$ is a Brownian motion.
From the expression of Z found in the previous question, it is clear that $0 \leq Z_s \leq e^{\frac{bs}{2}}$ for all $s \in [0,t]$. In particular, we have $\mathbb{E}\left[\sup_{s\in[0,t]}|Z_s|\right] < \infty$, so the local martingale Z is a martingale. By Girsanov's theorem, we know that the formula

$$\mathbb{Q}_t(A) \;:=\; \mathbb{E}[Z_t \mathbf{1}_A], \qquad A \in \mathcal{F}_t,$$

defines a probability measure on $(\Omega, \mathcal{F}_t)$ under which $(W_s)_{s\in[0,t]}$ is a Brownian motion.

5. Show that for all $t \geq 0$,

$$B_t \;=\; \int_0^t e^{b(u-t)}\,\mathrm{d}W_u.$$

By the stochastic integration by parts formula, we have

$$\mathrm{d}(e^{bt}B_t) \;=\; e^{bt}(\,\mathrm{d}B_t + bB_t\,\mathrm{d}t) \;=\; e^{bt}dW_t.$$

In integral form, this gives $e^{bt}B_t = \int_0^t e^{bu}\,\mathrm{d}W_u$, and the desired identity readily follows.

6. Determine the law of B_t under $\mathbb{Q}_t$ and deduce the formula

$$L_t(a,b) \;=\; \frac{1}{\sqrt{\cosh(bt) + \frac{2a}{b}\sinh(bt)}}.$$

Under $\mathbb{Q}_t$, the process $(W_u)_{u\in[0,t]}$ is a Brownian motion, so the integral $\int_0^t e^{bu}\,\mathrm{d}W_u$ is a Wiener integral. In particular, it is Gaussian with mean 0 and variance $\int_0^t e^{2bu}\,\mathrm{d}u = \frac{e^{2bt}-1}{2b}$. Dividing through by e^{bt}, we see that under $\mathbb{Q}_t$, the variable B_t is Gaussian with mean 0 and variance $\frac{1-e^{-2bt}}{2b}$. Using this together with Question 3 and 1, we get

$$e^{\frac{bt}{2}}L_t(a,b) \;=\; \mathbb{E}^{\mathbb{Q}_t}\left[e^{\left(\frac{b}{2}-a\right)B_t^2}\right] \;=\; \mathbb{E}\left[e^{-\alpha B_1^2}\right] \;=\; L_1(\alpha,0) \;=\; \frac{1}{\sqrt{1+2\alpha}},$$

where we have introduced the short-hand $\alpha := \frac{1-e^{-2bt}}{2b}\left(a-\frac{b}{2}\right)$. The result now readily follows from the observation that $(1+2\alpha)e^{bt} = \cosh(bt) + \frac{2a}{b}\sinh(bt)$.