Local matching indicators for concave transport costs

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Abstract

In this note, we introduce a class of indicators that enable to compute efficiently optimal transport plans associated to arbitrary distributions of N demands and N supplies in $\mathbf{R}$ in the case where the cost function is concave. The cost of these indicators is small and independent of N. Using them recursively according to a particular algorithm allows to find an optimal transport plan in less than N^2 evaluations of the cost function. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Dans cette note, nous introduisons une classe d'indicateurs permettant de calculer efficacement des plans de transport optimaux associés à des distributions arbitraires de N sources et de N puits sur la droite réelle dans le cas d'une fonction de coût concave. Ces indicateurs ont un coût de calcul faible et indépendant de N. Leur usage récursif permet, selon un certain algorithme, le calcul d'un plan de transport optimal en au plus N^2 opérations. *Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Version française abrégée

Cette note traite de la résolution numérique d'un problème de transport optimal associé à un coût de transport concave défini en 2.1. Le cadre adopté est celui de deux distributions de N sources et N puits unitaires arbitrairement répartis sur la droite $\mathbf{R}$ (Les cas non-unitaires seront abordés dans [3]). Après avoir rappelé la règle de non-croisement découlant directement de la concavité du coût –voir section 3–, nous indiquons brièvement en suivant [1] en quoi elle permet de se ramener au cas où les sources et les

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puits forment une distribution alternée –voir (2)– appelée *chaîne*.

Pour ce type de distribution, nous introduisons à la section 4 une classe d'indicateurs d'appariement locaux, voir la définition 4.1. Outre le fait qu'ils ont un coût de calcul très faible et indépendant de N, l'intérêt de ces indicateurs réside dans le théorème 4.2 qui indique que sous certaines hypothèses, la négativité d'un indicateur d'ordre k implique que k paires consécutives sont appariées dans un plan de transport optimal associé aux distributions considérées.

L'usage récursif de ces indicateurs débouche sur un algorithme efficace de calcul de plans de transport optimaux présenté à la section 5. La complexité de cet algorithme est N^2 dans le pire des cas et apparaît comme linéaire dans nos premiers tests numériques.

1. Introduction

It is well known that transport problems on the line involving convex cost functions have explicit solutions, consisting in a monotone rearrangement. Recently, an efficient method has been introduced to tackle this issue on the circle [4]. In this note we introduce an algorithm that enables to tackle optimal transport problems on the line (but actually also on the circle) with concave costs. Our algorithm complements the method suggested by McCann [2]. McCann considers general real values of supply and demand and shows how the problem can be reduced to convex optimization somewhat similar to the simplex method in linear programming. Our approach as presented here is developed for the case of unit masses and is closer to the purely combinatorial approach of [1], but extends it to a general concave cost function. The extension to integer masses will be presented in [3].

The method we propose is based on a class of local indicators, that allow to detect consecutive points that are matched in an optimal transport plan. Thanks to the low number of evaluations of the cost function required to apply the indicators, we derive an algorithm that finds an optimal transport plan in n^2 operations in the worst case. In practice, the computational cost of this method appears to behave linearly with respect to n.

Since the indicators apply locally, the algorithm can be massively parallelized and also allows to treat optimal transport problems on the circle. In this way, it extends the work of Aggarwal *et al.* [1] in which cost functions have a linear dependence in the distance.

2. Setting of the problem

For $N_0 \in \mathbf{N}^*$, consider $P = (p_i)_{i=1,\dots,N_0}$ and $Q = (q_i)_{i=1,\dots,N_0}$ two sets of points in $\mathbf{R}$ that represent respectively demand and supply locations. The problem we consider in this note consists in minimizing the transport cost

$$C(\sigma) = \sum_{i,j} c(p_i, q_{\sigma(i)}), \tag{1}$$

where σ is a permutation of $\{1, \dots, N_0\}$. This permutation forms a *transport plan*.

We focus on the case where the function c involves a concave function as stated in the next definition.

Definition 2.1 *The* cost function *in (1) is defined on* $\mathbf{R}$ *by* $c(p,q) = g(|p-q|)$ *with* $p, q \in \mathbf{R}$, *where* $g(\cdot)$ *is a concave non-decreasing real-valued function of a real positive variable such that* $g(0) := \lim_{x \to 0} g(x) \geq -\infty$.

Some examples of such costs are given by $g(x) = \log(x)$ with $g(0) = -\infty$, and $g(x) = \sqrt{x}$ or $g(x) = |x|$ is with $g(0) = 0$.

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Finally, we denote by $\sigma^\star$ the permutation associated to a given optimal transport plan between P and Q: for all permutation σ of $\{1, ..., N_0\}$, $C(\sigma^\star) \leq C(\sigma)$.

3. Chains

In this section, we present a way to build a particular partition of the set $P \cup Q$.

Consider two pairs of matched points $(p_i, q_{\sigma^\star(i)})$ and $(p_{i'}, q_{\sigma^\star(i')})$, say e.g. $p_i \leq q_{\sigma^\star(i)}$, $p_{i'} \leq q_{\sigma^\star(i')}$. It is easy to prove that the following alternative holds:

(i) $[p_i, q_{\sigma^\star(i)}] \cap [p_{i'}, q_{\sigma^\star(i')})] = \emptyset$,

(ii) $[p_i, q_{\sigma^\star(i)}] \subset [p_{i'}, q_{\sigma^\star(i')})]$ or $[p_{i'}, q_{\sigma^\star(i')})] \subset [p_i, q_{\sigma^\star(i)}]$.

This remark is a direct consequence of the concavity of the cost function and is often denominated as "the non-crossing rule" [1,2]. In the next section, we show how it allows decompose the initial situation in sub-problems where supply and demand points are alternated.

Because of the non-crossing rule in an optimal plan there are as many supply points as demand points between any pair of matched points p_i and $q_{\sigma(i)}$. For a given demand point p_i, define its *left neighbor* q_i' as the nearest supply point on the left of p_i such that the numbers of supply and demand points between q_i' and p_i are equal; define the *right neighbor* q_i'' of p_i in a similar way. Then define a *chain* as a maximal alternating sequence of supply and demand points $(p_{i_1}, q_{j_1}, p_{i_1}, ..., q_{j_k})$ or $(q_{j_1}, p_{i_2}, ..., p_{i_{k+1}})$ such that each q_{i_l} is the right neighbor of p_{i_l} and the left neighbor of $p_{i_{l+1}}$. An extension of the proof of Lemma 3 of [1] shows that the collection of chains forms a partition of the set $P \cup Q$. Note that construction of this collection only depends on relative positions of supply and demand points and does not involve any evaluation of the cost function. It can be done in $O(N_0)$ operations.

The non-crossing rule implies that all matched pairs of points in an optimal transport plan must belong to the same chain. We therefore restrict ourselves in the sequel, without loss of generality, to the case of a single chain

$$p_1 < q_1 < ... < p_i < q_i < p_{i+1} < q_{i+1} < ... < p_N < q_N, \tag{2}$$

for $N \in \mathbf{N}^*$ and keep these last notations throughout the rest of this paper.

4. Local matching indicators

Thanks to the non-crossing rule, one knows that there exists at least two consecutive points (p_i, q_i) or (q_i, p_{i+1}) that are matched in any optimal transport plan. Starting from this remark, we take advantage of the structure of a chain to introduce a class of indicators that enable to detect a priori such pairs of points.

Definition 4.1 (Local Matching Indicators of order k) *We define*

$$I_k^p(i) = c(p_i, q_{i+k}) + \sum_{\ell=0}^{k-1} c(p_{i+\ell+1}, q_{i+\ell}) - \sum_{\ell=0}^{k} c(p_{i+\ell}, q_{i+\ell}),$$

where k, i are such that $1 \leq k \leq N - 1$ and $1 \leq i \leq N - k$, and

$$I_k^q(i) = c(p_{i+k+1}, q_i) + \sum_{\ell=1}^{k} c(p_{i+\ell}, q_{i+\ell}) - \sum_{\ell=0}^{k} c(p_{i+\ell+1}, q_{i+\ell}),$$

for $k, i \in \mathbf{N}$, such that $1 \leq k \leq N - 2$ and $1 \leq i \leq N - k - 1$.

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The interest of these functions lies in the next result.

Theorem 4.2 (Negative Local Matching Indicators of order k) *Let $k_0 \in \mathbf{N}$ with $1 \le k_0 \le N-1$ and $i_0 \in \mathbf{N}$ (resp. $i_0' \in \mathbf{N}$), such that $1 \le i_0 \le N - k_0$ (resp. $1 \le i_0' \le N - k_0 - 1$).*
Assume that

  (i) $I_k^p(i) \ge 0$ for $k = 1, ..., k_0 - 1$, $1 \le i \le N - k$,
  (ii) $I_k^q(i') \ge 0$ for $k = 1, ..., k_0 - 1$, $1 \le i' \le N - k - 1$,
  (iii) $I_{k_0}^p(i_0) < 0$ (resp. $I_{k_0}^q(i_0') < 0$).
Then any permutation $\sigma^\star$ associated to an optimal transport plan satisfies $\sigma^\star(i) = i - 1$ for $i = i_0 + 1, ..., i_0 + k_0$ (resp. $\sigma^\star(i) = i$ for $i = i_0 + 1, ..., i_0 + k_0$).

In practice, these indicators allow to find pairs of neighbors that are matched in an optimal transport plan.

Before giving the proof, we state a basic result.

Lemma 4.3 *We keep the previous notations. Define*

$$\varphi_{k,i}^p(x,y) = g(x + y + p_{i+k} - q_i) + \sum_{\ell=0}^{k-1} c(p_{i+\ell+1}, q_{i+\ell}) - g(x) - g(y) - \sum_{\ell=1}^{k-1} c(p_{i+\ell}, q_{i+\ell}),$$

for $k, i \in \mathbf{N}$, such that $1 \le k \le N - 1$ and $1 \le i \le N - k$, and

$$\varphi_{k,i}^q(x,y) = g(x + y + p_{i+k+1} - q_i) + \sum_{\ell=1}^{k} c(p_{i+\ell}, q_{i+\ell}) - g(x) - g(y) - \sum_{\ell=1}^{k-1} c(p_{i+\ell+1}, q_{i+\ell}),$$

for $k, i \in \mathbf{N}$, such that $1 \le k \le N - 2$ and $1 \le i \le N - k - 1$. Both functions $\varphi_{k,i}^p(x,y)$ and $\varphi_{k,i}^q(x,y)$ are decreasing with respect to each of their two variables.

This lemma is a direct consequence of the concavity of the function g. We are now in the position to give the sketch of the proof of Theorem 4.2.

Sketch of the proof: We consider the case where $I_{k_0}^p(i_0) < 0$. The case $I_{k_0}^q(i_0') < 0$ can be treated the same way.

The proof consists in proving that Assumptions (i–iii) imply that neither demand nor supply points located between p_{i_0} and $p_{i_0+k_0+1}$ can be matched with points located outside this interval, i.e. that the set $\mathcal{S}_{i_0} = \{p_i, i_0 + 1 \le i \le i_0 + k_0\} \cup \{q_i, i_0 \le i \le i_0 + k_0 - 1\}$ is stable by an optimal transport plan. In this case, the result follows from Assumption (i–ii).

Suppose that $\mathcal{S}_{i_0}$ is not preserved by an optimal transport plan $\sigma^\star$. Three cases can occur:

a) There exists $i_1 \in \mathbf{N}$, such that $1 \le i_1 \le i_0$ and $i_0 \le \sigma^\star(i_1) \le i_0 + k_0 - 1$ and there exists $i_1' \in \mathbf{N}$, such that $\sigma^\star(i_1) + 1 \le i_1' \le i_0 + k_0$ and $i_0 + k_0 \le \sigma^\star(i_1') \le N$.

b) There exists $i_2 \in \mathbf{N}$, with $i_0 + 1 \le i_2 \le i_0 + k_0$ such that $1 \le \sigma^\star(i_2) \le i_0 - 1$.

c) There exists $i_2 \in \mathbf{N}$, with $i_0 + k_0 < i_2 \le N$ such that $i_0 \le \sigma^\star(i_2) < i_0 + k_0$.

We only prove that Case a) cannot occur. The fact that Cases b) and c) contradict the assumptions are actually not a consequence of the concavity of the cost, but of its non-decreasing property. Details will be given in [3].

In Case a), one can assume without loss of generality that $\sigma^\star(i_1)$ is the largest index such that $1 \le i_1 \le i_0$, $i_0 \le \sigma^\star(i_1) \le i_0 + k_0 - 1$ and that i_1' is the smallest index such that $\sigma^\star(i_1) + 1 \le i_1' \le i_0 + k_0$, $i_0 + k_0 \le \sigma^\star(i_1') \le N$. With such assumptions, the (possibly empty) subset $\{p_i, \sigma^\star(i_1) + 1 \le i \le i_1' - 1\} \cup \{q_i, \sigma^\star(i_1) + 1 \le i \le i_1' - 1\}$ is stable by $\sigma^\star$. Because of Assumptions (i–ii), no nesting can occur in this subset, and $\sigma^\star(i) = i$ for $i = \sigma^\star(i_1) + 1, ..., i_1' - 1$.

On the other hand, since $\sigma^\star$ is supposed to be optimal, one has:

$$c(p_{i_1}, q_{\sigma^\star(i_1)}) + c(p_{i_1'}, q_{\sigma^\star(i_1')}) + \sum_{i=\sigma^\star(i_1)+1}^{i_1'-1} c(p_i, q_i) \le c(p_{i_1}, q_{\sigma^\star(i_1')}) + \sum_{i=\sigma^\star(i_1)}^{i_1'-1} c(p_{i+1}, q_i).$$

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Thanks to Lemma 4.3, one deduces from this last inequality that:

$$c(p_{i_0}, q_{\sigma^\star(i_1)}) + c(p_{i'_1}, q_{i_0+k_0}) + \sum_{i=\sigma^\star(i_1)+1}^{i'_1-1} c(p_i, q_i) \leq c(p_{i_0}, q_{i_0+k_0}) + \sum_{i=\sigma^\star(i_1)}^{i'_1-1} c(p_{i+1}, q_i),$$

and then:

$$c(p_{i_0}, q_{\sigma^\star(i_1)}) + \sum_{i=i_0}^{\sigma^\star(i_1)-1} c(p_{i+1}, q_i) + c(p_{i'_1}, q_{i_0+k_0}) + \sum_{i=i'_1}^{i_0+k_0-1} c(p_{i+1}, q_i) + \sum_{i=\sigma^\star(i_1)+1}^{i'_1-1} c(p_i, q_i)$$

$$\leq c(p_{i_0}, q_{i_0+k_0}) + \sum_{i=i_0}^{i_0+k_0-1} c(p_{i+1}, q_i). \tag{3}$$

According to Assumption (i), $I^p_{\sigma^\star(i_1)-i_0}(i_0) \geq 0$ and $I^p_{i_0+k_0-i'_1}(i'_1) \geq 0$, so that:

$$\sum_{i=i_0}^{\sigma^\star(i_1)} c(p_i, q_i) \leq c(p_{i_0}, q_{\sigma^\star(i_1)}) + \sum_{i=i_0}^{\sigma^\star(i_1)-1} c(p_{i+1}, q_i)$$

$$\sum_{i=i'_1}^{i_0+k_0} c(p_i, q_i) \leq c(p_{i'_1}, q_{i_0+k_0}) + \sum_{i=i'_1}^{i_0+k_0-1} c(p_{i+1}, q_i).$$

Combining these last inequalities with (3) one finds that:

$$\sum_{i=i_0}^{i_0+k_0} c(p_i, q_i) \leq c(p_{i_0}, q_{i_0+k_0}) + \sum_{i=i_0}^{i_0+k_0-1} c(p_{i+1}, q_i),$$

which contradicts Assumption (iii). □

5. An algorithm for balanced chains

The recursive use of our indicators is on the basis of the next algorithm.
Algorithm: Set $\mathcal{P} = \{p_1, ..., p_N, q_1, ..., q_N\}$, $\ell^p = (1, ..., N)$, $\ell^q = (1, ..., N)$, and $k = 1$.
While $\mathcal{P} \neq \emptyset$ and $k \leq N - 1$ do
   (i) Compute $I^p_k(i)$ and $I^q_k(i')$ for $i = 1, ..., N - k$ and $i' = 1, ..., N - k - 1$.
   (ii) Define
$$\mathcal{I}^p_k = \{i_0, 1 \leq i_0 \leq N - k, I^p_k(i_0) < 0\},$$
$$\mathcal{I}^q_k = \{i_0, 1 \leq i_0 \leq N - k - 1, I^q_k(i_0) < 0\},$$
   and do
   (a) If $\mathcal{I}^p_k = \emptyset$ and $\mathcal{I}^q_k = \emptyset$, set $k = k + 1$.
   (b) Else do
       – for all i_0 in $\mathcal{I}^p_k$ and for $i = i_0 + 1, ..., i_0 + k$, do
           · define $\sigma^\star(\ell^p_i) = \ell^q_{i-1}$,
           · remove $\{p_{\ell^p_i}, q_{\ell^q_{i-1}}\}$ from $\mathcal{P}$,
           · remove ℓ^p_i and ℓ^q_i from ℓ^p and ℓ^q respectively.
       – for all i'_0 in $\mathcal{I}^q_k$ and for $i = i'_0 + 1, ..., i'_0 + k$, do
           · define $\sigma^\star(\ell^p_i) = \ell^q_i$,

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· remove $\{p_i, q_i\}$ from $\mathcal{P}$,

· remove ℓ_i^p and ℓ_i^q from ℓ^p and ℓ^q respectively.

– set $N = \frac{1}{2}Card(\mathcal{P})$, and rename the points in $\mathcal{P}$ such that $\mathcal{P} = \{p_1, ..., p_N, q_1, ..., q_N\}$,

$$p_1 < q_1 < ... < p_i < q_i < p_{i+1} < q_{i+1} < ... < p_N < q_N.$$

– reindex the integers of ℓ^p and ℓ^q in accordance to the previous step,

– set $k = 1$.

If $k = N - 1$, for $i = 1, ..., N$ set $\sigma^\star(\ell_i^p) = \ell_i^q$.

To test the efficiency of our algorithm, we have applied it to an increasing number N of pairs of points. For a fixed value of N, 100 samples of points have been chosen randomly in $[0, 1]$, and the mean of the number of evaluations of g has been computed. The results are shown in Fig.1.

The best case consists in finding a negative indicator at each step, and the worst corresponds to the case where all the indicators are positive. These two cases require respectively $N - 1$ and $(N - 1)^2$ evaluations of g.
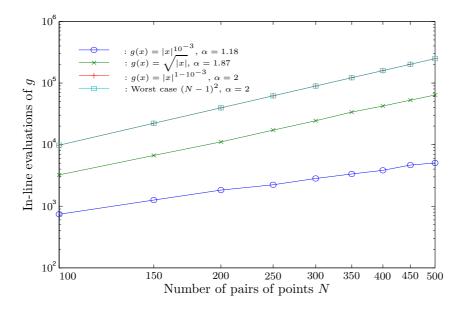

Figure 1. Number of in-line evaluations with respect to the number of pairs, for various cost functions. The number α is the slope of the log-log graphs.

Acknowledgments

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