Adaptive Bayes Test for monotonicity

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Contents

Introduction

Consider the regression problem

$$Y_i = f(i/n) + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0, \sigma^2) \tag{1}$$

where $f : [0, 1] \to \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ are unknown

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Aim

We want to test, for $0 < \alpha \leq 1$

$$H_0 : f \searrow \text{ versus } H_1 : f \text{ is not } \searrow \text{ and } f \in H_\alpha(L)$$

Introduction cont'd

More precisely we want a test that

- require no regularity assumptions under the null
- has good asymptotic properties (consistency)
- does not depend on the regularity α under the alternative
- achieves the "optimal" separation rate for a wide variety of α
- is easy to implement

Why ?

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Shape constrained estimation appears in a variety of models (Drug Response, Global Warming, Survival Analysis, . . .)
Many tests in the frequentist literature but no Bayesian results . . .

Our Model

We build a prior on f by considering a piecewise constant approximation

$$f_{\omega,k} = \sum_{i=1}^{k} \omega_i \mathbb{1}_{](i-1)/k, i/k]}$$

and a prior

$$\pi : \begin{cases} k & \sim \pi_k \\ \omega_i & \overset{iid}{\sim} g \\ \sigma & \sim h \end{cases}$$

Test in a nutshell

Idea for a test

We will test the monotonicity of the sequence $(\omega_i)_i$ and thus need

- A criteria for monotonicity
- Conditions on the prior such that $f_{\omega,k}$ concentrates around f

Bayes Test

We define

$$H(\omega, k) = \max_{1<i<j\leq k} (\omega_j - \omega_i)$$

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Test

We consider the test

$$\delta_n^\pi = \mathbb{1}\left\{\pi\left(H(\omega, k) > M_n^k | Y_n\right) > 1/2\right\}$$

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Remarks

- This is a modified version of the Bayes Factor
- We reject monotonicity if we have sufficiently strong evidence that f is not $\searrow$

Some definitions

Consistency

We will say that δ_n^π is consistent if

$$\begin{aligned} \sup_{f \searrow} \mathrm{E}_0^n(\delta_n^\pi) &= o(1) \\ \sup_{d(f,\searrow)>\rho, f\in H_\alpha(L)} \mathrm{E}_0^n(1-\delta_n^\pi) &= o(1) \end{aligned} \tag{2}$$

Separation rate

We define the separation rate ρ_n of our test as the minimal sequence r_n such that

$$\begin{aligned} \sup_{f \in \searrow} \mathrm{E}_0^n(\delta_n^\pi) &= o(1) \\ \sup_{d(f,\searrow)>r_n, f\in H_\alpha(L)} \mathrm{E}_0^n(1-\delta_n^\pi) &= o(1) \end{aligned} \tag{3}$$

Main Theorem

Theorem

Let $M_n^k = M_0\sqrt{k\log(n)/n}$ *and let* π *be a prior on* $f_{\omega,k}$ *such that* $\omega_i \overset{iid}{\sim} g$ *and* $k \sim \pi_k$, $\sigma \sim h$. *Assume that g and h put mass on* $\mathbb{R}$ *and* $\mathbb{R}^{+*}$ *respectively and that* π_k *is such that there exist positive constants* C_d *and* C_u *such that*

$$e^{-C_d kL(k)} \leq \pi_k(k) \leq e^{-C_u kL(k)}$$

Where $L(k)$ *is either* $\log(k)$ *or* 1. *Consider the test*

$$H_0 : f \searrow \text{ versus } H_1 : f \text{ not } \searrow, f \in H_\alpha(L)$$

let $\rho_n = M(n/\log(n))^{-\alpha/(2\alpha+1)}$, *then the test* δ_n^π *is consistent and achieve the separation rate* ρ_n.

Sketch of the proof

For a fixed k, let ω_0 be such that

$$\omega_0 = \operatorname{argmin}\{\omega, KL(f_{\omega,k}, f)\}$$

We first prove

Result

$$\pi\left(\max_i |\omega_i - \omega_i^0| \geq C\xi_n^k | Y^n\right) \leq 1/2 + o_{P_0^n}(1). \tag{4}$$

where ξ_n^k is such that

$$\xi_n^k := \begin{cases} (n/\log(n))^{-1/3} & \text{if } f \searrow \\ \rho_n & \text{if } f \text{ not } \searrow,\ f \in H_\alpha(L) \end{cases}$$

Sketch of the proof cnt'd

- For $f \searrow$

$$H(\omega, k) \leq 2 \max_i |\omega_i - \omega_i^0|$$

Sketch of the proof cnt'd

- For $f \searrow$

$$H(\omega, k) \leq 2 \max_i |\omega_i - \omega_i^0|$$

- For f not $\searrow$, and $f \in H_\alpha(L)$

$$H(\omega, k) \geq d_n(f_{\omega^0,k}, \searrow) - 2 \max_i |\omega_i - \omega_i^0|$$

$$\geq \rho_n - ||f_{\omega^0,k} - f||_n - 2 \max_i |\omega_i - \omega_i^0|$$

Sketch of the proof cnt'd

Results

- $||f_{\omega^0,k} - f||_n \leq \rho_n/4$ given some consistency results
- $M_n^k \leq \xi_n^k/4$ with posterior probability $\to 1$

Sketch of the proof cnt'd

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Under H_0

$$\pi\left(H(\omega, k) > M_n^k | Y^n\right) \leq \pi\left(\max|\omega_i - \omega_i^0| > \xi_n^k/8\right)$$

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Under H_0

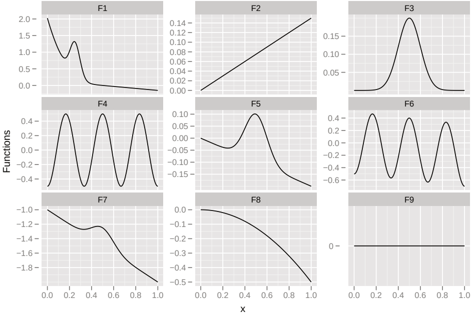
$$\pi\left(H(\omega,k) > M_n^k | Y^n\right) \leq \pi\left(\max|\omega_i - \omega_i^0| > \xi_n^k/8\right)$$

Under H_1

$$\pi\left(H(\omega,k) \leq M_n^k | Y_n\right) \leq \pi\left(\max_i |\omega_i - \omega_i^0| \geq \frac{\rho_n - ||f_{\omega^0,k} - f||_n - M_n^k}{4} | Y^n\right)$$

Simulated Examples

We run our test for nine functions adapted from the frequentist literature



Simulated Examples cnt'd

- We propose a generic choice for the prior
- We study the behaviour of our test for various values of n
- Compare our results with those obtained in the literature

Some specific choice of prior

We choose

- $k \sim \mathcal{P}(\lambda)$
- $\sigma | k \sim IG(\alpha, \beta)$
- $\omega_i | k, \sigma^2 \sim \mathcal{N}(m, \sigma^2/\mu)$

With specific choices for the hyper-parameters

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With specific choices for the hyper-parameters

- We get a closed form formulation for the posterior distribution.
- All the computations are straightforward

Results of the Simulation study

f	σ^2	Barraud et al. $n = 100$	Akakpo et al. $n = 100$	Bayes Test, n : 100	250	500	1000
f_1	0.01	99	99	100.0	100.0	100.0	100.0
f_2	0.01	99	100	99.1	100.0	100.0	100.0
f_3	0.01	99	98	99.6	100.0	100.0	100.0
f_4	0.01	100	99	100.0	100.0	100.0	100.0
f_5	0.004	99	99	99.6	100.0	100.0	100.0
f_6	0.006	98	99	100.0	100.0	100.0	100.0
f_7	0.01	76	68	20.1	40.5	61.6	85.4
f_8	0.01	-	-	0.5	1.0	0.6	0.4
f_9	0.01	-	-	6.2	6.2	4.4	3.4

Skip Real Data

Real Datas

We apply our procedure to Global Warming dataset

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Data

We model the annual temperature anomalies since the 1850's.

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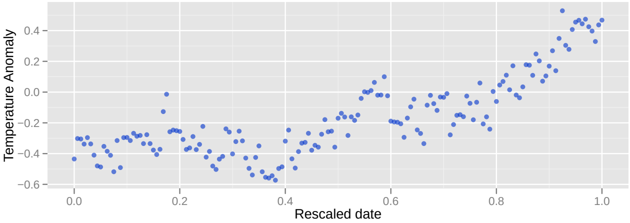
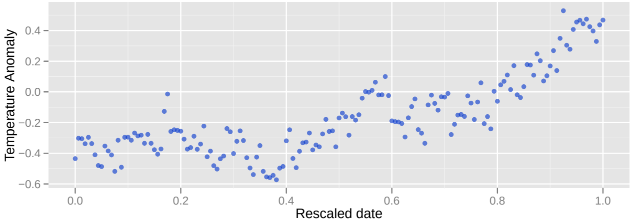
We model the annual temperature anomalies since the 1850's. Used in the frequentist literature to perform isotonic regression
$\rightarrow$ Is there a monotone trend ?

Real Data cnt'd



We run the MCMC sampler using $K = 10^5$ iterations

Real Data cnt'd



We run the MCMC sampler using $K = 10^5$ iterations $\rightarrow$ non monotonic trend (!)

Summary

We have

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Problems

- We don't control for the constant
- No idea how the BF behave