Adaptive Bayes Test for monotonicity

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Introduction

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Aim

We thus construct a Bayesian testing procedure which

- has good asymptotic properties
- is easy to implement and does not require heavy computations, even for large datasets

Why ?

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Many results in the frequentist literature

but no Bayesian results are known

Our Model

Given a partition $(I_i)_i$ of $(0,1)$ in k steps, we consider a piecewise constant approximation of the regression function

$$f_{\omega,k}(.) = \sum_{i=1}^{k} \omega_i \mathbb{1}_{I_i}(.)$$

and put a prior on f by choosing a prior on ω and k

Our Model cnt'd

When f is monotone, $f_{\omega,k}$ will be monotone

Idea for a test

We will thus test the monotonicity of the sequence $(\omega_i)_i$ and thus need

- A criteria for monotonicity
- Conditions on the prior such that $f_{\omega,k}$ concentrates around f

The testing procedure

We consider $H(\omega, k) = \max_{j>i}(\omega_j - \omega_i)$. We thus have a test

$$\delta_n^\pi = \mathbb{1}\left\{\pi\left(H(\omega, k) > M_n^k | Y_n\right) > 1/2\right\}$$

where M_n^k is a threshold such that our test is consistent.

Required asymp. properties

Let $\mathcal{F}$ be the set of monotone non increasing functions. We would like our test to be consistent

$$\sup_{f \in \mathcal{F}} \mathrm{E}_0^n(\delta_n^\pi) = o(1) \tag{1a}$$

$$\sup_{f, d(f,\mathcal{F}) > \rho, f \in H_\alpha(L)} \mathrm{E}_0^n(1 - \delta_n^\pi) = o(1) \tag{1b}$$

Required asymp. properties cnt'd

We would like our test to achieve an optimal separation rate

$$\sup_{f \in \mathcal{F}} \mathrm{E}_0^n(\delta_n^\pi) = o(1) \tag{2a}$$

$$\sup_{f, d(f, \mathcal{F}) > \rho_n, f \in H_\alpha(L)} \mathrm{E}_0^n(1 - \delta_n^\pi) = o(1) \tag{2b}$$

Main Theorem

Theorem

Let $M_n^k = M_0\sqrt{k\log(n)/n}$ *and let* π *a prior on* $f_{\omega,k}$ *such that* $\omega_i|k \overset{iid}{\sim} g$ *and* $k \sim \pi_k$. *Assume that* g *puts mass on* $\mathbb{R}$ *and that* π_k *is such that their exist positive constants* C_d *and* C_u *such that*

$$e^{C_d k L(k)} \leq \pi_k(k) \leq e^{C_u k L(k)}$$

Where $L(k)$ *is either* $\log(k)$ *or* 1. *Consider the test*

$$H_0 : f \in \mathcal{F} \text{ versus } H_1 : f \notin \mathcal{F}, f \in H_\alpha(L)$$

let $\rho_n = M\left(n/\log(n)\right)^{-\alpha/(2\alpha+1)}$, *then* δ_n^π *is consistent and achieve the separation rate* ρ_n

Some useful results

Some specific choices for the prior can be handy.

Prior

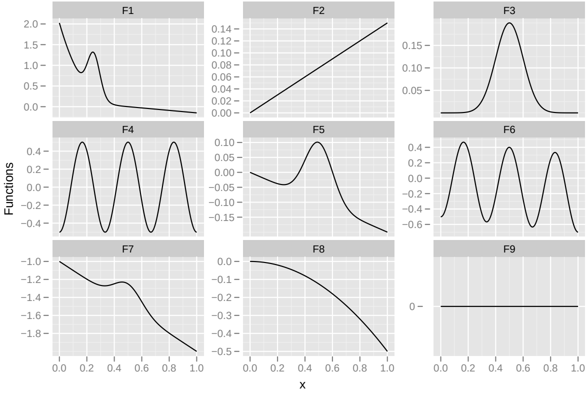
We take $k \sim \mathcal{P}(\lambda)$ and $\omega | k \sim \mathcal{N}(m, v^2)$. Then, if $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ we get

$$\pi_k(k|Y^n) = C(Y^n) e^{1/2 \sum_i \left(\frac{\mathbf{Y_i}/\sigma^2 + m/v^2}{ni/\sigma^2 + 1/v^2} \right)^2} \prod_{i=1}^{k} \left(n_i/\sigma^2 + 1/v^2 \right)^{1/2} \pi_k(k)$$

$$\omega_i | Y^n, k \sim \mathcal{N}\left(\frac{m/v^2 + \mathbf{Y_i}/\sigma^2}{n_i/\sigma^2 + 1/v^2}; \frac{1}{n_i/\sigma^2 + 1/v^2} \right)$$

Experimental design

We choose nine regression functions and generate N independent samples y_i, $i = 1 \ldots n$. For each sample we perform our test and approximate the proportion of rejection.

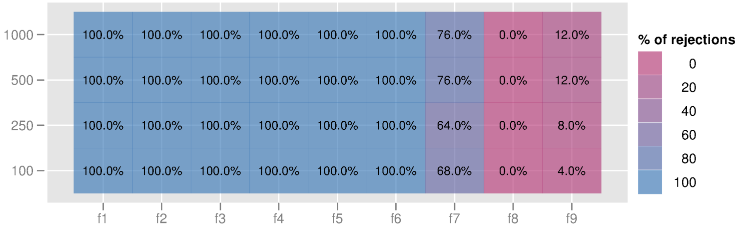


Experimental design cnt'd

For some empirically realistic value for the parameters, simulate $N = 25$ samples for $n = 100, 250, 500, 1000$.

 Still performing simulation for larger values of N and n

Results



Good results even for small sample size

Results cnt'd

We now compare our result with the existing procedures

Function	σ^2	Bayes	S_n^{reg}	T_B
f_1	0.01	1.00	0.99	0.99
f_2	0.01	1.00	1.00	0.99
f_3	0.01	1.00	0.98	0.99
f_4	0.01	1.00	0.99	1.00
f_5	0.004	1.00	0.99	0.99
f_6	0.006	1.00	0.99	0.98
f_7	0.01	0.68	0.68	0.76

Table: Comparison with the existing procedures for $n = 100$

Results cnt'd

Comparison with the Bayes Factor approach

We compute the following Bayes Factor

$$BF_{01} = \frac{\pi(H(\omega,k) \leq 0 | Y^n)}{\pi(H(\omega,k) > 0 | Y^n)} \times \frac{\pi(H(\omega,k) > 0)}{\pi(H(\omega,k) \leq 0)}$$

	n			
	100	250	500	1000
f_1	0.00	0.00	0.00	0.00
f_2	0.00	0.00	0.00	0.00
f_3	0.02	0.00	0.00	0.00
f_4	0.00	0.00	0.00	0.00
f_5	0.00	0.00	0.00	0.00
f_6	0.00	0.00	0.00	0.00
f_7	0.00	0.00	0.00	0.00
f_8	0.94	0.37	0.23	0.10
f_9	0.11	0.02	0.00	0.00

Table: B_{01}

Results cnt'd

Comparison with the Bayes Factor approach

Satisfying results under H_1, but our procedure perform better under H_0

	n			
	100	250	500	1000
f_1	0.00	0.00	0.00	0.00
f_2	0.00	0.00	0.00	0.00
f_3	0.02	0.00	0.00	0.00
f_4	0.00	0.00	0.00	0.00
f_5	0.00	0.00	0.00	0.00
f_6	0.00	0.00	0.00	0.00
f_7	0.00	0.00	0.00	0.00
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