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# Adaptive Bayes Test for monotonicity ENSAI-ENSAE

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## Introduction

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Testing for Monotonic	ity			
Introductio	n			

We consider the regression model with Gaussian residuals

$$Y_i = f(i/n) + \epsilon_i$$

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Testing for Monotonici	ty			
Introductio	n			

### We consider the regression model with Gaussian residuals

$$Y_i = f(i/n) + \epsilon_i$$

#### Aim

We want to test f is monotone non increasing versus f is not

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We consider the regression model with Gaussian residuals

 $Y_i = f(i/n) + \epsilon_i$ 

#### Aim

We thus construct a Bayesian testing procedure which

- has good asymptotic properties
- is easy to implement and does not require heavy computations, even for large datasets

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Motivations				
Why?				

# Monotonicity appears in many applications (drug response models for instance)

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Motivations				
Why?				

Monotonicity appears in many applications (drug response models for instance) Many results in the frequentist literature but no Bayesian results are known

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Our Model				
Our Model				

Given a partition  $(I_i)_i$  of (0, 1) in k steps, we consider a piecewise constant approximation of the regression function

$$f_{\omega,k}(.) = \sum_{i=1}^{k} \omega_i \mathbb{1}_{l_i}(.)$$

and put a prior on f by choosing a prior on  $\omega$  and k

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Our Model				
Our Model	cnt'd			

## When f is monotone, $f_{\omega,k}$ will be monotone

Idea for a test We will thus test the monotonicity of the sequence  $(\omega_i)_i$  and thus need

- A criteria for monotonicity
- Conditions on the prior such that  $f_{\omega,k}$  concentrates around f

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The testing proce	edure			
The test	ing procedu	re		

We consider  $H(\omega, k) = \max_{j>i} (\omega_j - \omega_i)$ . We thus have a test

$$\delta_n^{\pi} = \mathbb{1}\left\{\pi\left(H(\omega, k) > M_n^k | Y_n\right) > 1/2\right\}$$

where  $M_n^k$  is a threshold such that our test is consistent.

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The testing procedure				
Required as	symp. prop	erties		

Let  ${\mathcal F}$  be the set of monotone non increasing functions. We would like our test to be consistent

$$\sup_{f \in \mathcal{F}} \mathsf{E}_0^n(\delta_n^{\pi}) = o(1) \tag{1a}$$

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$$\sup_{f,d(f,\mathcal{F})>\rho,f\in H_{\alpha}(L)} \mathsf{E}_{0}^{n}(1-\delta_{n}^{\pi}) = o(1) \tag{1b}$$

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The testing procedure				
Required a	symp. pro	perties cnt'd		

We would like our test to achieve an optimal separation rate

$$\sup_{f \in \mathcal{F}} \mathsf{E}_0^n(\delta_n^\pi) = o(1) \tag{2a}$$

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$$\sup_{f,d(f,\mathcal{F})>\rho_n,f\in H_{\alpha}(L)} \mathsf{E}_0^n(1-\delta_n^{\pi}) = o(1) \tag{2b}$$

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Main Th	eorem			

#### Theorem

Let  $M_n^k = M_0 \sqrt{k \log(n)/n}$  and let  $\pi$  a prior on  $f_{\omega,k}$  such that  $\omega_i | k \stackrel{iid}{\sim} g$  and  $k \sim \pi_k$ . Assume that g puts mass on  $\mathbb{R}$  and that  $\pi_k$  is such that their exist positive constants  $C_d$  and  $C_u$  such that

 $e^{C_d k L(k)} \leq \pi_k(k) \leq e^{C_u k L(k)}$ 

Where L(k) is either log(k) or 1. Consider the test

 $H_0: f \in \mathcal{F}$  versus  $H_1: f \notin \mathcal{F}, f \in H_{\alpha}(L)$ 

let  $\rho_n = M(n/\log(n))^{-\alpha/(2\alpha+1)}$ , then  $\delta_n^{\pi}$  is consistent and achieve the separation rate  $\rho_n$ 

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Useful remark				
Some us	eful results			

Some specific choices for the prior can be handy.

#### Prior

We take  $k \sim \mathcal{P}(\lambda)$  and  $\omega | k \sim \mathcal{N}(m, v^2)$ . Then, if  $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ we get

$$\pi_{k}(k|Y^{n}) = C(Y^{n})e^{1/2\sum_{i}\left(\frac{\mathbf{Y}_{i}/\sigma^{2}+m/v^{2}}{ni/\sigma^{2}+1/v^{2}}\right)^{2}}\prod_{i=1}^{k}\left(n_{i}/\sigma^{2}+1/v^{2}\right)^{1/2}\pi_{k}(k)$$
$$\omega_{i}|Y^{n}, k \sim \mathcal{N}\left(\frac{m/v^{2}+\mathbf{Y}_{i}/\sigma^{2}}{n_{i}/\sigma^{2}+1/v^{2}}; \frac{1}{n_{i}/\sigma^{2}+1/v^{2}}\right)$$

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Experimental desig	n			
Experime	ental design			

We choose nine regression functions and generate N independent samples  $y_i$ ,  $i = 1 \dots n$ . For each sample we perform our test and approximate the proportion of rejection.



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For some empirically realistic value for the parameters, simulate N = 25 samples for n = 100, 250, 500, 1000.

Still performing simulation for larger values of N and n

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Results				
Results				



Good results even for small sample size

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Results				
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We now compare our result with the existing procedures

Function	$\sigma^2$	Bayes	S <sub>n</sub> reg	$T_B$
$f_1$	0.01	1.00	0.99	0.99
$f_2$	0.01	1.00	1.00	0.99
$f_3$	0.01	1.00	0.98	0.99
$f_4$	0.01	1.00	0.99	1.00
$f_5$	0.004	1.00	0.99	0.99
$f_6$	0.006	1.00	0.99	0.98
$f_7$	0.01	0.68	0.68	0.76

Table: Comparison with the existing procedures for n = 100

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Comparison with the				n	
Bayes Factor approach		100	250	500	1000
Ma compute the following	$f_1$	0.00	0.00	0.00	0.00
ve compute the following	$f_2$	0.00	0.00	0.00	0.00
Bayes Factor	$f_3$	0.02	0.00	0.00	0.00
	f <sub>4</sub>	0.00	0.00	0.00	0.00
$BF_{01} =$	$f_5$	0.00	0.00	0.00	0.00
$\pi(H(\omega,k)\leq 0 Y^n)$	$f_6$	0.00	0.00	0.00	0.00
$\overline{\pi(H(\omega,k)>0 Y^n)}^{\wedge}$	$f_7$	0.00	0.00	0.00	0.00
$\pi(H(\omega, k) > 0)$	$f_8$	0.94	0.37	0.23	0.10
$\overline{\pi(H(\omega, k) \leq 0)}$	<i>f</i> 9	0.11	0.02	0.00	0.00
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Results				
Results of	cnt'd			

Compa	arison	with	the
Bayes	Facto	r app	roach

Satisfying results under  $H_1$ , but our procedure perform better under  $H_0$ 

	n				
	100	250	500	1000	
$f_1$	0.00	0.00	0.00	0.00	
$f_2$	0.00	0.00	0.00	0.00	
$f_3$	0.02	0.00	0.00	0.00	
f <sub>4</sub>	0.00	0.00	0.00	0.00	
$f_5$	0.00	0.00	0.00	0.00	
$f_6$	0.00	0.00	0.00	0.00	
$f_7$	0.00	0.00	0.00	0.00	
f <sub>8</sub>	0.94	0.37	0.23	0.10	
f <sub>9</sub>	0.11	0.02	0.00	0.00	

Table: B<sub>01</sub>