A large deviation principle for a RWRC in a box

7th Cornell Probability Summer School

Michele Salvi

TU Berlin

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1. Large Deviations for dummies
   - What are Large Deviations?
   - A proper definition

2. Random Walk among Random Conductances
   - The model
   - The main theorem
   - Related fields
   - Some heuristics

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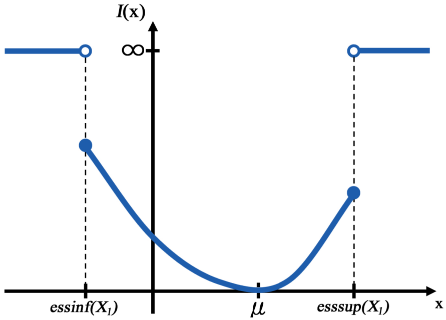
- Large Deviation Principle (LDP) (+ finite exponential moments):

$$\mathbb{P}\Big(\frac{1}{n}\sum_{j=1}^{n} X_j \geq x\Big) \approx \mathrm{e}^{-n\mathcal{I}(x)}, \qquad \forall x \geq 0.$$

Large Deviation Theory deals with asymptotic computation of small probabilities on an exponential scale.



I(x)
∞
essinf(X1)
μ
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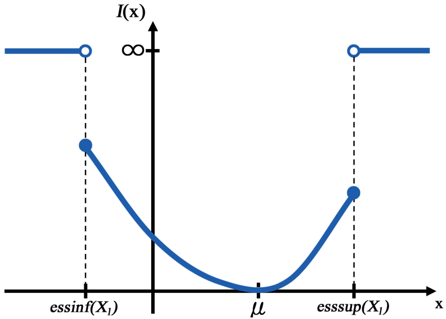
x



The function $\mathcal{I}(x)$

- is convex,
- has compact level sets ($\Longrightarrow$ is lower semi-continuous),
- $\mathcal{I}(x) \geq 0$ and equality holds iff $x = \mu = \mathbb{E}[X_1]$.

Definition

Let $\mathcal{X}$ be a Polish space. A function $\mathcal{I} : \mathcal{X} \to [0, \infty]$ is called rate function if

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1. *For every open set O,* $$\liminf_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(O) \geq - \inf_{x\in O} \mathcal{I}(x);$$
2. *For every closed set C,* $$\limsup_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(C) \leq - \inf_{x\in C} \mathcal{I}(x).$$

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5. in general no relation between LDP and CLT.

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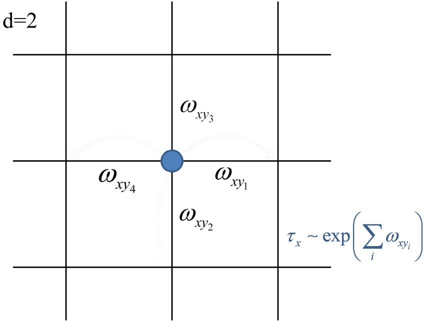
- $\omega_{x,e} = \omega_{x+e,-e}$ (symmetry),
- $\{\omega_{x,e}\}_{x\in\mathbb{Z}^d, e\in\mathcal{E}}$ are i.i.d.,
- $\omega_{x,e} \geq 0$ (positivity).



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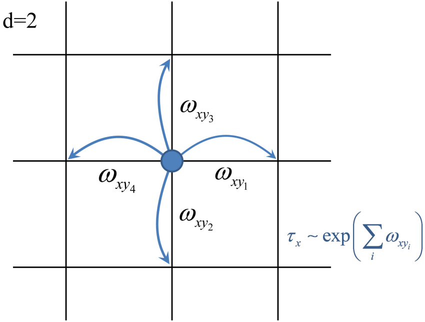

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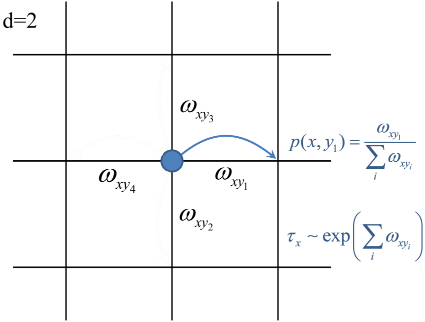

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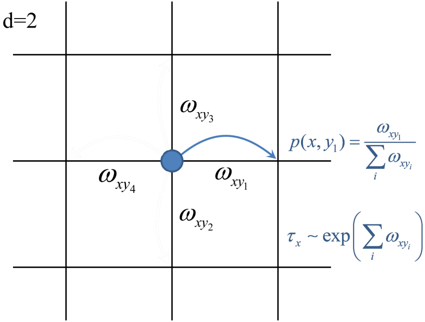

Definition

The Random Walk among Random Conductances *(RWRC) is the continuous-time process generated by*

$$\Delta^\omega f(x) := \sum_{x\in\mathbb{Z}^d, e\in\mathcal{E}} \omega_{x,e}\big(f(x+e) - f(x)\big).$$

	RWRE	RWRC
Time	Mostly discrete	Mostly continuous
Reversibility	No	Yes
Problems	CLT, SLLN, criteria for transience/recurrence, ballisticity	CLT, SLLN

Let $(X_t)_{t\in[0,\infty)}$ be the RWRC. For $x \in B \subseteq \mathbb{Z}^d$, B finite and connected set, define the local time

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We want to study the annealed behaviour of ℓ_t:

$$\left\langle \mathbb{P}_0^\omega \left(\frac{1}{t} \ell_t \sim g^2 \right) \right\rangle$$

where $g^2 \in \mathcal{M}_1(B)$ and $\langle \cdot \rangle$ is the expectation w.r.t. the conductances. The conductances can attain arbitrarily small values.

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Three noises:

- ★ the conductances;
- ★ the waiting times;
- ★ the embedded discrete-time RW.

Hypothesis:

- $B \subseteq \mathbb{Z}^d$ finite and connected;
- $\log \Pr\big(\omega_{x,e} < \varepsilon\big) \approx -\varepsilon^{-\eta}, \quad$ for $\varepsilon \downarrow 0$, $\eta > 1$.

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Theorem (joint work with Wolfgang König and Tilman Wolff)

The process of empirical measures $(\frac{1}{t}\ell_t)_{t\in\mathbb{R}^+}$ of the Random Walk among Random Conductances under the annealed law $\big\langle \mathbb{P}_0^\omega(\,\cdot\, \cap \{\mathrm{supp}(\ell_t) \subseteq B\})\big\rangle$ satisfies a large deviation principle on $\mathcal{M}_1(B)$ with speed $\gamma_t = t^{\frac{\eta}{\eta+1}}$ and rate function J given by

$$J(g^2) := C_\eta \sum_{z,e} |g(z+e) - g(z)|^{\frac{2\eta}{\eta+1}} = C_\eta \|\nabla g\|_{\frac{2\eta}{1+\eta}}^{\frac{2\eta}{1+\eta}}$$

for all $g^2 \in \mathcal{M}_1(B)$, where $C_\eta := \big(1 + \frac{1}{\eta}\big)\eta^{\frac{1}{1+\eta}}$.

This means

$$\left\langle \mathbb{P}_0^\omega\Big(\big\{ \frac{1}{t}\ell_t \sim g^2 \big\} \cap \big\{ \mathrm{supp}(\ell_t) \subseteq B \big\} \Big) \right\rangle \approx \mathrm{e}^{-\gamma_t C_\eta \sum_{z,e} |g(z+e)-g(z)|^{\frac{2\eta}{\eta+1}}}.$$

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In particular:

Corollary

The annealed probability of non-exit from the box B for the Random Walk among Random Conductances for $t \gg 0$ is

$$\log \left\langle \mathbb{P}_0^\omega \Big(\mathrm{supp}(\ell_t) \subseteq B \Big) \right\rangle \simeq -t^{\frac{\eta}{1+\eta}} \inf_{g^2 \in \mathcal{M}_1(B)} C_\eta \sum_{z,e} |g(z+e) - g(z)|^{\frac{2\eta}{\eta+1}}.$$

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- Parabolic Anderson model with random Laplace operator:

$$\begin{cases}\partial_t u(x,t) & =\Delta^\omega u(x,t)+\xi(x)u(x,t), \quad t\in(0,\infty),\ x\in\mathbb{Z}^d\\ u(x,0) & =\delta_0(x) \qquad\qquad\qquad\qquad\qquad x\in\mathbb{Z}^d.\end{cases}$$

Feynman-Kac formula gives $u(x,t)=\mathbb{E}_x^\omega\left[\mathrm{e}^{\int_0^t\xi(X_s)\mathrm{d}s}\delta_0(X_t)\right]$, where X_t is a RWRC.

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- "physicists' trick" ($t^{1-r} = t^{r\eta}$);
- optimization over the rescaled shape of the conductances.

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- Upper bound for closed sets.
  Problem: $t^r\omega$ is not bounded. We need a compactification argument for the space of rescaled conductances.

Thank you!