A Central Limit Theorem for the Effective Conductance

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Background

In nature, even materials with a very regular structure can exhibit impurities at a microscopic level. It may therefore appear surprising how well-known physical phenomena (such as heat diffusion and electric conductivity) can be described via differential equations with smooth coefficients. An explanation to this behaviour is offered by homogenization theory: at a macroscopic scale the imperfections are smoothened out. Nevertheless the influence of the impurities remains hidden in the coefficients of the equations. The rescaled effective conductance, representing the total electric current flowing through a resistor network, is a typical example where the randomness is 'homogenized' in this way.

The model

The Random Conductance Model on $\mathbb{Z}^d$ consists in assigning a positive random value a_{xy} (conductance) to each edge $\langle x,y\rangle$ of the square lattice.

For a set $\Lambda \subset \mathbb{Z}^d$ and its bonds $\mathbb{B}(\Lambda)$, the Dirichlet Energy associated to a potential function f is

$$Q_\Lambda(f) := \sum_{\langle x,y\rangle \in \mathbb{B}(\Lambda)} a_{xy}[f(y) - f(x)]^2.$$

The Effective Conductance with linear boundary conditions $t \in \mathbb{R}^d$ on a square box $\Lambda_L = [0,L)^d \cap \mathbb{Z}^d$ is

$$C_L^{eff}(t) = \min\{Q_{\Lambda_L}(f) : f(x) = t\cdot x, \forall x \in \partial\Lambda_L\}.$$

Prior work

In the case of conductances sampled from an ergodic, shift invariant, strongly elliptic environment, the deterministic limit $\lim_{L\to\infty} \frac{1}{L^d} C_L^{eff}(t)$ has been completely characterized in the 80's (e.g., in the works of Papanicolau and Varadhan, Kozlov, Künnemann).

After the leading order had been indentified, it was left to study the oscillations of C_L^{eff} around its mean. Only last year Gloria and Otto managed to match the lower bound due to Wehr (1997), showing that L^d is in fact the correct order of the variance in the case of an i.i.d. environment with some additional technical assumptions.

The problem

What is the exact behaviour of the fluctuations of C_L^{eff} around its mean? Does a Central Limit Theorem hold?

Main result

Assume i.i.d. conductances with sufficiently small ellipticity contrast. Then a CLT holds:

$$\frac{C_L^{eff}(t) - \mathbb{E}[C_L^{eff}(t)]}{L^{\frac{d}{2}}} \xrightarrow{law} \mathcal{N}(0,\sigma_t^2). \qquad (1)$$

Moreover, if $t \neq 0$ and the law of the conductances is non-degenerate, then $\sigma_t^2 > 0$.

Techniques

The proof of the theorem relies on a simple CLT for Martingales: rewrite the numerator of the l.h.s. of (1) as the sum of the increments of a Doob martingale (w.r.t. an *ad hoc* filtration). This is a function of the discrete gradient of Ψ_Λ, the minimizer of $Q_\Lambda(f)$; in order to use the ergodic theorem we have to substitute this gradient with a shift-invariant object. The candidate is the gradient of the harmonic coordinate, the infinite counterpart of Ψ_Λ.

For this, one needs to control the L^2 and the $L^{4+\varepsilon}$ norms of the difference of the two gradients. This is the main technical obstacle, and can be overcome via the so called Meyers estimates.

Future work

Generalizations to be addressed in a following paper:

- The square domain hypothesis can be dropped.
- Linear boundary conditions can be substituted with general mixed Dirichlet/Neumann conditions.
- The technical assumption of small ellipticity contrast must be replaced by strong ellipticity or even less.

Personal information

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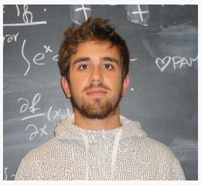

Other works:

- N. Berger, M. Salvi, *On the speed of random walks among random conductances*.
- W. König, M. Salvi, T. Wolff, *Large deviations for the local times of a random walk among random conductances*.

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