Law of large numbers for the Mott Variable Range Hopping model

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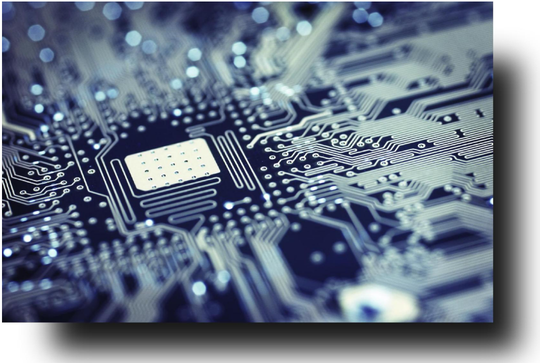
Doped semiconductors



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- $\{x_k\}$ simple point process in $\mathbb{R}^d$, law $\mathbb{P}$. Assume $x_0 = \bar{0}$.
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Transition probabilities:

$$p(x_j, x_k) = \frac{1}{Z_j} \mathrm{e}^{-|x_j - x_k|^\alpha - \beta u(E_j, E_k)}$$

- Z_j is the proper normalization.
- $\alpha \geq 1$. $\beta > 0$ inverse of the temperature.
- u is a symmetric bounded function
  (e.g., $u(E_j, E_k) = |E_j - E_k| + |E_j| + |E_k|$).

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Some more literature:

- Annealed CLT in $d \geq 2$ (A. Faggionato, H. Schulz-Baldes, D. Spehner (2006)).
- Quenched CLT in $d \geq 2$ (P. Caputo, A. Faggionato, T. Prescott (2013)).
- Behaviour for $\beta \to \infty$ (P. Caputo, A. Faggionato) and annealed CLT with weaker conditions in $d = 1$.

The external field

Take $\alpha = 1$, $\beta = 1$.

Assumption: $\quad |x_{k+1} - x_k| > d > 0 \quad$ a.s.

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$$p^{\lambda}(x_j, x_k) = \frac{1}{Z_j^{\lambda}} \mathrm{e}^{-|x_j - x_k| - u(E_j, E_k) + \lambda(x_k - x_j)}$$

Introduce an external field of intensity $\lambda \in (0, 1)$.

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Question

Does the walk have a limiting speed for all $\lambda > 0$?

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(ii) *If*

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Remarks

- For i.i.d. we have dichotomy:

$$\exists \lambda_c \geq 0 : \qquad \begin{cases} v(\lambda) > 0 & \forall \lambda > \lambda_c \\ v(\lambda) = 0 & \forall \lambda < \lambda_c. \end{cases}$$

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  Why? Big λ helps to overjump traps.
- Counterexample with $\mathbb{E}[\mathrm{e}^{(1-\lambda)x_1}] = +\infty$ and $v(\lambda) > 0$.

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Furthermore

$$0 < \gamma \leq \frac{d\mathbb{Q}^\infty}{d\mathbb{P}} \leq F\,, \qquad \mathbb{P}\text{–}a.s.$$

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Remarks

- $v(\lambda) = \mathbb{E}_{\mathbb{Q}^\infty}[E_0^\omega[X_1]]$.
- Not clear whether this is enough to obtain an Einstein Relation.
  Conjecture: When $\lambda_c = 0$,

$$\lim_{\lambda\to 0} \frac{v(\lambda)}{\lambda} = \sigma^2.$$

Techniques (case $v(\lambda) > 0$)

Many ideas from *F. Comets, S. Popov (2011)*.
Big difference: $p^{\lambda}(x_k, x_{k+1})$ is not bounded from below!

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STEP 1 (CUT-OFF)

$(X_n^\rho)_{n\in\mathbb{N}}$: Suppress jumps more than $\rho \in \mathbb{N}$ points away.

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$$(0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, ...)$$

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 & \uparrow & & & & \uparrow & & & & & & & \uparrow & & \\
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Couple Bernoullis and random walk s.t.

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The regeneration times are $T_{\ell_1\rho}$, $T_{\ell_2\rho}$, $T_{\ell_3\rho}$...

STEP 3 (INVARIANT MEASURE)

Note: $(\tau_{X_j^\rho}\omega : T_{\ell_k\rho} \leq j < T_{\ell_{k+1}\rho})$ is a stationary and ergodic sequence of random paths for the environment from the POV of the particle.

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Via comparison with n.n. walk ($\rho = 1$):

$$K_1 \cdot \mathcal{C}^1_{\text{eff}}(A \leftrightarrow B) \le \mathcal{C}^\rho_{\text{eff}}(A \leftrightarrow B) \le K_2 \cdot \mathcal{C}^1_{\text{eff}}(A \leftrightarrow B)$$

$\forall \rho \in \mathbb{N} \cup \{+\infty\}$, K_1, K_2 constants independent of $A, B \subset \mathbb{Z}$ and of ρ.

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We have

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where F is in $L^1(\mathbb{P})$.

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Thank you!