On the speed of Random Walks among Random Conductances

Probability seminars

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The model

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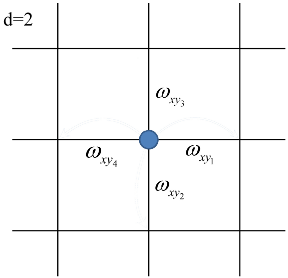
Consider the lattice $\mathbb{Z}^d$ and assign to any bond (x, y) a random weight $\omega_{x,y}$ such that

- $\omega_{x,y} = \omega_{y,x}$ (symmetry),
- $\omega_{x,y} \geq 0$ (positivity).



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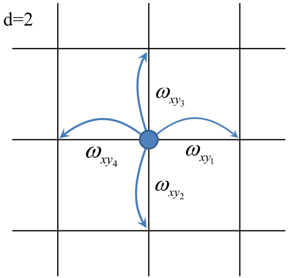

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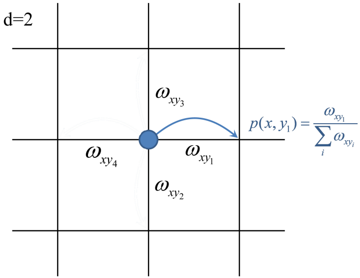

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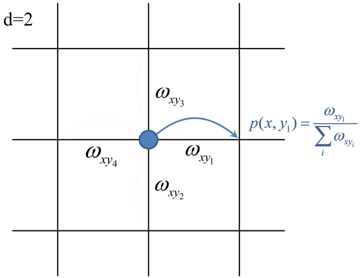

Definition

The discrete time Random Walk among Random Conductances (RWRC) $(X_n)_{n\in\mathbb{N}}$ has transition probabilities

$$P^{\omega}(x, y) = \frac{\omega_{x,y}}{\sum_{z\sim x} \omega_{x,y}}, \qquad \text{for } x \sim y \in \mathbb{Z}^d.$$

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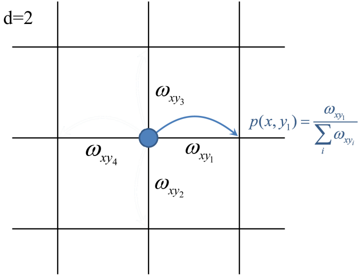

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! It is reversible w.r.t. $\pi(x) := \sum_{z\sim x} \omega_{x,z} \qquad x \in \mathbb{Z}^d$!

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Define the speed of the RWRC as

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What is its annealed behaviour(for $d = 2$)?

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- i.i.d. conductances

We know $\mathbb{P}\Big(\lim_{n\to\infty} \frac{X_n}{n} = 0\Big) = 1$ (*point of view of the particle*).

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  $\Longrightarrow$ Moment condition.

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Proposition

If $\exists \alpha > 1$ *s.t.* $\mathbb{E}[\log^{\alpha} \omega_{x,y}] < \infty$ *then* $\mathbb{P}\left(\lim_{n \to \infty} \frac{X_n}{n} = 0 \right) = 1$.

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Lemma (*Varopulos-Carne heat kernel estimates*)

L irreducible Markov transition kernel with reversible measure π, $d(x,y) = \min\{n : L^n(x,y) > 0\}$. Then for every x, y and n,

$$L^n(x,y) \leq 2\sqrt{\frac{\pi(y)}{\pi(x)}} \cdot e^{-\frac{d(x,y)^2}{2n}}.$$

If $\mathbb{E}[\log^\alpha \omega_{x,y}] < \infty$ for some $\alpha > 1 \Longrightarrow$ speed=0.

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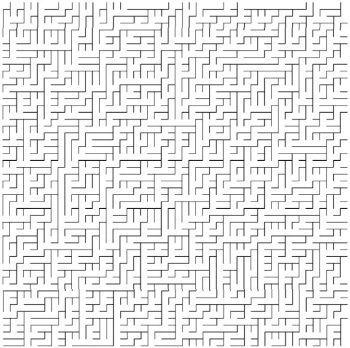

Note that

$$\mathbb{E}[\log^{\alpha} \omega_{x,y}] \simeq \sum_{k=1}^{\infty} Pr(h(y) > k)\, k^{A\alpha-1}, \quad 1 < A < \frac{1}{\alpha}$$

where $h(y) :=$ {distance from the farthest leaf in the branch of y}.

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Can we push α as close as we want to 1?
Is there a directed tree with $\mathbb{P}(h(y) \geq n) \leq \frac{C}{n}$?

The BZZ tree

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$\forall x \in \mathbb{Z}^2$ define random variables $L(x) > 0$ s.t.

$$P(L(x) > t) = \frac{\theta}{t^2} \qquad \text{for } t > t_0.$$

An umbrella of intesity $t > 0$ is

$$U_t = \bigcup_{i=1,2} \{y = (y_1, y_2) \in \mathbb{Z}^2 : y_i = 0, y_j \in (0, t], j \neq i\}$$

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- $(x, x + e_1) \in \mathit{Tree}$, if the strongest umbrella through x is horizontal;
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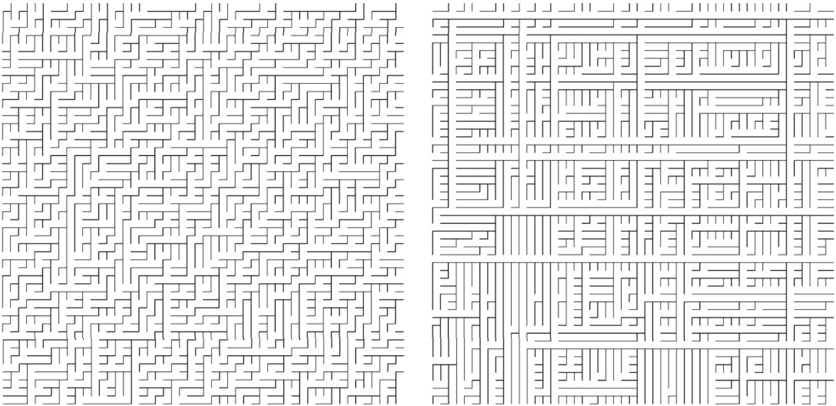
The conductances on the tree are:

$$\omega_{x,x+e_i} = \mathrm{e}^{(h(x)+1)^A} \qquad A > 1,\ i \in \{1, 2\}.$$



(picture stolen from *Shortest spanning trees and a counterexample for Random Walks in Random Environments*, by M. Bramson, O. Zeitouni and M. Zerner (2006))

Theorem (Bramson, Zerner and Zeitouni '06)

The BZZ Tree is such that

$$\limsup_{n\to\infty} nP(h(0) \geq n) < \infty \qquad \forall x \in \mathbb{Z}^2.$$

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Corollary

The RWRC with the environment on the BZZ Tree previously described is such that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{X_n}{n} = 0\right) = 0 \quad a.s..$$

But: the speed does not exist!

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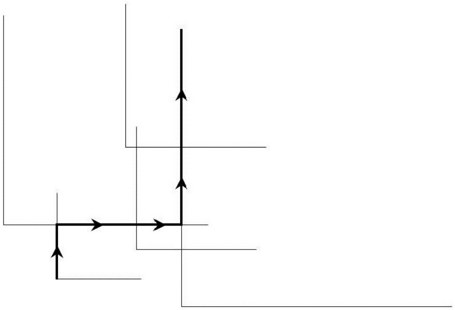

$$\mathbb{P}\Big(\text{The biggest umbrella met up to time } T \text{ is of order } O(T)\Big) > c > 0.$$

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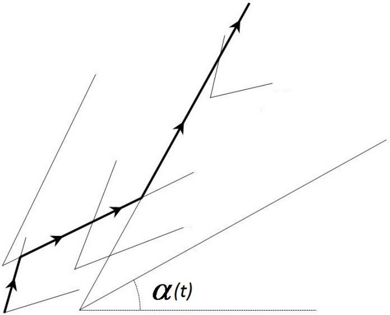

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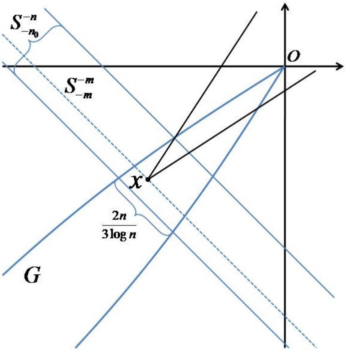

Theorem

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