THE HEAT EQUATION SHRINKS ISING DROPLETS TO POINTS

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Abstract. Let $D$ be a bounded, smooth enough domain of $\mathbb{R}^2$. For $L > 0$ consider the continuous time, zero-temperature heat bath stochastic dynamics for the nearest-neighbor Ising model on $(\mathbb{Z}/L)^2$ (the square lattice with lattice spacing $1/L$) with initial condition such that $\sigma_x = -1$ if $x \in D$ and $\sigma_x = +1$ otherwise. We prove the following classical conjecture [25, 5] due to H. Spohn: In the diffusive limit where time is rescaled by $L^2$ and $L \to \infty$, the boundary of the droplet of ”−” spins follows a deterministic anisotropic curve-shortening flow, such that the normal velocity is given by the local curvature times an explicit function of the local slope. Locally, in a suitable reference frame, the evolution of the droplet boundary follows the one-dimensional heat equation.

To our knowledge, this is the first proof of mean curvature-type droplet shrinking for a lattice model with genuine microscopic dynamics.

An important ingredient is our recent work [20], where the case of convex $D$ was solved. The other crucial point in the proof is obtaining precise regularity estimates on the deterministic curve shortening flow. This builds on geometric and analytic ideas of Grayson [16], Gage-Hamilton [15], Gage-Li [13, 14], Chou-Zhu [6] and others.

1. Introduction

A basic problem in non-equilibrium statistical mechanics is the following [25]: Take a microscopic statistical mechanics model at sufficiently low temperature so that there are, say, two pure thermodynamics phases. Assume that such system evolves according to a microscopic dynamics defined via local evolution rules. Then, the goal is to derive macroscopic, deterministic equations which describe, on large space-time scales, the evolution of spatial boundaries separating the two coexisting thermodynamic phases. An example to keep in mind is the nearest-neighbor Ising model on $\mathbb{Z}^d$, $d \geq 2$. Below the critical temperature, in the absence of an external magnetic field, there are two translation invariant equilibrium Gibbs measures (the “+” and the “−” thermodynamic phases). One can then easily define a Markov dynamics, the so-called Glauber dynamics, where individual spins are flipped with rates chosen so that the Gibbs measures are invariant and reversible. If at time zero a region of the space is occupied by the “+” phase and the rest by the “−” phase, we are interested in how the shape of these region will evolve with time.

If the dynamics does not conserve the order parameter (e.g. the total magnetization for the Ising model), it is well understood phenomenologically [22] that a droplet of one phase immersed in the opposite phase will shrink in order to decrease its surface tension until it disappears in finite time; also (roughly speaking) the normal velocity at a point of its boundary will be proportional to the local mean curvature. Based on this idea, one expects (“Lifschitz law”) that, if the initial droplet has diameter $L$, it will “evaporate” within a time of order $L^2$ (as would be the case for a sphere evolving via mean curvature motion). Moreover, the droplet evolution should become deterministic and follow some version of a mean curvature flow in the “diffusive limit” where $L \to \infty$, space is rescaled by $L$ (the initial droplet is then of size $O(1)$) and time is accelerated by $L^2$. The resulting large-scale deterministic evolution equation will in general be anisotropic.
(i.e. the normal velocity will depend also on the local orientation of the droplet boundary) when the microscopic model is defined on a lattice, as it is for the Ising model.

The main difficulty in implementing this program is that there is no obvious way how to separate the “fast modes” related to relaxation inside the bulk of the pure phases from the “slow modes”, related to the interface motion. Such problem is absent in so-called “effective interface models” of Ginzburg-Landau \( \nabla \phi \) type: for these models, under an assumption of strict convexity of the interaction, Funaki and Spohn \[12\] derived the full mean-curvature motion in the diffusive scaling. Another case \[8, 9, 10\] where mean-curvature motion is known to appear in the scaling limit are spin models with Kac-type interactions (the interaction range tends to infinity with the droplet size): in this case, however, the system is close to mean-field and the limiting deterministic flow is isotropic.

Results are much more incomplete for genuine lattice models: for instance, for the two-dimensional Ising model at low but non-zero temperature \( T \), it is only known that a droplet of “−” phase immersed in the “+” phase will disappear in a time of order at most \( L^{2/(T \log L)} \) \[21\], to be compared with the expected \( L^2 \) scaling.

In the present work, we study the two-dimensional nearest-neighbor Ising model on the square lattice, at zero temperature. Each spin variable \( \sigma_x = \pm 1 \) is updated on average once per time unit: after the update, the spin takes the same sign as the majority of its 4 neighbors, or the value \( \pm 1 \) with equal probabilities in case of a tie. Assume that the initial “−” droplet, when the lattice spacing tends to zero, converges to a smooth enough domain \( D \) of \( \mathbb{R}^2 \). Then, in the diffusive scaling limit the droplet boundary is expected to be given by a deterministic evolving curve \( \gamma(t) \). Such curve should move according to the following “(anisotropic) curve-shortening flow”: the normal velocity equals the local (signed) curvature, times a function \( a(\theta) \) with \( \theta \) the local tangent. The function \( a(\cdot) \) is explicitly given, cf. \((2.5)\).

This result was conjectured in \[25\] by Spohn, who gave some very reasonable supporting arguments, based on the local analysis of the droplet boundary evolution in terms of interacting particle systems. In \[5\], Cerf and Louhichi computed the “drift at time 0” of the droplet (for the non-modified dynamics), but their result does not allow to get any information on the evolution for positive time \( t > 0 \). The full convergence to the curve shortening flow for initial convex droplets \( D \) was recently obtained in \[20\]. The main result of the present work, Theorem \ref{thm:main}, is a proof of Spohn’s conjecture, for smooth enough initial droplets \( D \), without any convexity assumption. Smoothness of the initial condition is required essentially so that the limit flow is unambiguously defined.

As it was the case also in \[20\] for the convex initial condition, a preliminary but essential step before proving convergence of the stochastic evolution to the limit deterministic one is to show that the anisotropic curve-shortening flow does admit a global (in time) solution, and that it does not develop singularities before it shrinks to a point. For the isotropic case \( (a(\cdot) \equiv 1) \) this was proven by Grayson in a celebrated work \[16\]. In the anisotropic case, Grayson’s result has been extended (e.g. \[23, 7\]) under the assumption that \( a(\cdot) \) is at least \( C^2 \). The reason is very simple: the first step in the procedure is to write down the evolution equation satisfied by the curvature of \( \gamma(t) \), and in such equation a second derivative of \( a(\cdot) \) appears. In our case, the anisotropy function \( a(\cdot) \) is not even \( C^1 \) (which reflects singularities of the zero-temperature surface tension of the Ising model). To prove existence, uniqueness and regularity of the (classical) solutions of the curve-shortening flow (cf. Theorem \ref{thm:existence}), we first regularize the function \( a(\cdot) \) and then analyze the regularized equation following the ideas of \[16, 6, 7, 23\]. Of course, it is crucial to check that all the estimates we need are uniform in the regularization parameter, which is sent to zero in the end. Let us emphasize that the regularity estimates of \[7, 6, 23\] are far from being quantitative in terms of the smoothness of the anisotropy function \( a(\cdot) \).
Comparing our present result with that of [20], it is important to realize that dropping the convexity assumption is not at all a technical point. First of all, various monotonicity arguments that were used in [20] do not work here. The basic reason is that such ideas crucially relied on the fact that, in the convex case, the normal velocity is always directed inward (which is clearly false for non-convex droplets, at points where the curvature is negative). Secondly, proving existence and regularity of solution requires very different analytic and geometric arguments in the non-convex case with respect to the convex one (there, we were able to use ideas from [15, 13, 14]). At any rate, our previous result [20] is important in Section 6, where the evolution of the droplet boundary is controlled by locally comparing it with that of a suitable convex droplet.

Let us mention some recent related works by one of the authors. In [19] the issue of the evolution of a convex planar “minus droplet” in the presence of a positive magnetic field has been investigated. In this case the right time scaling is $L$ instead of $L^2$ for a droplet of size $O(L)$ and the scaling limit is given by the anisotropic eikonal equation: the drift of the interface loses its dependence on the curvature. Generalizing such a result in higher dimensions is a very challenging problem, see e.g. [24] for a non-rigorous attempt in this direction. In [18], the dynamical evolution of a half-droplet on a substrate that attracts the interface (a situation that also corresponds to “dynamical polymer pinning”) was studied, and the scaling limit was shown to be the solution of Stefan-type equation where the motion of the point of contact between the droplet and the substrate depends on the local curvature.

We close this introduction by mentioning a couple of intriguing open problems. First of all, one would like to know what are the finite-$L$ fluctuations of the droplet boundary around its limit shape $\gamma(t)$, along the evolution. Secondly, it is natural to wonder what happens for the zero-temperature dynamics of the three- (or higher-) dimensional Ising model. Recently, a weak version of the Lifshitz law was proven for the three-dimensional Ising model at zero temperature: the evaporation time of a “−” droplet is of order $L^2$, up to multiplicative logarithmic corrections [4]. An analogous upper bound was proven in higher dimensions [17]. However, it is still not clear (even at a heuristic level) what should be the precise macroscopic equation, analogous to (2.4), describing the droplet evolution in the diffusive limit.

2. Model and results

Given $L \in \mathbb{N}$ we consider the zero-temperature stochastic Ising model on $(\mathbb{Z}/L)^2$ (the square lattice with lattice spacing $1/L$). The state space is the set $\Omega = \{-1,+1\}^{(\mathbb{Z}/L)^2}$ of spin configurations $\sigma = (\sigma_x)_{x \in (\mathbb{Z}/L)^2}$ with $\sigma_x = \pm 1$. The dynamics is a Markov process $(\sigma(t))_{t \geq 0}$, with $\sigma(t) = (\sigma_x(t))_{x \in (\mathbb{Z}/L)^2} \in \Omega$. Each spin $\sigma_x$ is updated with unit rate: when the update occurs, $\sigma_x$ takes the value of the majority of its four neighbors, or takes values $\pm 1$ with equal probabilities if exactly two neighbors are $+1$ and two neighbors are $-1$.

We consider a compact, simply connected subset $D \subset [-1,1]^2$ whose boundary $\partial D$ is a Jordan curve of finite length. The initial condition of the stochastic dynamics will be set to be “−” inside $D$ and “+” outside:

$$\sigma_x(0) = \begin{cases} -1 & \text{if } x \in (\mathbb{Z}/L)^2 \cap D, \\ +1 & \text{otherwise.} \end{cases}$$

We want to compute the scaling limit of the set of “−” spins at positive times, when $L \to \infty$. In order to identify a set of “−” spins as a subset of $\mathbb{R}^2$, let for $x \in (\mathbb{Z}/L)^2$

$$C_x := x + [-1/(2L), 1/(2L)]^2$$

(2.2)
be the square of side $1/L$ centered at $x$ and define

$$M_L(t) := \bigcup_{\{y: \sigma_y(t) = -1\}} C_y,$$

which is the “− droplet” at time $t$ for the dynamics.

Our goal is to prove that, as $L \to \infty$, $M_L(L^2t)$ converges to the compact set $D_t$ whose boundary $\gamma(t) = \partial D_t$ is the solution of the anisotropic curve shortening flow

$$\partial_t \gamma = a(\theta) k \mathbf{N}$$

with initial condition $\gamma(0) := \partial D$. This equation has to be read as follows. The normal velocity at a point $p \in \gamma(t)$ is given by the curvature $k$ at point $p$ times $a(\theta(p))$, with

$$a(\theta) = \frac{1}{2(|\cos(\theta)| + |\sin(\theta)|)^2}, \quad 0 \leq \theta \leq 2\pi$$

and $\theta(p)$ the tangent angle to $\gamma(t)$ at $p$. The normal vector $\mathbf{N}$ at point $p$ points inward and the curvature is positive (resp. negative) at points of local convexity (resp. concavity) of $\gamma(t)$.

Since $a(\theta)$ is not differentiable for $\theta$ multiple of $\pi/2$, the existence of a solution for (2.4) does not follow from the standard literature, that assumes $a(\cdot)$ to be at least $C^2$ (see [23]). Our first result is an existence, uniqueness and regularity theorem for the solution of (2.4). Define

$$T = T(D) = \frac{\text{Area}(D)}{\int_0^{2\pi} a(\theta) \, d\theta} = \frac{\text{Area}(D)}{2}. \quad (2.6)$$

**Theorem 2.1.** Consider a domain $D \subset [-1,1]^2$ whose boundary is a Jordan curve $\gamma(0)$ of finite length, with curvature everywhere defined and $C^\infty$ as a function of the arc-length coordinate. Suppose moreover that $\gamma(0)$ has a finite number of inflection points.

There exists a unique solution $(\gamma(t))_{t \leq T(D)}$ of (2.4) that is a Jordan curve for $t < T$ and:

1. The area enclosed by $\gamma(t)$ is $\text{Area}(D) - 2t = 2(T-t)$ and $\gamma(t)$ shrinks to a point $X$ when $t \to T$.
2. For every $s < T$, the curvature function is equicontinuous on $[0,s]$ in the following sense: for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon,s,\gamma(0)) > 0$ such that if $t,t' \leq s$ and $p \in \gamma(t), p' \in \gamma(t')$ with $|t-t'| \leq \delta, |p-p'| \leq \delta$ then $|k(p,t) - k(p',t')| \leq \epsilon$.

Our main result gives convergence of the stochastic droplet $M_L(L^2t)$ to the deterministic flow $D_t$, that is the compact domain enclosed by $\gamma(t)$ (for $t > T$, we set by convention $D_t := \{X\}$).

We introduce some notations. For $\eta > 0$ let $\mathcal{B}(x,\eta)$ denote the ball of radius $\eta$ centered at $x \in \mathbb{R}^2$ and for any compact set $\mathcal{C} \subset \mathbb{R}^2$, we define

$$\mathcal{C}^{(\eta)} := \bigcup_{x \in \mathcal{C}} \mathcal{B}(x,\eta), \quad \mathcal{C}^{-\eta} := \left( \bigcup_{x \notin \mathcal{C}} \mathcal{B}(x,\eta) \right)^c.$$  

Finally, we will say that an event holds with high probability (w.h.p.) if its probability tends to 1 as $L$ tends to infinity.

**Theorem 2.2.** Consider $D$ such that $\gamma(0) = \partial D$ satisfies the assumptions of Theorem 2.1. Let us consider the zero temperature stochastic Ising model with initial condition (2.1). Then for any $\eta > 0$ the following holds w.h.p.:

1. for all $t \geq 0$,  
$$\mathcal{D}_t^{-\eta} \subset M_L(L^2t) \subset \mathcal{D}_t^{(\eta)}; \quad (2.8)$$
2. for all $t \geq T + \eta$, $M_L(L^2t)$ is empty.
We emphasize that the regularity estimates stated in Theorem 2.1 are not given just for the sake of completeness, but on the contrary are crucial in the proof of Theorem 2.2.

2.1. Generalizations and open problems. Let us mention a few immediate generalizations of our result, and an interesting open problem.

i. More general initial condition. Instead of (2.1), let us assume only that the (possibly random) initial droplet $M_L(0)$ converges w.h.p. in Hausdorff distance to $D$ as $L \to \infty$. Then, (2.8) still holds. Just note that, for any given $\epsilon > 0$, w.h.p. $D(\epsilon - \epsilon) \subset M_L(0) \subset D(\epsilon)$; then the claim follows from Theorem 2.2 plus monotonicity of the dynamics, cf. Section 6.1.

ii. Non-connected initial droplet. If $D$ is not connected but each of its connected components $D_i, i = 1, \ldots, k$ verifies the assumptions of Theorem 3.1, it is easy to see that Theorem 2.2 applies to each of the $k$ components of the stochastic droplet $M_L(t)$. Essentially, the various components evolve independently.

iii. Non-simply connected initial droplet. Suppose that $D$ is compact, connected but non-simply connected (say, an annulus). If each connected component $\gamma_i(0), i = 1, \ldots, k$ of $\gamma(0) = \partial D$ verifies the assumptions of Theorem 3.1, define $D_t$ to be the domain with boundary $\partial D_t = \bigcup_i \gamma_i(t)$. Then, it is not hard to see that Theorem 2.2 still holds. Again, roughly speaking the $k$ macroscopic components of the boundary of $M_L(L^2 t)$ evolve essentially independently and approach the deterministic evolution of the $k$ components of $\partial D_t$.

iv. $\partial D$ is not a simple curve. If $\partial D$ has self-intersections the situation is definitely more subtle. To fix ideas, consider the case of Figure 1. Note that $D$ can be seen either as the $\epsilon \to 0$ limit of a domain $F^\epsilon$ with Jordan boundary and an $\epsilon$-narrow pinch or as the limit of two $\epsilon$-close simple domains $G^\epsilon, H^\epsilon$. In this case, we expect that the evolution of $M_L(L^2 t)$ remains random in the $L \to \infty$ limit. More precisely, we expect that the Ising droplet will follow with some probability $p$ the deterministic evolution $\lim_{\epsilon \to 0} (F^\epsilon)_t$, and with probability $1 - p$ the deterministic evolution $\lim_{\epsilon \to 0} [(G^\epsilon)_t \cup (H^\epsilon)_t]$. Here, $(F^\epsilon)_t$ is the domain enclosed by the solution of (2.4) with initial condition $\partial F^\epsilon$, and similarly for $(G^\epsilon)_t, (H^\epsilon)_t$. More importantly, we expect that the law of the limit evolution (i.e. the probability $p$) depends crucially on the way the initial droplet $M_L(0)$ microscopically approximates $D$. This will be considered in future work.

![Figure 1](image_url)
3. Proof of Theorem 2.1

3.1. A few general facts on (anisotropic) curve shortening flows. Let us pretend for a moment that equation (2.4) is replaced by

$$
\partial_t \gamma(t) = A(\theta)k N
$$

(3.1)

where \(A(\cdot)\) is a smooth (at least \(C^2\)) function, that is strictly positive and \(\pi\)-periodic. Then, it is known from \([2]\) that (3.1) does admit a solution \(\gamma(t)\) that is a Jordan curve. Furthermore from \([23]\) the solution exists until a maximal time where it shrinks to a point. The function \(a(\cdot)\) in (2.5) is instead only Lipschitz, which is why we cannot apply known results to our case.

Call \(s\) the arc-length coordinate on \(\gamma(t)\). It is not important to establish what point of \(\gamma(t)\) is assigned the coordinate \(s = 0\), since we will always evaluate only derivatives or differences with respect to \(s\). In the following equations ((3.2) to (3.6)), \(\partial_s\) denotes derivation w.r.t. arc-length at fixed time, while \(\partial_t\) means derivation along the flux lines described by points \(p(t) \in \gamma(t)\) who move with velocity at time \(t\) given by \(k(p(t))A(\theta(p(t)))\) times the normal vector \(N\) to \(\gamma(t)\) at point \(p(t)\). It is well known (see \([16\text{, Lemma 1.4]}\) in the isotropic case) that the derivatives with respect to \(t\) and \(s\) do not commute, as motion affects the arc-length: in fact, we have

$$
\partial_t \partial_s = \partial_s \partial_t + Ak^2.
$$

(3.2)

We collect here a few useful formulas. First of all, define for \(p \in \gamma(t)\)

$$
g(p) = A(\theta(p))k(p)
$$

(3.3)

with \(\theta(p)\) the tangent angle at the point \(p\). Then, one has (see for instance \([7\text{, Ch. 1]}\); for the isotropic case \(A \equiv 1\), see \([15]\))

$$
\partial_t k = \partial_s^2 g + Ak^3
$$

(3.4)

$$
\partial_t g = \partial_s(A \partial_s g) + A^2 k^3
$$

(3.5)

$$
\partial_t \theta = \partial_s(Ak)
$$

(3.6)

$$
\partial_t \mathcal{L} = - \int_{\gamma} Ak^2 ds
$$

(3.7)

$$
\partial_t A = - \int_0^{2\pi} A(\theta) d\theta
$$

(3.8)
with $L(\gamma), A(\gamma)$ the length of $\gamma$ and the area enclosed by it. Remark also that $\partial_s \theta = k$ (this is a simple geometric fact that has nothing to do with the curve shortening flow).

3.2. The existence theorem. Let us call $(a^\omega(\cdot))_{\omega \in (0,1)}$ a family of $C^\infty$ regularizations of $a(\cdot)$ that are uniformly Lipschitz (this is possible because $a(\cdot)$ itself is 1-Lipschitz), have the same $\pi/2$-periodicity as $a(\cdot)$ and converge uniformly to $a(\cdot)$ when $\omega \to 0$. From the previous section, we know that (2.4) has a solution if $a$ is replaced by $a^\omega$. Let $\gamma^\omega(t)$ be this solution and $k^\omega = (k^\omega(p,t))_{t \geq 0, p \in \gamma^\omega(t)}$ denote its curvature function. The time when the curve shrinks to a point is given by (cf. (3.8))

$$T^\omega = \frac{A(\gamma(0))}{\int_0^{2\pi} a^\omega(\theta) d\theta}$$

and the maximal curvature is bounded for times smaller than $T^\omega$. Moreover, up to time $T^\omega$ the curve is $C^\infty$ [2].

Set for every $K \geq 0$

$$T_K := \sup \{ t \mid \lim_{\omega \to 0} \sup_{t' \leq t} \| k^\omega(\cdot, t') \|_\infty < K/2 \}, \quad T^* := \lim_{K \to \infty} T_K.$$  

Note that $T_K > 0$ for all $K \geq K_0(\gamma(0))$. Indeed, if $g^\omega = a^\omega k^\omega$ (cf. (3.3)), one has from (3.5), using $\partial_s \theta = k$ and dropping the argument $\omega$

$$\partial_s g = a^2 k^3 + \partial_s (a \partial_s g) = \frac{1}{a} g^3 + a \partial_s^2 g + k \partial_\theta a \partial_s g.$$  

From (3.11), recalling that $\partial_\theta a$ is uniformly bounded and calling $|g|_{\max}(t)$ the maximal value of $|g|$ along $\gamma(t)$, we obtain

$$\frac{d}{dt} |g|_{\max}(t) \leq \frac{1}{a_{\min}} (|g|_{\max}(t))^3$$  

and since $a_{\min}$ is bounded away from zero uniformly in $\omega$, we find that $|g|$ (and therefore the curvature) cannot explode instantaneously.

**Theorem 3.1.** Let the initial condition $\gamma(0)$ be a Jordan curve of finite length, whose curvature is everywhere defined and $C^\infty$ as a function of the arc-length coordinate. Suppose moreover that $\gamma(0)$ has a finite number of inflection points.

Fix $K > 0$. There exists a unique solution $(\gamma(t))_{t \leq T_K}$ of (2.4) that is a Jordan curve for $t \leq T_K$. The area enclosed by $\gamma(t)$ is $2(T(D) - t)$. The curvature function is equicontinuous in the following sense: for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, K, \gamma(0)) > 0$ such that if $t, t' \leq T_K$ and $p \in \gamma(t), p' \in \gamma(t')$ with $|t - t'| \leq \delta, |p - p'| \leq \delta$ then $|k(p, t) - k(p', t')| \leq \epsilon$.

Further regularity properties of the limit flow and precise the connection with the one-dimensional heat equation are given in Lemma 3.4 below.

Note that, while $T^\omega$ in (3.9) converges to $T$ (cf. (2.6)) for $\omega \to 0$, it is not guaranteed that $T^* = T$: in principle, the curvature of the regularized solution could take larger and larger values (when $\omega \to 0$) at some time $T^* < T$. The following result rules out this pathological behavior:

**Theorem 3.2.** One has $T^* = T(D)$ and the curve $\gamma(t)$ shrinks to a point when $t \to T(D)$.

Clearly, Theorems 3.1 and 3.2 imply Theorem 2.1.
3.3. Regularity and maximum principles. In this section we give some additional properties of the anisotropic curve shortening flow (2.4), in addition to those stated in Theorem 3.1. These will be crucial for the proof of Proposition 5.2.

Let \( \gamma(t) \) satisfy equation (3.1) and consider a portion of \( \gamma(t) \) that is the graph \( y(x,t) \) of a function in the coordinate frame \((e_{\theta_0}, e_{\theta_0+\pi/2})\) obtained by rotating by \( \theta_0 \) anti-clockwise the usual Cartesian frame. The curvature is given by

\[
k(x,t) = \frac{\partial^2_{x}y(x,t)}{(1 + (\partial_{x}y(x,t))^2)^{3/2}}.
\] (3.13)

The evolution of \( y \), of the angle \( \Theta = \arctan(\partial_x y) \) and of the “curvature” \( g = Ak \) are given by the parabolic equations

\[
\partial_t y := \partial^2_{x}y \cos^2(\Theta)A(\Theta + \theta_0)
\] (3.14)

\[
\partial_t \Theta = \cos^2(\Theta) \partial_x(A(\Theta + \theta_0)\partial_x \Theta)
\] (3.15)

\[
\partial_t g = \cos^2(\Theta)\partial_x(A(\Theta + \theta_0)\partial_x g) + \frac{g^3}{A(\Theta + \theta_0)},
\] (3.16)

\[
\partial_t k = \cos^2(\Theta)\partial^2_{x}(A(\Theta + \theta_0)k) + A(\Theta + \theta_0)k^3.
\] (3.17)

Equation (3.14) is obtained just by projecting Equation (3.1) in the chosen frame of coordinates. For (3.15) and (3.16), one just needs to compute the derivatives with some patience (see also [6, Equation (1.2)] with a different notation). Equation (3.16) can be derived from (3.5) by noticing that

\[
\partial_t|_{\text{along flow lines}} = \partial_t|_{\text{for fixed } x} - \frac{\partial_x y Ak}{\sqrt{1 + (\partial_x y)^2}} \partial_x + \partial_x = \frac{1}{\sqrt{1 + (\partial_x y)^2}} \partial_x.
\]

Hence

\[
\partial_t g(x,t) = \frac{1}{\sqrt{1 + (\partial_x y)^2}} \partial_x \left( \frac{A}{\sqrt{1 + (\partial_x y)^2}} \partial_x g \right) + A^2 k^3 + \frac{A \partial_x y \partial^2_x y}{(1 + (\partial_x y)^2)^2} \partial_x g,
\] (3.18)

and a line of computation leads to (3.16).

**Lemma 3.3** (Properties of the regularized evolution). Let \( \gamma(t) \) be a solution of (3.1) with \( A(\cdot) \) smooth.

(i) For a given choice of Cartesian coordinates \((e_{\theta_0}, e_{\theta_0+\pi/2})\), local minima for \( y \) and \( \theta \) strictly increase with time and local maxima strictly decrease with time.

(ii) The total curvature of an arc connecting two isolated inflection points is decreasing with time, and the \( \theta \)-intervals of tangent directions of such arc (angle span) strictly nest with time.

(iii) Local minima of \( |g| \) or \( |k| \) that are not inflection points are strictly increasing with time. In particular, isolated points with curvature zero that are not inflection points disappear instantaneously. Such points can arise only when two or more inflection points merge.

(iv) If \( t > 0 \), \( \gamma(t) \) does not contain flat pieces and the number of inflection points of the curve is non-increasing with time.

**Proof.** Point (i) is a consequence of the maximum principle [11, Ch. 7] applied to equation (3.14) and (3.15). These equations are non-linear, but since we know a priori that \( \gamma(t) \) is \( C^\infty \), one can treat e.g. the factor \( \cos^2(\Theta)A(\Theta + \theta_0) \) in (3.14) as a smooth positive coefficient. Point (ii) is a consequence of (i), since inflection points are local maxima or minima of \( \theta \) (see also Lemma 1.6 in [6]). Point (iii) is obtained from (3.16) and (3.17). Say for instance that one has a local minimum of \( g \) that is non-negative. Then, omit the term \((1/A)g^3\) that is locally positive
and apply the maximum principle to the remaining equation. Point (iv) is proven in Lemma 1.4 of [6] and is based on a theorem by Angenent [1] that says that the set of zeros of a parabolic equation is finite at all positive times and its cardinality is non-increasing with time.

When the regularization parameter \( \omega \) tends to zero, Theorem 3.1 says that (3.14) holds, with \( A(\cdot) \) replaced by \( a(\cdot) \), but the analog of Equations (3.15) to (3.17) is not guaranteed to hold. However, the following result gives extra information with respect to Theorem 3.1:

**Lemma 3.4 (Properties of the limit evolution).** Let \( \gamma(t) \) solve (2.4).

(i) In the Cartesian frame \( (e_{\theta_0}, e_{\theta_0+\pi/2}) \) with \( \theta_0 \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\} \), along the portions of \( \gamma(t) \) that are locally the graph of a 1-Lipschitz function \( y(x,t) \), the evolution of \( y \) is given by the one-dimensional heat equation

\[
\partial_t y = \frac{1}{4} \partial_x^2 y
\]

(which explains the title of the article).

(ii) For any positive time, \( \gamma(t) \) consists of a finite number of analytic arcs separated by points where the tangent angle belongs to \( \{0, \pi/2, \pi, 3\pi/2\} \).

(iii) The total curvature of an arc connecting two isolated inflection points is decreasing with time, and the \( \theta \)-intervals of tangent directions of such arc strictly nest with time. The number of inflection points is decreasing with time (they can merge).

(iv) Flat portions of \( \gamma(t) \) with zero curvature disappear instantaneously and can in principle be present only at a finite number of times \( t_i \).

(v) Inflection points evolve continuously in time (except possibly at the times \( t_i \) of point (iv)).

(vi) Isolated points where \( k = 0 \), that are not inflection points, can be present only at a finite set of times and disappear instantaneously.

**Proof.** We emphasize that this proof assumes the existence and regularity results of Theorem 2.1.

For point (i), just look at (3.14) and (2.5) and observe that, for \( \theta_0 \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\} \) and \( |u| \leq 1 \), one has

\[
\frac{1}{1+u^2} a(\arctan(u) + \theta_0) = \frac{1}{4}.
\]

Fix \( t_0 \), consider any point \( p \in \gamma(t_0) \) where the tangent angle is not a multiple of \( \pi/2 \) and rotate the Cartesian coordinate frame by a suitable multiple of \( \pi/4 \) so that locally the curve is the graph of a 1-Lipschitz function. Call \( p_1 \) the horizontal projection of \( p \) in this frame. There exists an open interval \( I \supseteq p_1 \) and \( t_1 > t_0 \) such that the evolution of the curve is described by the solution \( y(x,t) \) of the heat equation (3.19), for \( x \in I \) and \( t \in [t_0, t_1) \), with time-dependent boundary conditions at the endpoints of \( I \). Then [3, Theorem 10.5.1] tells us that this solution is analytic in \( x \) in the interior of the domain \( I \), for \( t_0 < t < t_1 \). This guarantees that the curve is composed of analytic arcs whose evolution in a certain coordinate frame is given by the heat equation (3.19). They are separated either by points or by flat segments where the tangent angle is a multiple of \( \pi/2 \). Point (ii) is proven.

For point (iii) it is sufficient to prove that local minima of \( \theta \) are increasing, and local maxima are decreasing (this also implies that the number of inflection points cannot increase). We remark that in the neighborhood of an extremum of \( \theta \) (i.e. of an inflection point) the curve can always be represented as the graph of a 1-Lipschitz function \( y(x) \) in the Cartesian frame \( (e_{\theta_0}, e_{\theta_0+\pi/2}) \) with \( \theta_0 \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\} \) (this is the case even when the tangent angle is a multiple of \( \pi/2 \)). The inflection point corresponds to a local extremum of \( \partial_x y \). As \( \partial_x y \) is the solution of the heat equation, the result follows.
If $\gamma(0)$ contains flat segments of slope not multiple of $\pi/2$, then by points (i)-(ii) they disappear instantaneously and cannot be created at later times. Flat segments with slope multiple of $\pi/2$ require a different argument. If such a segment (say, with slope 0) is present at some time $t_0$, then

- either it is locally a maximum/minimum for the curve in the usual Cartesian frame (see Fig. 3 (a)) and in that case it disappears immediately since the equation (3.14) is strictly parabolic;
- or it is not (see Fig. 3 (b)), in which case its evolution in the Cartesian frame rotated by $\pm\pi/4$ is locally described by (3.19), so for $t$ just after $t_0$ the curve is analytic and cannot contain flat segments.

The only way to create a flat segment (say with slope zero) is to have an analytic arc, connecting two isolated points of zero slope (“poles”) that degenerates at some time $t_i$ to a horizontal segment, see Fig. 3 (c). At time $t_i$, the number of poles in $\gamma(t)$ therefore decreases by 2. However, poles cannot be created (again because (3.14) is strictly parabolic), so this scenario can happen only a finite number of times ($\gamma(0)$ has a finite number of poles because we assumed it has a finite number of inflection points). This proves point (iv).

\[ (a) \quad (b) \]

\[ (c) \]

\[ t < t_i \quad t = t_i \]

**Figure 3.** Examples of flat segments on the curve corresponding to a local extremum in the local coordinate frame (a) and not corresponding to a local extremum (b). The third picture (c) describes the only possible way to create flat segments on $\gamma(t)$ at positive time

For point (v), continuity of the curvature in space and time obtained in Theorem 3.1 makes it impossible for the inflection points to jump, except possibly at times $t_i$ (at such times, the position of the inflection point on the flat segment is not well defined anyway).

For point (vi), let us call for simplicity “zero-curvature points” the points in question. It is immediate (from the heat equation) that zero-curvature points with slope not multiple of $\pi/2$ that are present in $\gamma(0)$ disappear immediately and cannot be created at later times. A zero-curvature point with (say) slope zero is a pole. If such a point existed for a time interval $[t_a, t_b)$, the pole would not move in $[t_a, t_b)$ (its velocity would be zero according to the curve-shortening flow), which is not possible since the equation (3.14) is strictly parabolic. At later times, zero-curvature points of zero slope can appear only when an odd number of poles merge (see Fig. 4): this can happen only a finite number of times. Isolated poles cannot become zero-curvature
points, as can be easily obtained from the fact that, for the regularized evolution, local minima of \( k \) where \( k > 0 \) are non-decreasing, cf. point \((iii)\) of Lemma 3.3.

\[ \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{three_poles_merge_to_single_pole.png}
\caption{Three poles (black dots) merge to create a single pole (right drawing). At the time of the merging, the curvature at the new pole is zero: this is because the curvature is zero at the two inflection points (white dots) and curvature is continuous w.r.t. space and time (Proposition 4.8).}
\end{figure}

\[ \]

4. Proof of Theorem 3.1

The scheme of the proof is the following. First we prove that \( \gamma^\omega(t) \) has a limit for every fixed \( t \), when \( \omega \) tends to zero (this is done in Section 4.2). Then in Sections 4.3 and 4.4 we prove that \( \gamma(t) \) satisfies the regularity properties stated in Theorem 3.1, and that it does solve equation (2.4).

4.1. Some preliminary remarks. Given a smooth simple curve \( \gamma \) we write \( \gamma + xN \) for the curve obtained by shifting each point of \( \gamma \) by an amount \( x \in \mathbb{R} \) in the (inward) normal direction at \( x \). Set

\[ m(\gamma) := \sup\{\eta : \forall| x | \leq \eta, \gamma + xN \text{ is a simple curve} \}. \tag{4.1} \]

\[ \]

Remark 4.1. If \( | x | < m(\gamma) \), then the function \( \gamma \mapsto \hat{\gamma} = \gamma + xN \) defined by \( \gamma \ni p \mapsto p + xN \in \hat{\gamma} \) is a bijection that preserves the normal direction \( N \). As a consequence, if \( D \) is the interior of \( \gamma \), then the interior of \( \hat{\gamma} \) is \( D(-x) \). Identifying the points that are in bijection, we have that the curvatures of \( \gamma \) and \( \hat{\gamma} \) are related by

\[ \hat{k} = \frac{k}{1 - kx}. \tag{4.2} \]

Lemma 4.2. For the evolution \( \gamma^\omega(t) \) set

\[ u^\omega(t) := \min\{|x - x'| > 0 : x \neq x' \in \gamma^\omega(t) \text{ such that } |x - x'| \text{ is a local minimum}\} \tag{4.3} \]

and \( r^\omega(t) \) to be the minimal value of the curvature radius \( |k^\omega(p)|^{-1} \) for \( p \in \gamma^\omega(t) \). Then

\[ m(\gamma^\omega(t)) = \min(u^\omega(t)/2, r^\omega(t)). \tag{4.4} \]

Furthermore \( t \mapsto u^\omega(t) \) is an increasing function.

Remark 4.3. What is meant precisely in the definition of \( u^\omega(t) \) is that \( x, x' \) are such that for all sufficiently small \( \delta > 0 \), the minimum \( |y - y'| \) with \( y \in \gamma^\omega(t) \cap B(x, \delta), y' \in \gamma^\omega(t) \cap B(x', \delta) \) is positive and realized for \( y = x, y' = x' \). If the set in (4.3) is empty as in the case where \( \gamma \) is convex then we set \( m(\gamma) \) to be infinite.
Proof of Lemma 4.2. We drop all superscripts $\omega$ in this proof. Let us consider the evolution of $\gamma(t) + xN$ for fixed $t$ when $x \geq 0$ increases. There are two ways for the curve to create self-intersections: either the maximal curvature explodes when $x \to x_0^-$ and a loop is created when $x > x_0$, or two distant arcs of the curves kiss when $x = x_0$. The first can occur when $x = r(t)$ if $r(t)$ is attained at a point of positive curvature and the second for $x = u(t)/2$ if the chord linking the two points where $u(t)$ is attained is enclosed in $\gamma$. A similar argument holds for the evolution of $\gamma(t) + xN$ for fixed $t$ when $x \leq 0$ decreases; hence our result.

To see that $u(t)$ is increasing, fix $t$ and consider two points $p_1, p_2 \in \gamma(t)$ realizing the minimum $u(t)$. Assume to fix ideas that the segment connecting them is vertical, enclosed in $\gamma$ and that $p_1$ is above $p_2$. If $k_{p_1}, k_{p_2}$ are the curvatures at $p_1, p_2$, then the definition of $u(t)$ (the distance being locally minimized by $p_1, p_2$) implies that

$$k_{p_i}u(t) \leq 1, i = 1, 2. \quad (4.5)$$

Let $\Gamma(t, s) := \gamma(t + s) + Nu(t)/2$ (cf. Remark 4.1, see Fig/ 5 for a graphical construction) and consider the points $p_1(t, s), p_2(t, s)$ on $\Gamma(t, s)$ with same horizontal coordinate as $p_1, p_2$, that tend to $p_1, p_2$ when $s \to 0$. For short times $s$, locally around $p_1(t, s)$ and $p_2(t, s)$, the curve $\Gamma(t, s)$ is the graph of functions $y_1(x, t, s)$ and $y_2(x, t, s)$. For $s = 0$ one has

$$y_1(\cdot, t, 0) \geq y_2(\cdot, t, 0)$$

by definition of $u(t)$. To prove that $u(t)$ is increasing, we just need to show that this inequality remains valid (locally around $p_i(t, s)$) for small positive $s$.

![Figure 5. Graphical representation of the curve $\Gamma(t, s)$ around the points $p_1$ and $p_2$.](image)

From Remark 4.1 one has for fixed $t$

$$\partial_s \Gamma = a(\theta) \frac{\hat{k}}{1 + k u(t)/2} \mathbf{N} = a(\theta) \left( \hat{k} - \frac{u(t)\hat{k}^2}{2 + u(t)\hat{k}} \right) \mathbf{N}$$
with $\hat{k}$ the curvature of $\Gamma$. The reason is that, if a point of $\Gamma(t, s)$ has slope $\theta$ and curvature $\hat{k}$, the point in $\gamma(t+s)$ that is in bijection with it has the same slope $\theta$ and curvature $\hat{k}/(1 + \hat{k}u(t)/2)$, see (4.2) with $x = u(t)/2$. See also Lemma 4.5 and in particular (4.9) (where one has to take $\varepsilon'(t) = 0$) for the more detailed proof of a similar result.

Projecting this equation vertically (like what we did to get (3.14)) we obtain differential equations for $y_1$ and $y_2$ and (keep in mind that the normal vector $N$ is pointing downward around $p_1(t, s)$ and upward around $p_2(t, s)$) we deduce that $z = y_1 - y_2$ satisfies the parabolic equation

$$\partial_s z(x, t) = b_1 \partial^2_x z(x, t) + b_2 \partial_x z(x, t) + b_3$$

where $b_1 = f(\partial_x y_1) = (1 + (\partial_x y_1)^2)^{-1}a(\arctan(\partial_x y_1)) > 0$ and $b_2 = \partial^2_x y_2(f(\partial_x y_1) - f(\partial_x y_2))/((\partial_x y_1 - \partial_x y_2)$ are two smooth functions and

$$b_3 = u(t) \left[ (1 + (\partial_x y_1)^2)^{1/2} a(\arctan(\partial_x y_1)) \hat{k}_1^2 \right] + (1 + (\partial_x y_2)^2)^{1/2} a(\arctan(\partial_x y_2)) \hat{k}_2^2 \right].$$

Here, $\hat{k}_i, i = 1, 2$ denotes the curvature along the portions of the curve $\Gamma$ whose graph is $y_i$. If we can prove that $b_3 \geq 0$, then from the (weak) maximum principle it follows that locally $z \geq 0$ for positive time and thus the result is proved. To see that $b_3$ is positive, note that (again thanks to (4.2)),

$$\frac{1}{2 + k_1 u(t)} = \frac{1}{2} \left( 1 - \frac{u(t)k_1}{2} \right)$$

with $k_1$ the curvature on $\gamma(t+s)$ of the point in bijection with the point of $\Gamma(t, s)$ with curvature $\hat{k}_1$. For $s = 0$ and looking at point $p_1$, the r.h.s. of (4.6) is larger than $1/4$ thanks to (4.5), since $k_1$ reduces to $k_{p_1}$. Thanks to continuity of curvature in space and time (the regularized evolution is $C^\infty$), the positivity continues to hold for $s$ small, close to $p_1(t, s)$. A similar argument shows that the second term in $b_3$ is positive. \hfill\Box

4.2. The limit flow. We first show that, for $t < T^*$ (cf. definition (3.10)), $\gamma^\omega(t)$ converges to a limit curve as $\omega \to 0$:

**Proposition 4.4.** Take any sequence $\omega_n \to 0$. For any fixed $K$, $\gamma^\omega_n(t)$ is a Cauchy sequence for the Hausdorff distance $d^\mathcal{H} \setminus$ uniformly for $t \leq T_K$.

**Proof of Proposition 4.4.** Let us consider $\omega$ and $\omega'$ close enough to zero so that the curvature of $\gamma^\omega(t), \gamma^{\omega'}(t)$ is bounded by $K$ uniformly on $[0, T_K]$.

**Lemma 4.5.** Let $\mathcal{D}^{\omega}(t), \mathcal{D}^{\omega'}(t)$ be the compact sets enclosed by the curves $\gamma^\omega(t), \gamma^{\omega'}(t)$. Consider $n < (1/2)\min(1/K, m(\gamma(0)))$ and set

$$\varepsilon(t) := n \exp(10K^2(t - T_K)),$$

For all $t \leq T_K$, for all $\omega, \omega'$ sufficiently small, the curves $\gamma^\omega(t) \pm \varepsilon(t)N$ are simple curves and are the boundaries of $(\mathcal{D}^\omega(t))_{(-\varepsilon(t))}$ (cf. Remark 4.1). Furthermore

$$\mathcal{D}^\omega(t) \setminus (\varepsilon(t)) \subset \mathcal{D}^{\omega'}(t) \subset (\mathcal{D}^\omega(t))_{(+\varepsilon(t))}.$$  \hfill(4.7)

**Remark 4.6.** We will see that it is sufficient to choose $\omega, \omega'$ sufficiently small so that they satisfy

$$\|a^\omega - a^{\omega'}\|_\infty \leq K \eta \exp(-10K^2T_K)$$

with $a^\omega(\cdot)$ the regularized version of $a(\cdot)$.  \hfill(4.8)
As a consequence of (4.7) and of $\varepsilon(t) \leq \eta$,
\[
\max_{t \leq T_K} d_H(\gamma^\omega(t), \gamma^{\omega'}(t)) \leq \eta
\]
and Proposition 4.4 follows. \hfill \Box

Proof of Lemma 4.5. To prove that $\gamma^\omega(t) \pm \varepsilon(t)N$ are simple curves for all $t \leq T_K$, it is sufficient to remark that by Lemma 4.2
\[
m(\gamma^\omega(t)) \geq \min(u(0)/2, 1/K) \geq \eta \geq \varepsilon(t).
\]
We prove the inclusions (4.7). Set
\[
\gamma^-(t) := \gamma^\omega(t) + \varepsilon(t)N \quad \text{and} \quad \gamma^+(t) := \gamma^\omega(t) - \varepsilon(t)N
\]
and observe that $\gamma^\pm(t)$ is the boundary of $((D^\omega(t))^{(\pm \varepsilon(t))}$, see Remark 4.1. Let $k^\pm$ be the respective curvature functions of $\gamma^\pm$, that are related to $k^{\omega}$ via $1/k^\pm = 1/k^{\omega} \pm \varepsilon(t)$, cf. (4.2).

The normal velocity of a point on $\gamma^-$ is given by
\[
v^- = a^\omega(\theta)k^{\omega}N + \varepsilon'(t)N,
\]
with $\varepsilon'(t)$ the $t$-derivative of $\varepsilon(t)$. Similar considerations concerning $\gamma^+$ implies that the curves $\gamma^\pm$ are solution of the following equations:
\[
\begin{align*}
\partial_t \gamma^- &= v^- = a^\omega(\theta)k^-N + \left(\varepsilon'(t) - \frac{a^\omega(\theta)\varepsilon(t)(k^-)^2}{1 + \varepsilon(t)k^-}\right)N, \\
\partial_t \gamma^+ &= v^+ = a^\omega(\theta)k^+N - \left(\varepsilon'(t) - \frac{a^\omega(\theta)\varepsilon(t)(k^+)^2}{1 - \varepsilon(t)k^+}\right)N.
\end{align*}
\] (4.9)

The curve $\gamma^-(t)$ is clearly enclosed by $\gamma^\omega(t)$ at initial time, since $\gamma^\omega(0) = \gamma(0)$. To show that this inclusion holds for all times, observe that the normal velocity associated to the evolution of $\gamma^-(t)$ can be written
\[
v^- = a^\omega(\theta)k^-N + N\Delta := a^\omega(\theta)k^-N + N\left((a^\omega(\theta) - a^{\omega'}(\theta))k^- + \varepsilon'(t) - \frac{a^\omega(\theta)\varepsilon(t)(k^-)^2}{1 + \varepsilon(t)k^-}\right).
\]
If $\Delta$ were zero, $\gamma^-(t)$ would just evolve with anisotropy function $a^{\omega'}$ (like $\gamma^\omega(t)$), so it would be a standard fact that $\gamma^-(t)$ would be included in $\gamma^\omega(t)$ for all times. This clearly remains true if we can show that $\Delta \geq 0$, since the term $N\Delta$ gives a further inward push to $\gamma^\omega(t)$.

As $|k^\omega| \leq K$ for all $t \leq T_K$ and $\varepsilon(t) \leq 1/(2K)$, from (4.2) we have $-K \leq k^- \leq 2K$. Hence
\[
\Delta \geq -2K|a^\omega - a^{\omega'}|(\theta) + \varepsilon(t) \left[10K^2 - \frac{a^\omega(\theta)4K^2}{1 - \eta K}\right]
\]
\[
\geq 2K^2\eta \exp(-10K^2T_K) - 2K\|a^\omega - a^{\omega'}\|_{\infty} \geq 0, \quad (4.10)
\]
where in the second inequality we used that $\eta K \leq \frac{1}{2}$ and that $a^\omega \leq 1$, and in the last one we used condition (4.8).

With a similar reasoning, the proof that $\gamma^{\omega'}(t)$ is enclosed by $\gamma^+(t)$ can be reduced to check that
\[
a^{\omega'}(\theta)k^+ - \left(\varepsilon'(t) - \frac{a^{\omega'}(\theta)\varepsilon(t)(k^+)^2}{1 - \varepsilon(t)k^+}\right) \leq a^{\omega'}(\theta)k^+ \quad (4.11)
\]
whose proof is analogous to that of $\Delta \geq 0$. \hfill \Box
4.3. Uniform regularity of the (regularized) flow. Here we prove that the curvature function is an equicontinuous function of arc-length, uniformly in \( \omega \), up to time \( T_K \). This is an essential step in establishing the regularity properties stated in Theorem 3.1, point (ii).

Recall the definition of \( g^\omega \) given just before (3.11).

**Proposition 4.7.** Given \( K < \infty \) there exists a constant \( f(K,\gamma(0)) \) such that, for every \( t \leq T_K \) and \( \omega \in (0,1) \),

\[
|g^\omega(s_1,t) - g^\omega(s_2,t)| \leq \sqrt{|s_2 - s_1|} f(K,\gamma(0)). \tag{4.12}
\]

This 1/2-Hölder regularity is not optimal (one can improve it to \( 1^- \)-Hölder regularity) but is sufficient for our purposes.

**Proof of Proposition 4.7.** It is sufficient to show that (omitting for simplicity the argument \( \omega \) everywhere)

\[
\psi(t) := \int_{\gamma(t)} (\partial_s g)^2 \, ds \leq f(K,\gamma(0))^2 \tag{4.13}
\]

(we will see that the dependence of \( f \) on \( \gamma(0) \) is only through the value \( \psi(0) \)). Indeed, (4.13) and Cauchy-Schwarz imply

\[
|g(s_1,t) - g(s_2,t)| = \left| \int_{s_1}^{s_2} \partial_s g \, ds \right| \leq \sqrt{|s_2 - s_1|} f(K,\gamma(0)). \tag{4.14}
\]

To show (4.13), write

\[
\frac{d}{dt} \psi(t) = \frac{d}{dt} \int_{\gamma(t)} (\partial_s g)^2 \, ds = \int_{\gamma(t)} \partial_t (\partial_s g)^2 \, ds - \int_{\gamma(t)} ak^2 (\partial_s g)^2 \, ds, \tag{4.15}
\]

where the second term comes from the fact that the length of \( \gamma \) decreases with time according to (3.7) (see also the proof of Proposition 2.7 in [16] for the formula in the case \( a \equiv 1 \) and \( a \) should be seen as \( a(\theta(s)) \), with \( \theta(s) \) the angle at the point of \( \gamma(t) \) with arc-length coordinate \( s \). Applying (3.5) and (3.2) with \( A(\cdot) \) replaced by \( a(\cdot) = a^\omega(\cdot) \), the right-hand side of (4.15) can be rewritten as

\[
\int_{\gamma(t)} ak^2 (\partial_s g)^2 \, ds + 2 \int_{\gamma(t)} \partial_s g \partial_s (a^2 k^3) + 2 \int_{\gamma(t)} (\partial_s g) \partial_s^2 (a^4) ds. \tag{4.16}
\]

The first term is bounded by \( C(K) \psi(t) \). For the second one, we observe that, as \( \partial_s \theta = k \),

\[
\partial_s (a^2 k^3) = 3ak^2 \partial_s g - ak^3 \partial_s a = 3ak^2 \partial_s g - ak^4 \partial_\theta a.
\]

Altogether, recalling that \( a^\omega \) is Lipschitz uniformly in \( \omega \), the second term is bounded by \( C(K)[\psi(t) + \sqrt{\psi(t)}] \).

The third term in (4.16) looks more problematic. However, by integration by parts it equals

\[
-2 \int_{\gamma(t)} (\partial^2 g) \partial_s (a^2) ds = -2 \int_{\gamma(t)} a (\partial_s g)^2 ds - 2 \int_{\gamma(t)} k (\partial_s g) (\partial_s g)(\partial_\theta a) ds. \tag{4.17}
\]

\[
\int_{\gamma(t)} (\partial_s g)^2 \, ds \leq \int_{\gamma(t)} (\partial_s g)^2 \, ds + \int_{\gamma(t)} ak^2 (\partial_s g)^2 \, ds.
\]
Now we use that $a$ is bounded away from zero and that $\partial_y a$ is bounded away from infinity (uniformly in $\omega$): \eqref{eq:4.17} is then upper bounded by

\[
-2a_{\min} \int_{\gamma(t)} (\partial^2 \rho) ds + C(K) \sqrt{\int_{\gamma(t)} (\partial^2 \rho)^2 ds + \int_{\gamma(t)} (\partial \rho)^2 ds} = -2a_{\min} \sqrt{\int_{\gamma(t)} (\partial^2 \rho)^2 ds + C(K) \int_{\gamma(t)} (\partial \rho)^2 ds \psi(t)} = -2a_{\min} \left[ \sqrt{\int (\partial^2 \rho)^2 ds - C'(K) \sqrt{\psi(t)}} \right]^2 + C''(K) \psi(t) \leq C''(K) \psi(t). \tag{4.18}
\]

Altogether we have obtained

\[
\frac{d}{dt} \psi(t) \leq c(K)(\psi(t) + \sqrt{\psi(t)}) \tag{4.19}
\]

and, since $\psi(0) < \infty$ ($\gamma(0)$ was assumed to have a $C^\infty$ curvature) we are done. \hfill \Box

### 4.4. Proof of Theorem 3.1: conclusion

Uniqueness is trivial, so we concentrate on existence. Proposition 4.4 implies that we can define $\gamma(t) = \lim_{n \to \infty} \gamma^{\omega_n}(t)$ and actually that the limit does not depend on the chosen sequence $\omega_n \to 0$. We have to prove that $\gamma(t)$ does solve equation \eqref{eq:2.4} and that it has the desired regularity properties.

The first step is:

**Proposition 4.8.** Fix $K > 0$.

(i) Given a sequence of points $p^{\omega} \in \gamma^{\omega}(t)$, for $t \leq T_K$, that converges to a point $p \in \gamma(t)$, the curvature and the tangent angle of $\gamma^{\omega}(t)$ at $p^{\omega}$ converge to the curvature and tangent angle of $\gamma(t)$ at $p$.

(ii) The curvature function $k$ and the angle function $\theta$ of the family $(\gamma(t))_{t \leq T_K}$ are equicontinuous in the sense that for $\epsilon > 0$ there exists $\delta(\epsilon, K, \gamma(0)) > 0$ such that if $p \in \gamma(t), p' \in \gamma(t')$ with $t, t' \leq T_K$, $|t - t'| \leq \delta$, $|p - p'| \leq \delta$ then $|k(p, t) - k(p', t')| \leq \epsilon$ and $|\theta(p, t) - \theta(p', t')| \leq \epsilon$.

**Proof of Proposition 4.8.**

(i) The angle $\theta^{\omega}$ at $p^{\omega}$ converges when $\omega \to 0$ to a limit $\theta$, as a simple consequence of the convergence in Hausdorff distance of $\gamma^{\omega}(t)$ to $\gamma(t)$, plus the fact that the curvature of $\gamma^{\omega}(t)$ is bounded by $K$. Assume to fix ideas (and without loss of generality) that $\theta \in [-\pi/3, \pi/3]$ (if this is not the case, the Cartesian coordinate frame below has to be rotated by a multiple of $\pi/2$). Then, using Proposition 4.4:

(1) there exists $c = c(K)$ such that for $\omega$ small enough $\gamma^{\omega}(t)$ is locally the graph of a function $x \mapsto y^{\omega}(x, t)$ in the usual Cartesian coordinate frame, for $x$ in an interval $I = [p_1 - c, p_1 + c]$, where $p_1$ is the horizontal coordinate of $p$;

(2) the same holds for the limit curve $\gamma(t)$ and the function $y^{\omega}(\cdot, t)$ converges to $y(\cdot, t)$ uniformly on $I$; more than that,

\[
\max_{x \in I} |y^{\omega}(x, t) - y(\cdot, t)| \leq \epsilon(\omega, K, \gamma(0)) \tag{4.20}
\]

where the right-hand side does not depend on $t$ as long as $t \leq T_K$, where $\epsilon$ tends to zero when $\omega \to 0$;

(3) the $x$-derivative of $y^{\omega}(\cdot, t)$ is bounded by 2 on $I$, for $\omega$ small enough, where the value 2 is chosen simply because $|\tan(\pi/3)| < 2$. 


From (3.13), the curvature of $\gamma^\omega(t)$ at the point with horizontal coordinate $x \in I$ is
\[
k^\omega(x, t) = \frac{\partial_x^2 y^\omega(x, t)}{(1 + |\partial_x y^\omega(x, t)|^2)^{3/2}}
\] (4.21)
from which we infer that the second derivative of $y^\omega(\cdot, t)$ is uniformly bounded. Then, the Ascoli-Arzelà Theorem implies that $\partial_x y^\omega(x, t)$ converges to $\partial_x y(x, t)$ uniformly on $I$, and in particular that $\theta$ is the tangent angle of $\gamma(t)$ at $p$. Similarly, since $k^\omega$ is uniformly continuous w.r.t. arc-length (Proposition 4.7), $k^\omega$ converges uniformly and the limit (that is a continuous function) is the curvature function of $\gamma(t)$. Let us emphasize that
\[
\max_{x \in I} |k^\omega(x, t) - k(x, t)| \leq \epsilon(\omega) = \epsilon(\omega, K, \gamma(0))
\] (4.22)
i.e. the estimates are uniform in $t \leq T_K$, otherwise one would easily find a contradiction with (4.20).

(ii) Take $t \leq T_K$ and $p \in \gamma(t)$. As in point (i), assume without loss of generality that $\theta \in [-\pi/3, \pi/3]$, so that the curve is locally the graph of a function $y(x, t)$ with $x$ in some interval $I$ of width depending only on $K$.

As long as $\omega > 0$ we know that the evolution is smooth, in particular for $t'$ very close to $t$ the curve $\gamma^\omega(t')$ is still the graph of a function in $I$ and (see below for a bit more of detail)
\[
|\partial_t \theta^\omega(x, t)| \leq c(\omega), |\partial_t k^\omega(x, t)| \leq c(\omega)
\] (4.24)
for some $c(\omega) = c(\omega, K, \gamma(0))$ that may diverge as $\omega \to 0$. Then, for $t'$ very close to $t$
\[
|k^\omega(x, t) - k^\omega(x, t')| \leq c(\omega)|t - t'|
\] (4.25)
\[
|\theta^\omega(x, t) - \theta^\omega(x, t')| \leq c(\omega)|t - t'|.
\] (4.26)
Assuming for a moment (4.24) we have from (4.22)
\[
|k(x, t) - k(x, t')| \leq \inf_{\omega > 0} \left(2\epsilon(\omega) + c(\omega)|t - t'|\right).
\] (4.27)
The right-hand side clearly tends to zero with $|t - t'|$ (choose a sequence $\omega_n$ tending to zero. If $c(\omega_n)$ does not diverge we are done. Otherwise, compute the right-hand side for $\omega = \omega_n$ with the largest value of $n$ such that $c(\omega_n) \leq |t - t'|^{-1/2}$). A similar argument gives that
\[
|\theta(x, t) - \theta(x, t')| = o(1) \quad \text{as} \quad |t - t'| \to 0
\] (4.28)
uniformly for $t, t' \leq T_K$. To conclude the proof of point (ii), take $p, p'$ close to each other, with $p \in \gamma(t'), p' \in \gamma(t)$. Just write
\[
|k(p, t) - k(p', t')| \leq |k(p, t) - k(p'', t')| + |k(p'', t') - k(p', t')|
\]
with $p''$ the point on $\gamma(t')$ close to $p$ with the same horizontal coordinate as $p$. We have just proven that the first term in the right-hand side is $o(1)$ as $|t - t'|$ goes to 0; as for the second one, it vanishes when $|p - p'| \to 0$, since we have shown in the proof of point (i) (using Proposition 4.7) that the curvature function of $\gamma(t')$ is uniformly continuous w.r.t. arc-length. Similarly one proves the continuity statement for the angle function $\theta$.

It remains to prove (4.24). From [2, Theorem 3.1], we know that, since the regularized anisotropy function $\omega^\omega(\cdot)$ is $C^\infty$, the curve $\gamma^\omega(t)$ is also $C^\infty$ at all times $t < T^\omega$ (cf. (3.9)) and thus (4.24) follows from (3.5) and (3.6).
Back to the proof of Theorem 3.1, we still have to prove that the limit flow solves (2.4). As in the proof of Proposition 4.8, let us concentrate on a portion of the curve that is locally described by the graph of a function \( y^\omega(x,t), x \in I \). We have from (3.14) (with \( \theta_0 = 0 \)), for \( t_2 > t_1 \),

\[
y^\omega(x,t_2) - y^\omega(x,t_1) = \int_{t_1}^{t_2} k^\omega(x,u) a^\omega(\theta^\omega(x,u))/\cos(\theta^\omega(x,u)) du,
\]
with \( k^\omega \) as in (4.21) and \( \theta^\omega(x,u) = \arctan(\partial_x y^\omega(x,u)) \). From the (uniform) convergence of \( k^\omega, a^\omega, \theta^\omega \) shown above, plus the uniform bound \( |k^\omega| \leq K \) that allows to dominate bounded convergence, we have for the limit curve \( \gamma(t) \)

\[
y(x,t_2) - y(x,t_1) = \int_{t_1}^{t_2} k(x,u) a(\theta(x,u))/\cos(\theta(x,u)) du.
\]
Continuity of \( k \) and \( \theta \) with respect to time (Proposition 4.8, point (ii)) allows to deduce that

\[
\partial_t y(x,t) = \frac{k(x,t) a(\theta(x,t))}{\cos(\theta(x,t))},
\]
hence \( \gamma(t) \) does solve (2.4). Finally, from (3.8) and the uniform convergence of \( a^\omega(\cdot) \) to \( a(\cdot) \), we see that the area of \( \gamma(t) \) is \( A(\gamma(0)) - t \int_0^{2\pi} a(\theta)d\theta = A(\gamma(0)) - 2t \).

5. Proof of Theorem 3.2

The basic ingredients of the proof are the following two results, that say that when the curvature diverges, it necessarily does so on arcs of total curvature at least \( \pi \). This says that the curve cannot develop “corners”.

Proposition 5.1. Let \( T^* \) be as in (3.10). For any given \( K > 0 \) there exists \( t < T^* \) and an arc of \( \gamma(t) \) of total curvature at least \( \pi \) on which the curvature has a constant sign and is larger than \( K \) in absolute value.

Proposition 5.2. There exists a constant \( \tilde{K} \) (which depends on \( \gamma(0) \)) such that, for \( t \leq T^* \), if the curvature at \( p \in \gamma(t) \) is larger than \( \tilde{K} \) in absolute value then \( p \) belongs to an arc on which the curvature has constant sign, and whose total curvature is at least \( \pi \).

Propositions 5.1 and 5.2 are the analog of Theorem 2.1 and Lemma 3.5 of [16]; however, due to anisotropy (\( a(\cdot) \not\equiv 1 \) in our case) and to the need to regularize \( a(\cdot) \) to \( a^\omega(\cdot) \) while obtaining bounds that are uniform in \( \omega \), the proofs require many non-trivial modifications. This is done in detail in Sections 5.1 and 5.2 below.

Given these two propositions, the proof of Theorem 3.2 becomes essentially identical to the proof of the main theorem of [16] (that says that under isotropic curve shortening flow \( a(\cdot) \equiv 1 \) Jordan curves do not develop singularities before shrinking to a point). Let us just remind the general strategy (giving details would amount to repeating the argument of [16]).

Let \( \Theta \) be the supremum of the angles \( \theta \) such that there exist sequences \( t_i \to T^*, \epsilon_i \to 0, \theta_i \to \theta \) and arcs \( C_i \subset \gamma(t_i) \) satisfying:

- the diameter of \( C_i \) is at most \( \epsilon_i \);
- the curvature \( k \) has constant sign on \( C_i \) and is pointwise larger than \( \tilde{K} \) in absolute value, with \( \tilde{K} \) the constant of Proposition 5.2;
- the total curvature \( \int_{C_i} |k| ds \) is \( \theta_i \).

Proposition 5.1 implies directly that \( \Theta \geq \pi \) (observe that a non-self-intersecting arc of curvature pointwise larger than \( K \) has diameter \( O(1/K) \)). One then excludes the following two possibilities:
(Case 1) $\Theta > \pi$ and $\gamma(t)$ does not shrink to a point as $t \to T^*$;

(Case 2) $\Theta = \pi$.

Case 1 can be ruled out following the argument of [16, Theorem 4.1] and Case 2 is dealt with exactly like in [16, Section 5]. More precisely:

- whenever the author of [16] invokes his Theorem 2.1 (resp. Lemma 3.5), one should apply our Proposition 5.1 (resp. Proposition 5.2) instead;
- Lemma 3.2 of [16] (the “δ-whisker Lemma”) holds also in our case (with the same proof), since it is based only on the maximum principle and on the fact that locally $\gamma(t)$ evolves according to a strictly parabolic equation.

The only remaining possibility is that $\gamma(t)$ does shrink to a point when $t \to T^*$. Since the area of $D_t$ is $2(T(D) - t)$ up to $T^*$ (Theorem 3.1), we obtain that $T^* = T(D)$, that is the claim of Theorem 3.2.

As a side remark, for the isotropic curve shortening flow Grayson [16] proves also that $\gamma(t)$ becomes convex at a time strictly smaller than the time when it shrinks to a point (so that $\Theta = 2\pi$), and becomes asymptotically a circle (of radius going to zero).

5.1. Proof of Proposition 5.1. Recall the definition $g^\omega = a^\omega k^\omega$; in this section, for lightness of notation, we drop the argument $\omega$ in $g^\omega, \gamma^\omega(t)$, etc. We let $g_{\max}(t)$ (resp. $g_{\min}(t)$) denote the maximum (resp. minimum) of $g$ along $\gamma(t)$. Let us first prove a weaker result, that is, that the curvature explodes on an arc of total curvature larger than $\pi/2$.

**Proposition 5.3.** Suppose that $|g|$ is uniformly bounded by $K$ at time zero and that

$$g_{\max}(t) \geq A(K) := 3K \exp(8K\mathcal{L}(\gamma(0))/a_{\min})$$

(respectively, $g_{\min}(t) \leq -A(K)$) for some $t > 0$. We recall that $\mathcal{L}(\gamma)$ is the length of $\gamma$. Then there exists $u \leq t$ and a sub-arc of $\gamma(u)$ with total curvature at least $\pi/2$ on which $g \geq K$ (respectively, $g \leq -K$).

We will consider only the case $g_{\max}(t) \geq A(K)$, the proof of the other statement being essentially identical. We are going to rely on the following Lemma, whose proof is given at the end of this section:

**Lemma 5.4.** Let us call $R(t) = \bigcup_i R_i(t)$ the subset of $\gamma(t)$ on which $g \geq K$: it is a union of arcs of positive curvature and we assume that each $R_i(t)$ has an angle span $[a_i(t), b_i(t)]$ (the angle span can be larger than $2\pi$). The two following statements hold:

(i) If at time $t$ one has $\max_i (b_i(t) - a_i(t)) \leq \pi/2$ then

$$\frac{d}{dt} \int_{\gamma(t)} a \log \left( \frac{g}{K} \right) 1_{g \geq K} \, d\theta := \frac{d}{dt} \sum_i \int_{R_i(t)} a(\theta) \log \left( \frac{g}{K} \right) \, d\theta \leq$$

$$- (g_{\max}(t) - K)_+^2 - 2K \frac{d}{dt} \mathcal{L}(\gamma(t)). \quad (5.1)$$

(ii) If at time $t$ one has $\max_i (b_i(t) - a_i(t)) \leq \pi$ then

$$\frac{d}{dt} \int_{\gamma(t)} a \log(g/K) 1_{g \geq K} \, d\theta \leq -2K \frac{d}{dt} \mathcal{L}(\gamma(t)). \quad (5.2)$$

**Proof of Proposition 5.3.** Suppose that $g_{\max}(t) \geq A := A(K)$ and, by contradiction, that for all $u \leq t$ there are no sub-arcs of length at least $\pi/2$ on which $g \geq K$. Integration of (i) of Lemma
on \([0, t]\) gives
\[
0 \leq \int_{\gamma(t)} a \log(g/K)1_{g \geq K}d\theta \leq -\int_0^t (g_{\max}(u) - K)\, du + 2KL(\gamma(0)).
\] (5.3)
Here we have used the fact that
\[
\int_{\gamma(0)} a \log(g/K)1_{g \geq K}d\theta = 0,
\] (5.4)
since at time zero \(g\) is dominated by \(K\) by assumption.
Moreover from the assumption \(g_{\max}(t) \geq A(K)\) and from (3.12) one has
\[
g_{\max}(u) \geq \left[ A(K)^{-2} + \frac{2}{a_{\min}}(t - u) \right]^{-1/2}
\] (5.5)
for any \(u \leq t\). Hence when \((t - u) \leq a_{\min}/(16K^2)\), we have \(g_{\max}(u) \geq 2K\) (i.e. \(g_{\max}(u) - K \geq g_{\max}(u)/2\)) and thus
\[
\int_0^t (g_{\max}(u) - K)^2\, du \geq \frac{1}{4} \int_0^{a_{\min}/(16K^2)} g_{\max}(t - u)^2\, du
\]
\[
\geq \frac{1}{4} \int_0^{a_{\min}/(16K^2)} \left[ A^{-2} + \frac{2}{a_{\min}}u \right]^{-1}\, du
\]
\[
= \frac{a_{\min}}{8} \log\left(1 + \frac{A^2}{8K^2}\right) > \frac{a_{\min}}{4}\log(A/(3K)) = 2KL(\gamma(0))
\] (5.6)
which contradicts (5.3).

**Proposition 5.5.** Suppose that \(|g|\) is uniformly bounded by \(K\) at time zero and that \(g_{\max}(t) \geq B(K) := A(K_1)\) (resp. \(g_{\min}(t) \leq -B(K)\)), where \(K_1 = K\exp(2L(\gamma(0))K/a_{\min})\) and \(A(\cdot)\) is as in Proposition 5.3. Then there exists \(u \leq t\) and a sub-arc of \(\gamma(u)\) with curvature at least \(\pi\) on which \(g \geq K\) (resp. \(g \leq -K\)).

**Proof.** Assume to fix ideas that \(g_{\max}(t) \geq A(K_1)\). From Proposition 5.3, there exists a time \(t_1 \leq t\) such that
\[
\int_{\gamma(t_1)} a \log(g/K)1_{g \geq K}d\theta \geq \frac{\pi}{2}a_{\min}\log(K_1/K) \geq \pi L(\gamma(0))K.\] (5.7)
If one assumes that until time \(t_1\) there is no arc with curvature at least \(\pi\) on which \(g \geq K\), from Lemma 5.4 (ii), we have (using also (5.4))
\[
\int_{\gamma(t_1)} a(\theta)\log(g/K)1_{g \geq K}d\theta \leq 2KL(\gamma(0)),\] (5.8)
which contradicts (5.7).

**Proof of Proposition 5.1 (Conclusion).** When \(t\) approaches \(T^*\), the maximum of \(|k|\) and therefore the maximum of \(|g|\) diverges (by definition of \(T^*\)). Therefore, for any \(K\) there is \(t < T^*\) such that either \(g_{\max}(t) > B(K)\) or \(g_{\min}(t) \leq -B(K)\) and Proposition 5.5 allows to conclude.

**Proof of Lemma 5.4.** One has
\[
\frac{d}{dt} \int_{R_i(t)} a(\theta)\log(g/K)d\theta = \int_{R_i(t)} a(\theta)\partial_t(\log(g/K))d\theta,
\] (5.9)
as on the (moving) boundary of $R_i(t)$ one has $\log(g/K) = 0$. We know from \[\text{[14, Lemma 2.1]}\]
that, on arcs where curvature has a constant sign,
\[a\partial_t \log g = (g_{t\theta} + g)g.\] (5.10)
Then the left-hand side of (5.1) is equal to
\[
\sum_i \int_{R_i(t)} (g_{t\theta} + g)g d\theta = \sum_i \int_{R_i(t)} (-g_{\theta}^2 + g^2) d\theta + K \sum_i [g_{\theta}(b_i) - g_{\theta}(a_i)].
\] (5.11)
The last term is negative as $g_{\theta}(b_i) \leq 0$, $g_{\theta}(a_i) \geq 0$ by assumption. We have to control the integral. For this we use Wirtinger’s inequality (which can be obtained by a Fourier decomposition of $f$ on a base of eigenfunctions of the Laplacian) for $(g - K)$ for each arc:

**Lemma 5.6** (Wirtinger’s inequality). Let $f$ be a $C^1$ function on an interval $[a, b]$. If $f(a) = f(b) = 0$ with $a \leq b$, then
\[
\int_a^b f^2(x) dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b \left( \frac{df}{dx} \right)^2 dx.
\]
Under assumption $(ii)$ we thus have for all $i$
\[
\int_{R_i(t)} (-g_{\theta}^2 + (g - K)^2) d\theta \leq 0.
\] (5.12)
Therefore,
\[
\int_{R_i(t)} (-g_{\theta}^2 + g^2) d\theta \leq \int_{R_i(t)} 2Kg d\theta \leq 2K \int_{\gamma(t)} 2k^2 ds = -2K \frac{d}{dt} \mathcal{L}(\gamma(t))
\] (5.13)
using $g = ak$ and $\partial_t \theta = k$ in the second inequality and (3.7) in the last step. Under assumption $(i)$ we do the same thing, except for the arc where the maximal curvature is attained if it is larger than $K$. On this arc (call it $R_i(t)$) one has, using the fact that its total curvature is smaller than $\pi/2$ and Wirtinger’s inequality,
\[
\int_{R_i(t)} (-g_{\theta}^2 + (g - K)^2) d\theta \leq -\frac{3}{4} \int_{R_i(t)} g_{\theta}^2 d\theta \leq -\frac{6}{\pi}(g_{\max} - K)^2:
\] (5.14)
the last step just uses the fact that if $|R_i(t)|$, the angle span of $R_i(t)$, satisfies $|R_i(t)| \leq \pi/2$, then
\[
\int_{R_i(t)} g_{\theta}^2 d\theta \geq |R_i(t)|^{-1} \left( \int_{R_i(t)} |g_{\theta}| d\theta \right)^2 \geq \frac{2}{\pi} [2(g_{\max} - K)]^2
\]
(apply Cauchy-Schwartz for the first inequality).

**5.2. Proof of Proposition 5.2.** To fix ideas we will assume that the curvature at the point $p$ mentioned in Proposition 5.2 is positive. Decompose $\gamma(t)$ into minimal arcs, i.e. arcs where the curvature has constant sign, and which are delimited by inflection points that evolve continuously (see again Lemma 3.4). We want to find $K$ such that the curvature on minimal arcs of total curvature smaller than $\pi$ is necessarily bounded by $K$. By choosing $K$ large, we can restrict ourselves to an arbitrarily small time neighborhood of $T^*$.

Recall from Lemma 3.4 that the number of inflection points of $\gamma(t)$ is decreasing with time and that points of zero curvature that are not inflection points can be present only for a finite set of times. Hence we can, without loss of generality, consider a time interval $(T^* - \varepsilon, T^*)$ where the number of inflection points is constant while points of the latter type are absent. Note that on the interval $(T^* - \varepsilon, T^*)$, by point $(iii)$ of Lemma 3.4, the total curvature of minimal arcs
is strictly decreasing. As the number of such arcs is finite, we can suppose (one might need to take \( \varepsilon \) smaller still) that after time \( T^* - \varepsilon \), on all such arcs, the total curvature is smaller than \( \pi - 2\varepsilon \). We emphasize that \( \varepsilon > 0 \) depends only (but in a very implicitly way) on the initial condition \( \gamma(0) \). In the rest of the proof one shows that the curvature remains bounded on each one of these arcs. We look at one of them, that we call \( C(t) \).

Let us call \( C^1(t) \), \( C^2(t) \) the two minimal arcs of negative curvature that are connected to the endpoints of \( C(t) \). It may happen that \( C^1(t) = C^2(t) \), when \( \gamma(t) \) has only two inflection points.

By definition of \( T^* \), there exists a \( K \) such that \( T^* \geq T_K \). Hence we can assume that at time \( T^* - \varepsilon \) the curvature of \( \gamma \) and of all the curves \( \gamma^\omega \) for \( \omega \) sufficiently small are bounded above by \( K \).

For \( K' > K \) fixed, for \( t \in (T^* - \varepsilon, T_{K'}) \), when \( \omega \) is small enough, we want to approximate \( C(t) \) by an arc \( C^\omega(t) \) of \( \gamma^\omega(t) \). What we do first is finding minimal arcs \( C^1_\omega(t) \) and \( C^2_\omega(t) \) of \( \gamma^\omega(t) \), of negative curvature, approximating \( C^1(t) \) and \( C^2(t) \). At the midpoint \( M_1(t) \) of \( C^1(t) \), the curvature is negative and bounded away from zero in \( (T^* - \varepsilon, T_{K'}) \). Hence, from Proposition 4.8, if \( \omega \) is sufficiently small, all the points of \( \gamma^\omega(t) \) that lie in the vicinity of \( M_1(t) \) have negative curvature, hence they belong to a common minimal arc \( C^1_\omega(t) \) which has negative curvature. A similar construction gives \( C^2_\omega(t) \).

There are two arcs that connect \( C^1_\omega(t) \) and \( C^2_\omega(t) \): we call \( C_\omega(t) \) the one whose distance from \( C(t) \) as \( \omega \to 0 \) tends to zero. Note that this is not necessarily a minimal arc. We claim however that \( C_\omega(t) \) converges in Hausdorff distance to \( C(t) \) as \( \omega \to 0 \), uniformly for \( t \in (T^* - \varepsilon, T_{K'}) \).

Indeed by Proposition 4.8 its endpoints \( p^1_\omega(t), p^2_\omega(t) \), on which curvature is zero, must converge to points of zero curvature and the only possible options are the inflection points \( p^1(t), p^2(t) \) that delimit \( C(t) \).

![Figure 6. Scheme of approximation of \( C(t) \) by an arc of \( \gamma^\omega(t) \).](image)

**Lemma 5.7.** Fix \( K' > K \).

(i) For \( \omega \) small enough, the integral of \( |k^\omega| \) along \( C_\omega(t) \) is bounded by \( \pi - \varepsilon \) for \( t \in (T^* - \varepsilon, T_{K'}) \).

(ii) For all \( t \leq T_{K'} \) the curvature on \( C_\omega(t) \) is uniformly smaller than \( K(3/\varepsilon)^m \) where \( 2m \) is the number of changes of sign of the curvature along \( \gamma(0) \).
Proof. For point (i), remark that we have
\[ \int_{C_\omega(t)} |k^\omega| \mathrm{d}s = \int_{C_\omega(t)} k^\omega \mathrm{d}s + 2 \int_{C_\omega(t)} k^\omega \mathrm{d}s, \]
where \( k^\omega = \max(0, -k^\omega) \). We first show that the second integral in the r.h.s., i.e. the total curvature \( C_\omega(t) \) restricted to sub-arcs where the curvature is negative, is smaller than \( \epsilon/4 \).

As \( C_\omega(t) \) converges to \( C(t) \) and since curvature also converges, the total length of arcs with negative curvature in \( C_\omega(t) \) is vanishing. The curvature is bounded (by \( K' \)) and this is sufficient to conclude.

The first integral is the angle between the tangents at the endpoints of \( C_\omega(t) \). Because the curvature is bounded (by \( K' \)) the convergence of \( C_\omega(t) \) to \( C(t) \) implies that the tangent directions at the extremities also converge. Hence \( \int_{C_\omega(t)} k^\omega \mathrm{d}s \) converges to the total curvature of \( C(t) \) which is smaller than \( \pi - 2\epsilon \). Therefore, the first integral is smaller than \( \pi - (3/2)\epsilon \) for \( \omega \) small enough.

For point (ii), we use the ideas from [16, Lemma 3.7]. We observe that \( C_\omega(t) \) is in general composed of adjacent minimal arcs \( c_i(t), 1 \leq i \leq 2n+1 \), for some \( n = n(t) \geq 0 \), with curvatures of alternating sign and \( c_1(t) \) having positive curvature. The number \( n(t) \) is decreasing in time, as arcs disappear when two inflection points merge, and is in any case upper bounded by \( m \), the number of positive curvature arcs on \( \gamma(0) \).

**Lemma 5.8.** Let us consider an arc \( c(t) \subset \gamma^\omega(t) \) where the curvature is non-negative and whose endpoints are inflection points. Assume that, at some time \( s \),

(a) \( g \) is bounded by \( K_1 \) on \( c(s) \);
(b) the total curvature of \( c(s) \) is bounded by \( \pi - \epsilon \).

Then until the first time \( t_1 \) when an inflection point at one of the extremities of \( c(t) \) disappears, \( g \leq K_1/\sin(\epsilon/2) \) on \( c(t) \).

We apply Lemma 5.8 for each arc \( c_{2i+1}(t) \) of positive curvature, starting at time \( s = T^* - \epsilon \), and iterate it when two such arcs merge. Since the number of such mergings cannot exceed \( m \), we obtain that \( g^\omega \) is bounded above by \( K/\sin(\epsilon/2)^m \leq K(3/\epsilon)^m \). Recall that \( \epsilon \) is positive and depends only on the initial condition \( \gamma(0) \).

We conclude the proof of Proposition 5.2 by noting that, when \( \omega \) tends to zero, the curvature on \( C(t) \) is well approximated by the one of \( C_\omega(t) \) (Proposition 4.8), and that \( K' \) in Lemma 5.7 is arbitrary. Therefore, the curvature on \( C(t) \) is pointwise bounded by say \( \bar{K} := 2K(3/\epsilon)^m \) for \( t \in (T^* - \epsilon, T^*) \) and we are done; we recall that \( \epsilon > 0 \) is a positive constant that depends (very implicitly) on \( \gamma(0) \).

**Proof of Lemma 5.8.** Recall that an arc of positive curvature can be parametrized by the tangent direction \( \theta \) and (cf. (5.10)) that \( g^\omega(\theta, t) := a^\omega(\theta) k^\omega(\theta, t) \) satisfies (cf. [14, Lemma 2.1])

\[ \partial_\theta g^\omega = \frac{1}{a^\omega} (g^\omega)^2 (g^\omega_{\theta\theta} + g^\omega). \]  

(5.15)

With no loss of generality we can consider that the initial angle span \( I \) of \( c \) (at time \( s \)) is included in \( [\epsilon/2, \pi - \epsilon/2] \) and from Lemma 3.3 (ii) we know that this remains true up to \( t_1 \). Then we remark that initially for all \( \theta \) in \( I \)

\[ g^\omega(\theta, 0) \leq \frac{K_1}{\sin(\epsilon/2)} \sin(\theta), \]  

(5.16)
and that the r.h.s. is a stationary solution of (5.15). This, via the maximum principle, implies that $q^\omega(\cdot, t) \leq \frac{K_1}{\sin(\epsilon/2)} \sin(\theta)$ for all $t \leq t_1$. \hfill \Box

6. Scaling limit of the droplet evolution: proof of Theorem 2.2

6.1. Monotonicity. We will use at various places (often implicitly) the well-known monotonicity or attractiveness of the dynamics, that we can formulate as follows. Consider the dynamics in a domain $V \subset (\mathbb{Z}/L)^2$, with boundary conditions $\tau$ on $\partial V = \{ x \in (\mathbb{Z}/L)^2 \setminus V : d(x, V) = 1/L \}$. In general, $\tau$ can be time-dependent, $\tau = (\tau_x(t))_{x \in \partial V, t \geq 0}$. Let $\sigma^{\bar{\gamma}, \tau}(t)$ denote the configuration at time $t$, when the initial condition is $\eta$ and the boundary conditions are $\tau$. Also introduce in the space of spin configurations the partial order $\preceq$ where $\sigma \preceq \sigma'$ if $\sigma_x \leq \sigma'_x$ for every $x$. Then, it is possible to define a global coupling $\mathbb{P}$ such that, if $\eta \preceq \eta'$ and $\tau \preceq \tau'$ one has

$$\sigma^{\bar{\gamma}, \tau}(t) \preceq \sigma^{\bar{\gamma}', \tau'}(t) \quad \text{for every } t \geq 0$$

with $\mathbb{P}$-probability 1. Here, $\tau \preceq \tau'$ means $\tau(t) \preceq \tau'(t)$ for every $t \geq 0$.

6.2. Proof of Theorem 2.2. The way to prove the result is slightly indirect. We first show that if the initial droplet includes a strict neighborhood of $D$ then $D_t$ is w.h.p. a lower bound for the droplet $M_L(L^2 t)$ (we also prove an analogous upper bound).

**Proposition 6.1.** Let $D$ be a compact set such that $\gamma(0) = \partial D$ satisfies the assumptions of Theorem 2.1.

1. If for some $\eta > 0$, the initial condition of the stochastic dynamics satisfies $M_L(0) \supset D^{(\eta)}$ then, w.h.p.,

$$M_L(L^2 t) \supset D_t \quad \text{for all } 0 \leq t \leq T(D) - \eta; \quad (6.1)$$

2. If instead $M_L(0) \subset D^{(-\eta)}$ then, w.h.p.,

$$M_L(L^2 t) \subset D_t \quad \text{for all } 0 \leq t \leq T(D^{(-\eta)}). \quad (6.2)$$

**Proof of Theorem 2.2 given Proposition 6.1.**

Lower bound. We prove that $M_L(L^2 t) \supset D_{t-\eta}^{(-\eta)}$ for $0 < t \leq \tilde{T}$, where $\tilde{T}$ is the time when $D_{t-\eta}^{(-\eta)}$ becomes empty. Take $\epsilon > 0$ and let $D_t^-$ be the deterministic flow started from initial condition $D^{(-\epsilon)}$ and $\gamma^-(t) := \partial D_t^-$. We assume $\epsilon$ is small enough (cf. Remark 4.1) so that the boundary of $D^{(-\epsilon)}$ is a simple curve and satisfies the assumptions of Theorem 2.1. Proposition 6.1 gives $M_L(L^2 t) \supset D_{t-\eta}^-$ for all times up to $T(D^{(-\epsilon)}) - \epsilon$, which is larger than $\tilde{T}$ for $\epsilon$ small enough. The domain $D_t^-$ is included in $D_t$ for all times. We want to show that $D_{t-}^+ \supset D_{t-\eta}^-$. Let us call $d(t) = \sup\{d(p, \gamma(t)) : p \in \gamma^-(t)\}$ with $d(p, \gamma)$ the distance from $p$ to $\gamma$. Assume for definiteness that there is a unique pair of points $(p_t, q_t) \in \gamma^- \times \gamma(t)$ that is exactly at distance $d(t)$ (the general case is analogous): then necessarily the slopes $\theta_t$ of $\gamma^-(t), \gamma(t)$ at these points are equal and the segment $[p_t, q_t]$ is normal to both $\gamma^-(t)$ and $\gamma(t)$. Moreover, since $\gamma(t), \gamma^-(t)$ are smooth curves, we can locally approximate them by arcs of circles of radii $1/k(q_t), 1/k^-(p_t)$ respectively, with $k(q_t)$ the curvature of $\gamma(t)$ at $q_t$ and $k^-(p_t)$ the curvature of $\gamma^-(t)$ at $p_t$. The fact that $q_t$ realizes the infimum of $d(p_t, q)$ for $q$ ranging on $\gamma_t$ implies that $K_1 d(t) \leq 1$. On the other hand, the fact that $p_t$ maximizes $d(p, \gamma(t))$ for $p$ ranging on $\gamma^-(t)$ implies

$$k^-(p_t) \leq \frac{k(q_t)}{1 - k(q_t)d(t)}. \quad (6.3)$$

Both inequalities are easy to check if $\gamma(t), \gamma^-(t)$ are replaced by arcs of circles.
A look at (2.4) then shows that, as long as \(2d(t)k_{\max} < 1\),
\[
\frac{d}{dt} d(t) = a(\theta_t)(-k(q_t) + k^-(p_t)) \leq 2a_{\max}k_{\max}^2 d(t)
\]  
(6.4)
where \(k_{\max} < \infty\) is the maximal curvature of \(\gamma(t)\) up to time \(\tilde{T}\). Observe that \(d(0) = O(\epsilon)\). Then, choosing \(\epsilon\) sufficiently small (as a function of \(\eta\)), (6.4) ensures that \(d(t)\) remains smaller than \(\min(1/(2k_{\max}), \eta/2)\) up to \(\tilde{T}\). As a consequence, \(D_t^+ \supset D_t^{(-\eta)}\). The inclusion \(M_L(L^2t) \supset D_t^{(-\eta)}\) is therefore proven up to time \(\tilde{T}\), as we wished.

**Upper bound.** Let \(D_t^+\) be the deterministic flow started from initial condition \(D(t)\) and fix \(\xi\) small. It follows from Proposition 6.1 that, until time \(T(D)\),
\[
M_L(L^2t) \subset D_t^+.
\]

By the same argument that showed that \(D_t^- \supset D_t^{(-\eta)}\) we have that \(D_t^+ \subset D_t^{(\eta/4)}\) until time \(T - \xi\), provided \(\epsilon\) is small enough. This proves the upper bound up to time \(T(D) - \xi\).

Then we notice that, if \(\xi\) has been chosen small enough, \(D_t^{(\eta/4)}\) is included in \(B(X, \eta/2)\), a ball of radius \(\eta/2\) centered at \(X\), the point to which \(\gamma(t)\) shrinks as \(t \to T(D)\). From [20, Theorem 2.2] we know that, w.h.p., an initial droplet contained in \(B(X, \eta/2)\) disappears within time \(O(L^2\eta^2)\), and is included at all times in \(B(X, 2\eta/3)\). On the other hand, always for \(\xi\) small, \(B(X, 2\eta/3) \subset D_t^{(\eta)}\) for all \(t \geq T - \xi\). This concludes the proof of both claims. \(\square\)

6.3. **Proof of Proposition 6.1.** To prove Proposition 6.1 one needs two ingredients. The first says essentially that in the diffusive scaling the speed of evolution of the random droplet boundary is finite: in a time \(L^2\varepsilon\), it moves at most a distance \(O(\varepsilon)\).

**Proposition 6.2.** Suppose that \(\mathcal{D}\) satisfies the assumptions of Theorem 2.1, so that in particular \(m = m(\mathcal{D}) > 0\) (cf. definition (4.1)), and consider the dynamics starting from initial condition (2.1). There exists \(\varepsilon_0(m) > 0\) and \(C_1(m) < \infty\) such that for any \(\varepsilon \in (0, \varepsilon_0)\), with high probability
\[
\mathcal{D}^{(-C_1\varepsilon)} \subset M_L(L^2t) \subset \mathcal{D}^{(C_1\varepsilon)} \text{ for every } t \in [0, \varepsilon].
\]  
(6.5)
The constants \(C_1\) and \(1/\varepsilon_0\) can be chosen to be decreasing in \(m\).

The second ingredient is a control of the droplet after a time small on the diffusive scale \(L^2\).

**Proposition 6.3.** Let \(\mathcal{D}\) satisfy the assumptions of Theorem 2.1 and fix \(c > 0\). There exists \(\lambda_0(c, \mathcal{D}) > 0\) and, for \(0 < \lambda < \lambda_0\), there exists \(\varepsilon_1 = \varepsilon_1(\lambda, c, \mathcal{D}) > 0\) such that for all for \(\varepsilon \leq \varepsilon_1\), for all integer \(j\) with \(j \varepsilon \leq T(\mathcal{D}) - c\),

1. If the initial condition (which might be random) satisfies w.h.p. \(M_L(0) \supset \mathcal{D}^{(\lambda(T - j\varepsilon))}_j\) then w.h.p.
\[
M_L(L^2\varepsilon) \supset \mathcal{D}^{(\lambda(T - (j+1)\varepsilon))}_j.
\]  
(6.6)
2. If the initial condition satisfies w.h.p. \(M_L(0) \subset \mathcal{D}^{(-\lambda(T - j\varepsilon))}_j\) then w.h.p.
\[
M_L(L^2\varepsilon) \subset \mathcal{D}^{(-\lambda(T - (j+1)\varepsilon))}_j.
\]  
(6.7)

Recall that \(\mathcal{D}^{(\lambda(T - j\varepsilon))}_j\) is the deterministic domain \(\mathcal{D}\) at time \(j\varepsilon\), expanded by \(\lambda(T - j\varepsilon)\).

**Proof of Proposition 6.1 given Propositions 6.2 and 6.3.** Remark that we can assume that \(\eta\) is small: indeed, if the claim of Proposition 6.1 holds for small values of \(\eta\), then it clearly holds also for larger values.
Claim (1). Choose $c = \eta, \lambda = \min(\eta / T(\mathcal{D}), \lambda_0(c, \mathcal{D}))$ and $\varepsilon$ smaller than $\varepsilon_1(\lambda, c, \mathcal{D})$, with $\lambda_0$ and $\varepsilon_1$ from Proposition 6.3. Iterating the first claim of Proposition 6.3, we obtain w.h.p. for all $j$ such that $j \varepsilon \leq T - c$

$$\mathcal{M}_L(L^2j \varepsilon) \supset \mathcal{D}^{(\lambda(\mathcal{T} - j \varepsilon))}_{j \varepsilon} \supset \mathcal{D}_{j \varepsilon}. \quad (6.8)$$

This is the desired statement, but only at the discrete set of times $j \varepsilon L^2$.

Then, in order to also have an inclusion bound in the time intervals $[j \varepsilon, (j + 1)\varepsilon]$, we use Proposition 6.2. First of all, define $m_{\text{min}} = \min_{t \leq T - c} m(\mathcal{D}^{(\lambda(t - t))}_t)$ (recall (4.1)). One has $m_{\text{min}} > 0$. If $\lambda$ were zero, this would follow immediately from Lemma 4.2, since the inverse maximal curvature $r(t)$ is bounded away from zero up to time $T(\mathcal{D}) - c$. For $\lambda$ small (i.e. $\eta$ small), instead, $m_{\text{min}} > 0$ follows from Remark 4.1 and Lemma 4.2, which relate explicitly the curvature function of $\mathcal{D}_t$ with that of $\mathcal{D}_t^{(x)}$ for small $x$.

Assume then that $\varepsilon$ was chosen smaller than $\varepsilon_0(m_{\text{min}})$, with $\varepsilon_0$ as in Proposition 6.2. Starting at time $j \varepsilon$ with an initial condition “$-$” in $\mathcal{D}^{(\lambda(T - j \varepsilon))}_{j \varepsilon}$ and “$+$” outside, we get that w.h.p.

$$\mathcal{M}_L(L^2t) \supset \mathcal{D}^{(\lambda(T - \varepsilon + t/e) - C_1 \varepsilon)}_{\varepsilon t/e} \quad (6.9)$$

for all $t \in [j \varepsilon, (j + 1)\varepsilon]$, and therefore (repeating the argument for all values of $j$) for all $t \leq T - c$. The set on the right hand side contains $\mathcal{D}^{(\lambda(T - \varepsilon + t/e) - C_1 \varepsilon)}_{\varepsilon t/e}$, provided that $C_2 \geq C_1 + k_{\text{max}}$ (we used that $a(\cdot) < 1$). In turn, $\mathcal{D}^{(\lambda(T - \varepsilon + t/e) - C_1 \varepsilon)}_{\varepsilon t/e} \supset \mathcal{D}_t$ for all $t \leq T - c$ if $\varepsilon$ is sufficiently small.

Claim (2). This is proven analogously, taking $c \leq T(\mathcal{D}) - T(\mathcal{D}^{(\eta)})$. \hfill \square

6.4. Proof of Proposition 6.2. We are going to use a simple consequence of [20, Theorem 2.2], that gives the convergence of the stochastic evolution $\mathcal{M}_L(L^2t)$ to the curve-shortening flow (2.4) in the case of a smooth convex initial droplet.

Lemma 6.4. Set $r > 0$, $x \in \mathbb{R}^2$ and $\varepsilon > 0$. Consider the zero-temperature stochastic Ising model starting from an initial condition which satisfies

$$\mathcal{M}_L(0) \supset \mathcal{B}(x, r).$$

Then w.h.p. for all $t \in [0, \varepsilon]$

$$\mathcal{M}_L(L^2t) \supset \mathcal{B}(x, \sqrt{r^2 - 4a_{\text{max}} \varepsilon})$$

where $\mathcal{B}(x, r)$ denotes the open ball of center $x$ and radius $r$ if $r > 0$ and the empty set otherwise.

Proof. According to [20, Theorem 2.2], for any positive $\eta$, w.h.p. for all $t \leq \varepsilon$

$$\mathcal{M}_L(L^2t) \supset (\xi^{r, x}_t)^{(-\eta)}.$$ 

where $(\xi^{r, x}_t)_{t \geq 0}$ is the solution of (2.4) with initial condition $\mathcal{B}(x, r)$. The boundary of the ball $(\mathcal{B}(x, \sqrt{r^2 - 4a_{\text{max}} \varepsilon}))_{t \geq 0}$ is instead solution of the isotropic curve-shortening flow

$$\partial_t \gamma := 2a_{\text{max}} k \mathbf{N}. \quad (6.10)$$

The domain $\xi^{r, x}_t$ is convex at all times [20], so the velocity always points inward. Since $a(\theta)$ is strictly smaller than $2a_{\text{max}}$, equation (6.10) shrinks convex domains strictly faster than the original equation (2.4). Therefore, one can find $\eta$ (depending on $\varepsilon$) such that for all $t \leq \varepsilon$,

$$\xi^{r, x}_t^{(-\eta)} \supset \mathcal{B}(x, \sqrt{r^2 - 4a_{\text{max}} \varepsilon}).$$

\hfill \square
Proof of Proposition 6.2. We first prove the lower bound of (6.5). Define \( r := m(D)/2 \). Recalling Remark 4.1, note that for every \( 0 < a < b \leq r \) one has
\[
D^{(-b-a)}(x,a) = \bigcup_{x \in D^{(-b)}} B(x,a).
\]
Choose \( \varepsilon_0(r) \) sufficiently small so that any \( \varepsilon < \varepsilon_0(r) \) satisfies
\[
\sqrt{r^2 - 4a_{\max}e} \geq r - \frac{3a_{\max}e}{r}.
\]
In practice, think of \( \varepsilon_0(r) \ll r^2 \). Set \( C_1 := 4a_{\max}/r \). We have
\[
D^{(-C_1\varepsilon)} \subset \bigcup_{x \in D^{(-r)}} B(x, \sqrt{r^2 - 4a_{\max}e}) \tag{6.12}
\]
from (6.11), that is applicable since
\[
C_1\varepsilon \leq 4a_{\max}r\varepsilon_0(r) \ll r.
\]
As \( D^{(-C_1\varepsilon)} \) is a compact set, one can extract a finite subset \( J \subset D^{(-r)} \) which satisfies
\[
D^{(-C_1\varepsilon)} \subset \bigcup_{x \in J} B(x, \sqrt{r^2 - 4a_{\max}e}). \tag{6.13}
\]
For each \( x \in J \) we have \( M_L(0) \supset B(x, r) \), thus we can apply Lemma 6.4 and a union bound to obtain that w.h.p. for all \( t \leq \varepsilon \)
\[
M_L(\sqrt{2}t) \supset \bigcup_{x \in J} B(x, \sqrt{r^2 - 4a_{\max}e}) \supset D^{(-C_1\varepsilon)}. \tag{6.14}
\]

The proof of the other inclusion of (6.5), \( M_L(\sqrt{2}t) \subset D^{(C_1\varepsilon)} \), is similar since the roles of “+” and “−” spins are symmetric; we just have to take care to work with compact sets while the set of “+” spins is not compact. For \( \varepsilon < \varepsilon_0(r) \), let \( R \) be a rectangle, with sides parallel to the coordinate axes, that contains \( D^{(C_1\varepsilon)} \). From the definition of the dynamics, the spins outside \( R \) remain “+” at all times, since they always have four “+” neighbors. Let \( U \) be the closure of \( (R \setminus D^{(C_1\varepsilon)}) \) (\( U \) is compact). We have to prove that all spins in \( U \) stay “+” up to time \( \varepsilon \). Similarly to (6.13) one can find a finite subset \( J' \subset R \setminus D^{(r)} \) which satisfies
\[
U \subset \bigcup_{x \in J'} B(x, \sqrt{r^2 - 4a_{\max}e}), \tag{6.15}
\]
and then the conclusion follows from Lemma 6.4 applied to the dynamics where the roles of “+” and “−” are reversed. \( \square \)

6.5. Proof of Proposition 6.3. We give full details only for the proof of the first claim, the proof of second one being very similar. For notational convenience we prove the result for \( j = 0 \), and explain briefly in Remark 6.9 why the proof remains valid for all \( j \) such that \( j\varepsilon \leq T - \kappa \).

Given \( \lambda > 0 \) we set \( \mathcal{G}_t := D^{(\lambda(T-t))} \). Let \( \chi(t) \) denote the boundary of \( \mathcal{G}_t \) and \( \kappa \) the curvature function associated to it. By monotonicity, it is sufficient to prove that if one starts with the initial condition “−” in \( \mathcal{G}_0 \) and “+” outside, then for \( \varepsilon \) smaller than \( \varepsilon_1(\lambda, c, D) \), w.h.p.
\[
M_L(\sqrt{2}t) \supset \mathcal{G}_{\varepsilon}. \tag{6.16}
\]
The parameter \( \lambda \) will be chosen small in Lemma 6.8. We can use Proposition 6.2 which says that w.h.p. \( \mathcal{G}_0^{(-C_1\varepsilon)} \) is filled with “−” spins at time \( \varepsilon \). Hence we just have to prove that w.h.p.
\[
M_L(\sqrt{2}t) \supset \mathcal{G}_\varepsilon \setminus \mathcal{G}_0^{(-C_1\varepsilon)}. \tag{6.17}
\]
We denote by \( V_\varepsilon \) the compact closure of \( G_\varepsilon \setminus G_0^{(-C_1 \varepsilon)} \). Roughly speaking, \( V_\varepsilon \) is the collection of points in \( G_\varepsilon \) that are at distance \( C_1 \varepsilon \) from the boundary of \( G_0 \).

**Sketch of the strategy.** The main idea to prove (6.17) is to control the motion of the boundary of \( M_L(L^2 t) \) around a given point \( x \in \chi(0) \) via a comparison with \( M^x_L(L^2 t) \): this is the stochastic droplet starting from a circular shape \( P^x \) which is tangent to \( \chi(0) \) at \( x \) and whose curvature is close to that of \( \chi(0) \) at \( x \). Roughly speaking, \( P^x \) will sit in the interior of \( \chi(0) \) when the curvature at \( x \) is positive and outside it when the curvature is negative (some care will be needed when the curvature is close to zero). Given that the two initial curves \( \chi(0), \partial P^x \) have the same slope and almost the same curvature at \( x \), with \( \chi(x) \) needed when the curvature is close to zero). Given that the two initial curves \( \chi(0), \partial P^x \) have the same slope and almost the same curvature at \( x \), at time zero and close to \( x \) the boundaries of \( M_L(L^2 t) \) and \( M^x_L(L^2 t) \) feel almost the same drift. Our work will then consist in proving that, locally around \( x \) and for small positive times, they do remain close. On the other hand, \( P^x \) being convex, the evolution of \( M^x_L(L^2 t) \) in the scaling limit for every \( t \geq 0 \) is precisely controlled by Theorem 2.2 of [20].

As we will see, in practice we need \( P^x \) to be slightly more curved (in the \( N \) direction) than \( \chi(0) \): we define therefore \( P^x \) to be the disk tangent to \( \chi(0) \) and whose curvature vector at \( x \) is equal to \( \kappa(x) N \), with \( \kappa \) defined by

\[
\bar{\kappa}(x) := \begin{cases} 
\kappa(x) + \lambda/100 & \text{if } |\kappa(x) + \lambda/100| \geq \lambda/200 \\
\lambda/200 & \text{if } |\kappa(x) + \lambda/100| < \lambda/200.
\end{cases}
\]  

(6.18)

The second condition is there to prevent the curvature radius of \( P^x \) from diverging. Note that with this definition \( \kappa \) and \( \bar{\kappa} \) do not necessarily have the same sign but we have in any case

\[
\lambda/100 \leq \bar{\kappa}(x) - \kappa(x) \leq \lambda/50.
\]  

(6.19)

We set

\[
P^x := \begin{cases} 
P^x & \text{if } \bar{\kappa}(x) > 0, \\
(P^x)^c & \text{if } \bar{\kappa}(x) < 0
\end{cases}
\]

and we note that \( P^x \) is unbounded if \( \bar{\kappa}(x) < 0 \). Define then \( \sigma^x(t) \) to be the dynamics starting from “-” in \( P^x \) and “+” in \( (P^x)^c \) at time zero, and \( M^x_L(t) \) as in (2.3) with \( \sigma(t) \) replaced by \( \sigma^x(t) \). As it is not true in general that \( P^x \subset G_0 \), there is no immediate comparison between \( M^x_L(t) \) and \( M_L(t) \). However, the comparison works if we restrict ourselves to a sufficiently small box around \( x \). This is what we do next.

Consider \( B_x \) to be a square of side-length \( 2d > 0 \), centered at \( x \in \chi(0) \), whose sides are parallel to the normal and tangent vectors to \( \chi(0) \) at \( x \). If \( d \) is chosen small enough (depending on \( m(\mathcal{D}) \), \( \lambda \), the sup norm and the modulus of continuity of \( \kappa \), but neither on \( \varepsilon \) nor \( x \) ), the intersection of \( \chi(0) \) with \( B_x \) has only one connected component and

\[
(G_0 \cap B_x) \supset (P^x \cap B_x).
\]  

(6.20)

Furthermore, if \( \varepsilon \) is small enough (depending on \( \lambda, d \) and on the constant \( C_1 \) of Proposition 6.2) then

\[
((P^x)^{C_1 \varepsilon} \cap \partial B_x) \subset (G_0^{(-C_1 \varepsilon)} \cap \partial B_x).
\]  

(6.21)

These properties, together with monotonicity of the dynamics (cf. Section 6.1), allow us to establish a local comparison, that holds w.h.p. between the two stochastic droplets \( M_L(L^2 t) \) and \( M^x_L(L^2 t) \):

**Lemma 6.5.** If \( d \) and \( \varepsilon \) are such that (6.20) and (6.21) are satisfied for the constant \( C_1 \) corresponding to Proposition 6.2 then for all \( x \) in \( \chi(0) \), w.h.p.

\[
(M^x_L(L^2 \varepsilon) \cap B_x) \subset (M_L(L^2 \varepsilon) \cap B_x).
\]  

(6.22)
Let $\mathcal{P}_t^x$ denote the evolution of $\mathcal{P}^x$ by the anisotropic curve shortening flow (2.4) and set
\[
\mathcal{P}_t^x = \begin{cases} 
\mathcal{P}_t^x & \text{if } \bar{\kappa}(x) > 0, \\
(\mathcal{P}_t^x)^c & \text{if } \bar{\kappa}(x) < 0.
\end{cases}
\tag{6.23}
\]

The second important ingredient of the proof is to show:

**Lemma 6.6.** For $\varepsilon$ small enough, there exists $\eta$ such that
\[
V_\varepsilon \subset \bigcup_{x \in \chi(0)} \left( (\mathcal{P}_\varepsilon^x)^c \cap B_x \right),
\tag{6.24}
\]
where $V_\varepsilon$ was defined just after (6.17). Furthermore the inclusion remains valid if the closed sets on the right-hand side are replaced by their interiors.

**Proof of Proposition 6.3, Claim (1), from Lemma 6.5 and 6.6.** As $V_\varepsilon$ is a compact set, in the inclusion (6.24) we can extract a finite set $J \subset \chi(0)$ which satisfies
\[
V_\varepsilon \subset \bigcup_{x \in J} \left( (\mathcal{P}_\varepsilon^x)^c \cap B_x \right).
\tag{6.25}
\]
Then combining Lemma 6.5 and [20, Theorem 2.2], that guarantees that $\mathcal{M}_{L^2}(L^2 t) \supset (\mathcal{P}_\varepsilon^x)^c$, we notice that w.h.p. for every $x \in J$,
\[
\mathcal{M}_L(L^2 \varepsilon) \supset (\mathcal{M}_{L^2}(L^2 \varepsilon) \cap B_x) \supset \left( (\mathcal{P}_\varepsilon^x)^c \cap B_x \right).
\tag{6.26}
\]
As a side remark, note that when $\bar{\kappa}(x) < 0$ we need to reverse the role of “–” and “+” spins when applying [20, Theorem 2.2]: this is because the bounded initial droplet $\mathcal{P}^x$ is filled with “+” and not “–” spins. Hence, as $J$ is finite, we can take the union over $x \in J$ to get that w.h.p.
\[
\mathcal{M}_L(L^2 \varepsilon) \supset V_\varepsilon.
\tag{6.27}
\]

**Proof of Lemma 6.5.** We observe the dynamics $\sigma^x(t)$ and $\sigma(t)$ restricted to the square window $B_x$. We let $\tau^x(t)$ and $\tau(t)$ denote the restriction of $\sigma^x(t)$ and $\sigma(t)$ to the boundary $\partial B_x$ of the
box (i.e. the set of lattice sites outside $B_x$ that are at distance $1/L$ from some lattice site in $B_x$). From monotonicity of the dynamics, to show (6.22) it is sufficient to show that w.h.p.

$$\{M_L^x(0) \cap B_x\} \subset \{M_L(0) \cap B_x\},$$

$$\tau_y(s) \leq \tau_y^x(s) \quad \text{for every} \quad s \in [0, \varepsilon], y \in \partial B_x.$$  

The first point (domination between the initial conditions) is just an immediate consequence of (6.20). The second one (domination between time-dependent boundary conditions) is a consequence of Proposition 6.2 combined with (6.21). □

**Proof of Lemma 6.6.** For all $x$ in $\chi(0)$, we define a local coordinate system $S_x = (x, T, N)$. Given $\delta$ (which will depend on $\varepsilon$) we define $M_x$ to be the rectangular box $[-\delta, \delta] \times [-d, d]$ in the frame $S_x$ (it is much narrower than the $2d \times 2d$ square $B_x$, i.e. we have to think $\delta \ll d$). Note that given $d$, if $\varepsilon$ is small enough, for all $\delta > 0$ we have

$$V_\varepsilon \subset \bigcup_{x \in \chi(0)} M_x.$$  

Hence to prove (6.24) it is sufficient to prove that for all $x \in \chi(0)$,

$$G_\varepsilon \cap M_x \subset \left( (\mathcal{P}_\varepsilon^x)^{-\eta} \cap M_x \right).$$

We first rewrite this inclusion as an inequality between functions. If $d$ and $\varepsilon$ are sufficiently small (with $\varepsilon$ that depends on $d$), the restriction of $\chi(t) = \partial G_t$ and of $\partial \mathcal{P}_t^x$ to $M_x$ can be considered as the graphs of functions in $S_x$, for all $t \in [0, \varepsilon]$. We denote these functions by $f^x(u, t)$ and $g^x(u, t)$ respectively, with $u \in [-\delta, \delta]$. Recall that the reference frame $S_x$ is such that the $y$-axis is directed along the inward pointing normal vector $N$. The proof of (6.31) will thus be complete if we can prove the following:

**Lemma 6.7.** There exists $\varepsilon_0(D, \lambda)$ so that for all $\varepsilon < \varepsilon_0$ there exists $\delta(\varepsilon)$ and $\eta(\varepsilon)$ such that for all $x \in \chi(0)$

$$g^x(u, \varepsilon) + \eta \leq f^x(u, \varepsilon), \quad \forall u \in [-\delta, \delta].$$

**Proof of Lemma 6.7.** In order to simplify notations we drop the exponent $x$ in the following.

The core of the proof is to show that the drift of $\partial G_t$ in the $N$ direction is stronger than the one of $\partial \mathcal{P}_t$. Thus, even though $g$ starts above $f$ initially (cf. (6.20)), it has time to catch up. Note that, since $\partial \mathcal{P}$ is more curved than $G$ at time zero (cf. (6.19)), it would look like the drift of $\partial \mathcal{P}$ should be larger: however, we will see that the boundary of $G_t$ solves the curve shortening flow (2.4) with an extra term, proportional to $\lambda$, in the normal velocity. This extra term guarantees the desired inequality between drifts.

Observe first of all that, provided that $\delta$ is small enough, for all $u \in [-\delta, \delta]$

$$\begin{align*}
g(u, 0) &= \frac{1}{2} \kappa(x) u^2 + O(u^4) \\
f(u, 0) &\geq \frac{1}{2} (\kappa(x) - \lambda/50) u^2 + O(u^4).
\end{align*}$$

The $O(u^4)$ term is uniform in $x$ and just depends on the maximal curvature. We have simply approximated locally the curves with suitable parabolas, and the second inequality is valid provided that $\kappa > \kappa(x) - \lambda/50$ on $\chi(0) \cap M_x$ (this is true if $\delta$ is small). Hence

$$f(u, 0) \geq g(u, 0) + \frac{1}{2} (\kappa(x) - \lambda/50 - \kappa(x)) u^2 + O(u^4) \geq g(u, 0) - \frac{\lambda}{40} u^2,$$

where the second inequality holds if $\delta$ is chosen small enough.
What we want to show is that for all $u \in [-\delta,\delta]$, $t \in [0,\varepsilon]$
\[ \partial_t f(u, t) - \partial_t g(u, t) \geq \lambda/10. \] (6.35)
Then the equation (6.32) will easily be derived by integrating (6.35) starting from (6.34), provided that
\[ \frac{\lambda\varepsilon}{10} \geq \eta + \frac{\lambda}{40} \delta^2 \] (6.36)
(for instance one can take $\delta = \sqrt{\varepsilon}$ and $\eta = 3\lambda\varepsilon/40$).

To have an estimate on the time derivatives we need to use the equation that are satisfied by $\chi(t) = \partial_t G_t$ and $\partial_t P_t^x$ respectively.

**Lemma 6.8.** The curve $\chi(t)$ is solution of the modified curve-shortening equation
\[ \partial_t \chi = \left[ a(\theta)\kappa + \lambda \left( 1 + \frac{a(\theta)(T - t)\kappa^2}{1 - \lambda(T - t)\kappa} \right) \right] N, \] (6.37)
where $N$ is the normal vector oriented inside the curve and $\theta, \kappa$ are the slope and curvature at the point in question.

On the other hand, $\partial_t P_t^x$ solves the usual curve shortening flow (2.4), that does not depend on $\lambda$. Using simple trigonometry we then obtain that
\[ \partial_t f(u, t) = \frac{1}{\cos(\theta - \theta_x)} \left( a(\theta)\kappa + \lambda \left( 1 + \frac{a(\theta)(T - t)\kappa^2}{1 - \lambda(T - t)\kappa} \right) \right), \] (6.38)
\[ \partial_t g(u, t) = \frac{1}{\cos(\theta - \theta_x)} a(\theta) \bar{\kappa} \] (6.39)
where $\kappa$ (resp. $\bar{\kappa}$) denotes the curvature on $\chi$ (resp. $\partial_t P_t^x$), $\theta$ (resp. $\bar{\theta}$) the tangent angle on $\chi$ (resp. $\partial_t P_t^x$) at the point of coordinate $u$, and $\theta_x$ the tangent angle to $\chi(0)$ at $x$.

By continuity of curvature in space and time (cf. Proposition 4.8), if one takes $\varepsilon$ and $\delta$ small enough, $\theta$ can be assumed to be arbitrarily close to $\theta_x$, $\kappa$ arbitrarily close to $\kappa(x)$ and $\bar{\kappa}$ arbitrarily close to $\bar{\kappa}(x)$. Hence one has for all $u \in [-\delta,\delta]$, $t \in [0,\varepsilon]$
\[ \partial_t f(u, t) \geq a(\theta_x)\kappa(x) + 2\lambda/3, \] (6.40)
\[ \partial_t g(u, t) \leq a(\theta_x)\bar{\kappa}(x) + \lambda/3. \] (6.41)
Then we conclude the proof of (6.35) using (6.19) and the fact that $a(\cdot) \leq 1$. The proof of Lemma 6.6 is also concluded.

**Proof of Lemma 6.8.** As mentioned in Remark 4.1, there is a natural bijection between $\gamma(t) = \partial_t \mathcal{D}_t$ and $\chi(t) = \partial_t \mathcal{D}_t^{(\lambda(T - t))}$, given by $\gamma(t) \ni x \mapsto x - \lambda(T - t)N \in \chi(t)$. It is here that one needs to have $\lambda$ smaller than some $\lambda_0(\mathcal{D})$, to guarantee that $\lambda(T - t) < m(\mathcal{D}_t)$. As observed in (4.2), for points that are in correspondence one has
\[ k = \frac{\kappa}{1 - \kappa(T - t)\lambda} \] (6.42)
where we remind that $k$ is the curvature of $\gamma(t)$ and $\kappa$ the curvature of $\chi(t)$. Moreover Equation (2.4) and the definition of $\chi(t)$ give
\[ \partial_t \chi := (k + \lambda)N. \] (6.43)
The desired equation (6.37) immediately follows from (6.42) and (6.43).
Remark 6.9. The only properties of $\mathcal{D}$ we used in the proof of the $(j = 0) \Rightarrow (j = 1)$ step of Proposition 6.3 are $m(\mathcal{D}) > 0$ and the fact that the curvature function is bounded and uniformly continuous. Since this continues to hold up to time $T(\mathcal{D}) - c$, the proof for $j > 0$ such that $j\varepsilon < T(\mathcal{D}) - c$ works exactly the same, and the small parameters $\varepsilon$ and $\lambda$ can be chosen to be independent of $j$.

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