Tutorial n°1 – Probability Basics

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Exercise 1 *(Probability or Statistics?).* Let $X_1, \dots, X_n$ be *i.i.d.* random variables and $x_1, \dots, x_n$ be realisations or observations of the later random variables. Which of the following quantities are random?

1. $\max\{x_1, \dots, x_n\}$,

2. the sample size n,

3. $\frac{1}{n}\sum_{i=1}^n X_i$,

4. $\min\{X_1, \dots, X_n\}$,

5. $\frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x}_n)^2$.

Exercise 2 *(True or False?).* Let X and Y be integrable random variables. Which of the following statements are correct? Justify your answer with a brief proof or a counterexample.

1. If X is symmetric with respect to 0, then $\mathbb{E}[X] = 0$.

2. $\mathbb{E}\left[\frac{1}{X}\right] = \frac{1}{\mathbb{E}[X]}$.

3. $\mathbb{E}[X]^2 \leq \mathbb{E}\left[X^2\right]$.

4. $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

5. $\mathbb{V}\text{ar}[X + Y] = \mathbb{V}\text{ar}[X] + \mathbb{V}\text{ar}[Y]$.

Exercise 3. Let X be a real random variable with density

$$f_X(x) = \frac{5}{x^2}\mathbb{1}_{\{x>5\}}$$

Compute the following quantities:

1. $\mathbb{P}[X > 20]$,

2. $F_X(t)$, for all $t \in \mathbb{R}$,

3. $\mathbb{E}[X]$.

Exercise 4. Let X be a random variable following the uniform distribution on $[-\pi/2, \pi/2]$. Determine the distribution of $\tan(X)$.

Exercise 5 *(Multinomial distribution).* A population is divided into K groups. We denote $p_1, \dots, p_K$ the proportions of individuals in each group, with $p_1, \dots, p_K \in [0,1]$ and $\sum_{i=1}^K p_i = 1$. We draw in this population n individuals with replacement. Let denote N_i the number of individuals belonging to group i, $i = 1, \dots, K$, among the n individuals drawn.

1. Give, with a justification, the distribution of $(N_1, \dots, N_K)$.

2. Give the marginal distribution of N_i, $i = 1, \dots, K$.

3. Give the R command to use to run this experiment.

Exercise 6. We consider a system made of two different machines working in series, that is the system works as long as both machines work. Let denote X_1 and X_2 the lifetime of the two machines and Z the lifetime of the system. We assume that the random variables X_1 and X_2 are independent and follow an exponential distribution with respective parameters λ_1 and λ_2.

1. Compute the probability that the system breaks down after time $t \geq 0$ and deduce the distribution of Z.
2. Compute the probability that the break down is due to a failure of machine 1.
3. Let Y be a random variable such that $Y = 1$ if the failure is due to machine 1 and $Y = 0$ otherwise.
   - **(a)** Compute $\mathbb{P}[Z > t, Y = 1]$ for all $t \geq 0$.
   - **(b)** Deduce that Z and Y are independent.

⋄ To do at Home ⋄

Exercise 7 *(Characteristic function).*

1. Let Y be a real random variable and Z a random variable, independent of Y, such that

$$\mathbb{P}[Z = 1] = \mathbb{P}[Z = -1] = \frac{1}{2}.$$

   - **(a)** Show that the law of $X = ZY$ is symmetric.
   - **(b)** Compute the characteristic function of X according to the characteristic function of Y.

2. Let X be a random variable following the standard Laplace law:

$$f_X(x) = 0.5 \exp(-|x|).$$

   Show that, for every real t, $\Phi_X(t) = (1 + t^2)^{-1}$.

Exercise 8 *(Gamma distribution).* Given $\alpha > 0$ and $\lambda > 0$, the gamma distribution $\gamma(\alpha, \lambda)$ is defined by the density with respect to the Lebesgue measure

$$f(x; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \exp(-\lambda x) x^{\alpha-1} \mathbb{1}_{\{x \geq 0\}}, \quad \text{where} \quad \Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} \mathrm{d}t.$$

1. Check that $f(\cdot; \alpha, \lambda)$ is a probability density function.
2. Compute the expectation of the gamma distribution $\gamma(\alpha, \lambda)$.
3. Let $X_1, \ldots, X_n$ be *i.i.d.* random variables with distribution $\mathscr{E}(\lambda)$. Show that $X_1 + \ldots + X_n$ is distributed according to the gamma distribution $\gamma(n, \lambda)$.
4. Let X and Y be independent random variables with distribution $\gamma(\alpha, \lambda)$.
   - **(a)** Show that $X + Y$ and $X/X+Y$ are independent random variables.
   - **(b)** Give the distributions of $X + Y$ and $X/X+Y$.