Tutorial n°2 – Convergence of random variables

stoehr@ceremade.dauphine.fr

Exercise 1 *(True or False?).* Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables and X a random variable. Are the following statements correct? Justify your answer.

1. If $(X_n)_{n\geq 1}$ converges in probability to X, then $(X_n)_{n\geq 1}$ converges almost surely to X.

2. If $(X_n)_{n\geq 1}$ converges in distribution to X, then for any function f

$$\mathbb{E}\left[f(X_n)\right] \underset{n\to+\infty}{\longrightarrow} \mathbb{E}\left[f(X)\right].$$

3. If $(X_n)_{n\geq 1}$ converges almost surely to X, then it converges in the p-th mean to X.

Exercise 2. Let $(X_n)_{n\geq 1}$ be a sequence of random variables defined by

$$\mathbb{P}\left[X_n = \sqrt{n}\right] = \frac{1}{n} \quad \text{and} \quad \mathbb{P}\left[X_n = 0\right] = \frac{n-1}{n}$$

Study the convergence in quadratic mean, in probability and in distribution of $(X_n)_{n\geq 1}$.

Exercise 3. Let $(X_n)_{n\geq 1}$ be a sequence of real random variables, such that for $n \geq 1$, X_n is distributed according to an exponential distribution with parameter $\lambda = n$. Let define

$$Y_n = \sin\left(\lfloor X_n \rfloor \frac{\pi}{2}\right), \quad \text{where } \lfloor\cdot\rfloor \text{ is the floor function.}$$

1. **(a)** Find the distribution of the random variable Y_n.

(b) Compute $\mathbb{E}[Y_n]$ and $\mathbb{V}\text{ar}[Y_n]$.

2. Show that the sequence $(Y_n)_{n\geq 1}$ converges in distribution to a constant random variable Y.

3. Show that the sequence $(Y_n)_{n\geq 1}$ also converges in probability to Y.

Exercise 4 *(CLT and asymptotic variance).* Let $(X_n)_{n\geq 1}$ be a sequence of *i.i.d.* random variables with distribution $\mathbb{P}$ such that $\mathbb{E}\left[X_1^2\right] < \infty$. Denote $\mu = \mathbb{E}[X_1]$ and $\sigma^2 = \mathbb{V}\text{ar}[X_1]$.

1. Suppose σ^2 is unknown. Derive a sequence $(a_n)_{n\geq 1}$ of random variables independent of σ^2 such that

$$\sqrt{\frac{n}{a_n}}\left(\overline{X}_n - \mu\right) \underset{n\to+\infty}{\longrightarrow} \mathcal{N}(0,1).$$

2. Suppose that σ^2 is a function of μ, *i.e.* it can be written as $\sigma^2(\mu)$. Find a differentiable function g such that $g'(\mu) \neq 0$ and

$$\sqrt{n}\left[g(\overline{X}_n) - g(\mu)\right] \xrightarrow[n\to+\infty]{d} \mathcal{N}(0,1).$$

3. Find $(a_n)_{n\geq 1}$ and g in the following particular cases

(a) $\mathscr{P} = \left\{\mathscr{B}(p), 0 < p < 1/2\right\}$. **(b)** $\mathscr{P} = \{\mathscr{E}(\lambda), \lambda > 0\}$.

◇ To do at Home ◇

Exercise 5. Let $(X_n)_{n\geq 1}$ be a sequence of mutually independent random variables defined by

$$\mathbb{P}\left[X_n = \frac{1}{2^n}\right] = \mathbb{P}\left[X_n = \frac{-1}{2^n}\right] = \frac{1}{2}.$$

Denote for all integer $n \geq 1$, $S_n = \sum_{i=1}^{n} X_i$.

1. Compute the characteristic function of the uniform distribution on $[-1,1]$.

2. Show that $(S_n)_{n\geq 1}$ converges in distribution to a random variable S with a distribution you will precise.

Exercise 6. Let $(X_n)_{n\geq 1}$ be a sequence of *i.i.d.* random variables distributed according to a Bernoulli distribution with parameter $p \in]0,1[$. For all integer $n \geq 1$, denote $Y_n = X_n X_{n+1}$ and $S_n = \sum_{i=1}^{n} Y_i$.

1. What is the distribution of Y_n, $n \geq 1$.

2. Given integers $1 \leq n < m$, under what conditions on n and m are random variables Y_n and Y_m independent?

3. Compute $\mathbb{E}[Y_n Y_m]$ and $\mathbb{E}[S_n/n]$ for all integers n and m such that $1 \leq n < m$.

4. Show that there exists a real constant C such that $\mathbb{V}\text{ar}[S_n] \leq Cn$.

5. Show that $(S_n/n)_{n\geq 1}$ converges in probability to a constant you will precise.

Exercise 7. In a desintegration process, electrons are emitted at an angle θ such that $\cos\theta$ is distributed according to the following density

$$f(x \mid \alpha) = \frac{1}{2}(1+\alpha x)\mathbb{1}_{\{x\in[-1,1]\}}, \quad \text{with} \quad |\alpha| < 1.$$

1. Compute $\mathbb{E}[X]$ and deduce an estimator $\widehat{\alpha}_n$ of α that converges in probability to α.

2. Find a sequence c_n independent of α such that

$$c_n(\widehat{\alpha}_n - \alpha) \xrightarrow[n\to+\infty]{d} \mathcal{N}(0,1).$$

3. Find a function g invertible on $]-1,1[$ such that

$$\sqrt{n}[g(\widehat{\alpha}_n) - g(\alpha)] \xrightarrow[n\to+\infty]{d} \mathcal{N}(0,1).$$