

Tutorial n°2 – Convergence of random variables

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Exercise 1 (*True or False?*). Let $(X_n)_{n \geq 1}$ be a sequence of independent random variables and X a random variable. Are the following statements correct? Justify your answer.

1. If $(X_n)_{n \geq 1}$ converges in probability to X , then $(X_n)_{n \geq 1}$ converges almost surely to X .
2. If $(X_n)_{n \geq 1}$ converges in distribution to X , then for any function f

$$\mathbb{E}[f(X_n)] \xrightarrow{n \rightarrow +\infty} \mathbb{E}[f(X)].$$

3. If $(X_n)_{n \geq 1}$ converges almost surely to X , then it converges in the p -th mean to X .

Exercise 2. Let $(X_n)_{n \geq 1}$ be a sequence of random variables defined by

$$\mathbb{P}[X_n = \sqrt{n}] = \frac{1}{n} \quad \text{and} \quad \mathbb{P}[X_n = 0] = \frac{n-1}{n}$$

Study the convergence in quadratic mean, in probability and in distribution of $(X_n)_{n \geq 1}$.

Exercise 3. Let $(X_n)_{n \geq 1}$ be a sequence of real random variables, such that for $n \geq 1$, X_n is distributed according to an exponential distribution with parameter $\lambda = n$. Let define

$$Y_n = \sin\left(\lfloor X_n \rfloor \frac{\pi}{2}\right), \quad \text{where } \lfloor \cdot \rfloor \text{ is the floor function.}$$

1. (a) Find the distribution of the random variable Y_n .
 (b) Compute $\mathbb{E}[Y_n]$ and $\text{Var}[Y_n]$.
2. Show that the sequence $(Y_n)_{n \geq 1}$ converges in distribution to a constant random variable Y .
3. Show that the sequence $(Y_n)_{n \geq 1}$ also converges in probability to Y .

Exercise 4 (*CLT and asymptotic variance*). Let $(X_n)_{n \geq 1}$ be a sequence of *i.i.d.* random variables with distribution $\mathbb{P}$ such that $\mathbb{E}[X_1^2] < \infty$. Denote $\mu = \mathbb{E}[X_1]$ and $\sigma^2 = \text{Var}[X_1]$.

1. Suppose σ^2 is unknown. Derive a sequence $(a_n)_{n \geq 1}$ of random variables independent of σ^2 such that

$$\sqrt{\frac{n}{a_n}} (\bar{X}_n - \mu) \xrightarrow{n \rightarrow +\infty} \mathcal{N}(0, 1).$$

2. Suppose that σ^2 is a function of μ , *i.e.* it can be written as $\sigma^2(\mu)$. Find a differentiable function g such that $g'(\mu) \neq 0$ and

$$\sqrt{n} [g(\bar{X}_n) - g(\mu)] \xrightarrow{n \rightarrow +\infty} \mathcal{N}(0, 1).$$

3. Find $(a_n)_{n \geq 1}$ and g in the following particular cases

1. $\mathcal{P} = \{\mathcal{B}(p), 0 < p < 1/2\}$.

2. $\mathcal{P} = \{\mathcal{E}(\lambda), \lambda > 0\}$.

◇ To do at Home ◇

Exercise 5. Let $(X_n)_{n \geq 1}$ be a sequence of mutually independent random variables defined by

$$\mathbb{P}\left[X_n = \frac{1}{2^n}\right] = \mathbb{P}\left[X_n = \frac{-1}{2^n}\right] = \frac{1}{2}.$$

Denote for all integer $n \geq 1$, $S_n = \sum_{i=1}^n X_i$.

1. Compute the characteristic function of the uniform distribution on $[-1, 1]$.
2. Show that $(S_n)_{n \geq 1}$ converges in distribution to a random variable S with a distribution you will precise.

Exercise 6. Let $(X_n)_{n \geq 1}$ be a sequence of *i.i.d.* random variables distributed according to a Bernoulli distribution with parameter $p \in]0, 1[$. For all integer $n \geq 1$, denote $Y_n = X_n X_{n+1}$ and $S_n = \sum_{i=1}^n Y_i$.

1. What is the distribution of Y_n , $n \geq 1$.
2. Given integers $1 \leq n < m$, under what conditions on n and m are random variables Y_n and Y_m independent?
3. Compute $\mathbb{E}[Y_n Y_m]$ and $\mathbb{E}[S_n/n]$ for all integers n and m such that $1 \leq n < m$.
4. Show that there exists a real constant C such that $\text{Var}[S_n] \leq Cn$.
5. Show that $(S_n/n)_{n \geq 1}$ converges in probability to a constant you will precise.

Exercise 7. In a desintegration process, electrons are emitted at an angle θ such that $\cos \theta$ is distributed according to the following density

$$f(x | \alpha) = \frac{1}{2}(1 + \alpha x) \mathbb{1}_{\{x \in [-1, 1]\}}, \quad \text{with } |\alpha| < 1.$$

1. Compute $\mathbb{E}[X]$ and deduce an estimator $\hat{\alpha}_n$ of α that converges in probability to α .
2. Find a sequence c_n independent of α such that

$$c_n(\hat{\alpha}_n - \alpha) \xrightarrow[n \rightarrow +\infty]{d} \mathcal{N}(0, 1).$$

3. Find a function g invertible on $] -1, 1[$ such that

$$\sqrt{n}[g(\hat{\alpha}_n) - g(\alpha)] \xrightarrow[n \rightarrow +\infty]{d} \mathcal{N}(0, 1).$$