Université Paris Dauphine
Département MIDO
Licence 3 – Statistical modelling 2022–2023

Tutorial n°5 – Estimation

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Exercise 1. Let consider the statistical model $\mathscr{P} = \{\mathcal{N}\left(a, a^2\sigma^2\right) \mid a \in \mathbb{R}_+^*\}$.

1. Write the likelihood $L(a \mid x_1, \dots, x_n)$ associated to a n-sample $(x_1, \dots, x_n)$ from this model.

2. Compute $\widehat{a}_n$ the maximum of $\log L(a \mid x_1, \dots, x_n)$ and show that $\widehat{a}_n$ converges in probability to a.

Exercise 2. Let X be a real random variable taking values in $[-0.5, 0.5]$ whose density parametrised by $\theta \in]-1, 1[$ is given by

$$f(x \mid \theta) = (1-\theta)^{\mathbb{1}\{-0.5 \leq x < 0\}}(1+\theta)^{\mathbb{1}\{0 \leq x < 0.5\}} = \begin{cases} 1-\theta, & \text{if } x \in [-0.5, 0[\\ 1+\theta, & \text{if } x \in [0, 0.5[. \end{cases}$$

Let $X_1, \dots, X_n$ be a n-sample from X.

1. Find $\widehat{\theta}_n$ the maximum likelihood estimator for θ.

2. **(a)** Show that $\widehat{\theta}_n$ is unbiased.

(b) Show that $T(x_1, \dots, x_n) = \sum_{i=1}^n \mathbb{1}_{\{x_i \in [0, 0.5]\}}$, for $x_1, \dots, x_n \in [-0.5, 0.5]$, is sufficient and complete.

(c) Deduce that $\widehat{\theta}_n$ is the unbiased estimator with uniformly minimum variance.

3. **(a)** Compute the Fisher information contained in X_1 for θ and deduce a sequence c_n independent of θ such that

$$c_n(\widehat{\theta}_n - \theta) \xrightarrow[n \to +\infty]{d} \mathcal{N}(0, 1).$$

(b) Given $\alpha \in]0, 1[$, build an asymptotic confidence interval for θ with $(1-\alpha)$ confidence level.

Exercise 3. Let consider the statistical model $\mathscr{P} = \{\mathcal{U}([-\theta, \theta]), \theta > 0\}$ and a n-sample $X_1, \dots, X_n$ from this model.

1. Find $\widehat{\theta}_n$ the maximum likelihood estimator for θ.

2. **(a)** Show that $\widehat{\theta}_n$ is biased.

(b) Provide a R code to estimate the bias in practice.

3. Justify that $\widehat{\theta}_n$ converges in probability to θ

4. **(a)** Show that $n(\theta - \widehat{\theta}_n) \xrightarrow[n \to +\infty]{d} \mathscr{E}(1/\theta)$.

(b) Given $\alpha \in]0, 1[$, build an asymptotic confidence interval for θ with $(1-\alpha)$ confidence level.

5. Give $\widehat{\theta}_n^\star$ the uniformly minimum variance unbiased estimator.

6. We assume that $\tilde{\theta}_n = |X|_{(1)} + |X|_{(n)}$, where $|X|_{(1)} = \min(|X_1|, \dots, |X_n|)$ and $|X|_{(n)} = \max(|X_1|, \dots, |X_n|)$ is an unbiased estimator of θ.

(a) Compute the conditional distribution of $|X|_{(1)}$ knowing $|X|_{(n)} = x_n$.

(b) Deduce that the Rao-Blackwell improved version of $\tilde{\theta}_n$ is $\widehat{\theta}_n^\star$.

⋄ To do at Home ⋄

Exercise 4. Let consider the statistical model $\mathscr{P} = \{f(\cdot \mid \theta); \theta \in \Theta$ associated to the Weibull distribution $\mathscr{W}(\theta, 1/2)$ whose density is given by

$$f(x \mid \theta) = \frac{1}{2\theta\sqrt{x}} \exp\left(-\frac{\sqrt{x}}{\theta}\right) \mathbb{1}_{\{x>0\}}$$

1. Show that this density belongs to a minimal and regular exponential family. For which set Θ is this density properly defined?

2. Write the likelihood $L(\theta \mid x_1, \dots, x_n)$ associated to a n-sample $(x_1, \dots, x_n)$ from this model.

3. Compute $\widehat{\theta}_n$ the maximum of $\log L(\theta \mid x_1, \dots, x_n)$ and show that $\widehat{\theta}_n$ converges in probability to θ.

Exercise 5. Let consider the statistical model $\{\mathscr{P}(\lambda), \lambda > 0\}$ and a n-sample $X_1, \dots, X_n$ from this model. We aim at estimating $\theta = \mathbb{P}[X = 0] = e^{-\lambda}$. We set

$$\widehat{\theta}_n = \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{X_k=0\}}.$$

1. Show that $\widehat{\theta}_n$ is an unbiased estimator of θ.

2. Justify that $T = \sum_{k=1}^n X_k$ is a sufficient statistic.

3. Compute $\mathbb{P}_\lambda[X_1 = 0 \mid T = t]$, for all $t \in \mathbb{N}$.

4. Using the Rao-Blackwell theorem, provide an improved version $\widehat{\theta}_n^\star$ of $\widehat{\theta}_n$.

5. Show that $\widehat{\theta}_n^\star$ is the uniformly minimum variance unbiased estimator.

Exercise 6. Let X be a Poisson random variable with parameter $\theta > 0$ and $X_1, \dots, X_n$ a n-sample from X. We consider

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad S_{n-1}^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2.$$

1. Show that $\overline{X}_n$ and S_{n-1}^2 are unbiased estimators of θ.

2. Show that $\overline{X}_n$ is the uniformly minimum variance unbiased estimator of θ and deduce that $\mathbb{V}\mathrm{ar}_\theta\left[\overline{X}_n\right] \leq \mathbb{V}\mathrm{ar}_\theta[S_{n-1}^2]$.