

Tutorial n°5 – Estimation

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Exercise 1. Let consider the statistical model $\mathcal{P} = \{\mathcal{N}(a, a^2\sigma^2) \mid a \in \mathbb{R}_+^*\}$.

1. Write the likelihood $L(a \mid x_1, \dots, x_n)$ associated to a n -sample $(x_1, \dots, x_n)$ from this model.
2. Compute $\hat{a}_n$ the maximum of $\log L(a \mid x_1, \dots, x_n)$ and show that $\hat{a}_n$ converges in probability to a .

Exercise 2. Let X be a real random variable taking values in $[-0.5, 0.5]$ whose density parametrised by $\theta \in]-1, 1[$ is given by

$$f(x \mid \theta) = (1 - \theta)\mathbb{1}_{\{-0.5 \leq x < 0\}}(1 + \theta)\mathbb{1}_{\{0 \leq x < 0.5\}} = \begin{cases} 1 - \theta, & \text{if } x \in [-0.5, 0[\\ 1 + \theta, & \text{if } x \in [0, 0.5[. \end{cases}$$

Let $X_1, \dots, X_n$ be a n -sample from X .

1. Find $\hat{\theta}_n$ the maximum likelihood estimator for θ .
2. (a) Show that $\hat{\theta}_n$ is unbiased.
 - (b) Show that $T(x_1, \dots, x_n) = \sum_{i=1}^n \mathbb{1}_{\{x_i \in [0, 0.5]\}}$, for $x_1, \dots, x_n \in [-0.5, 0.5]$, is sufficient and complete.
 - (c) Deduce that $\hat{\theta}_n$ is the unbiased estimator with uniformly minimum variance.
3. (a) Compute the Fisher information contained in X_1 for θ and deduce a sequence c_n independent of θ such that

$$c_n(\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow +\infty]{d} \mathcal{N}(0, 1).$$

- (b) Given $\alpha \in]0, 1[$, build an asymptotic confidence interval for θ with $(1 - \alpha)$ confidence level.

Exercise 3. Let consider the statistical model $\mathcal{P} = \{\mathcal{U}([-\theta, \theta]), \theta > 0\}$ and a n -sample $X_1, \dots, X_n$ from this model.

1. Find $\hat{\theta}_n$ the maximum likelihood estimator for θ .
2. (a) Show that $\hat{\theta}_n$ is biased.
 - (b) Provide a R code to estimate the bias in practice.
3. Justify that $\hat{\theta}_n$ converges in probability to θ
4. (a) Show that $n(\theta - \hat{\theta}_n) \xrightarrow[n \rightarrow +\infty]{d} \mathcal{E}(1/\theta)$.
 - (b) Given $\alpha \in]0, 1[$, build an asymptotic confidence interval for θ with $(1 - \alpha)$ confidence level.
5. Give $\hat{\theta}_n^*$ the uniformly minimum variance unbiased estimator.

6. We assume that $\tilde{\theta}_n = |X|_{(1)} + |X|_{(n)}$, where $|X|_{(1)} = \min(|X_1|, \dots, |X_n|)$ and $|X|_{(n)} = \max(|X_1|, \dots, |X_n|)$ is an unbiased estimator of θ .
- (a) Compute the conditional distribution of $|X|_{(1)}$ knowing $|X|_{(n)} = x_n$.
- (b) Deduce that the Rao-Blackwell improved version of $\tilde{\theta}_n$ is $\hat{\theta}_n^*$.

◇ To do at Home ◇

Exercise 4. Let consider the statistical model $\mathcal{P} = \{f(\cdot | \theta); \theta \in \Theta\}$ associated to the Weibull distribution $\mathcal{W}(\theta, 1/2)$ whose density is given by

$$f(x | \theta) = \frac{1}{2\theta\sqrt{x}} \exp\left(-\frac{\sqrt{x}}{\theta}\right) \mathbb{1}_{\{x>0\}}$$

1. Show that this density belongs to a minimal and regular exponential family. For which set Θ is this density properly defined?
2. Write the likelihood $L(\theta | x_1, \dots, x_n)$ associated to a n -sample $(x_1, \dots, x_n)$ from this model.
3. Compute $\hat{\theta}_n$ the maximum of $\log L(\theta | x_1, \dots, x_n)$ and show that $\hat{\theta}_n$ converges in probability to θ .

Exercise 5. Let consider the statistical model $\{\mathcal{P}(\lambda), \lambda > 0\}$ and a n -sample $X_1, \dots, X_n$ from this model. We aim at estimating $\theta = \mathbb{P}[X = 0] = e^{-\lambda}$. We set

$$\hat{\theta}_n = \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{X_k=0\}}.$$

1. Show that $\hat{\theta}_n$ is an unbiased estimator of θ .
2. Justify that $T = \sum_{k=1}^n X_k$ is a sufficient statistic.
3. Compute $\mathbb{P}_\lambda[X_1 = 0 | T = t]$, for all $t \in \mathbb{N}$.
4. Using the Rao-Blackwell theorem, provide an improved version $\hat{\theta}_n^*$ of $\hat{\theta}_n$.
5. Show that $\hat{\theta}_n^*$ is the uniformly minimum variance unbiased estimator.

Exercise 6. Let X be a Poisson random variable with parameter $\theta > 0$ and $X_1, \dots, X_n$ a n -sample from X . We consider

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{and} \quad S_{n-1}^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

1. Show that $\bar{X}_n$ and S_{n-1}^2 are unbiased estimators of θ .
2. Show that $\bar{X}_n$ is the uniformly minimum variance unbiased estimator of θ and deduce that $\text{Var}_\theta[\bar{X}_n] \leq \text{Var}_\theta[S_{n-1}^2]$.