Practical n°2

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Objectives. This sheet illustrates convergence theorems useful in statistics.

Exercise 1 *(Law of Large Numbers).*

Theorem

If $X_1, \dots, X_n$ are *i.i.d.* random variables with well-defined expectation $\mathbb{E}[X_1]$, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \underset{n\to+\infty}{\longrightarrow} \mathbb{E}[X_1] \quad a.s.$$

1. Write a function `cum_mean` which returns the cumulative average of a sample.
2. Using a scatterplot, illustrate the Law of Large Numbers on a sequence of 1000 *i.i.d.* random variables distributed according to a Poisson distribution with mean 0.5.
3. What do you observe if you consider a sequence of 1000 *i.i.d.* random variables distributed according to the standard Cauchy distribution?

Exercise 2 *(Central Limit Theorem).*

Theorem

If $X_1, \dots, X_n$ are *i.i.d.* random variables with well-defined expectation $\mathbb{E}[X_1]$ and a finite variance $\mathbb{V}\mathrm{ar}[X_1] = \sigma^2$, then

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}[X_1]\right) \underset{n\to+\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(0, \sigma^2).$$

1. Consider a matrix $\mathbf{X} = (X_{ij}) \in \mathcal{M}_{n,p}(\mathbb{R})$ such that X_{ij} are *i.i.d.* random variables distributed according to the uniform $\mathcal{U}([0,2])$. The theorem states that, for all $j \in [\![1, p]\!]$, the random variables

$$Z_j = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_{ij} - \mathbb{E}[X_{11}]\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_{ij} - \sqrt{n} \underset{n\to+\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(0, 1/3). \tag{1}$$

   (a) Generate a realisation of $\mathbf{X}$ with $n = 50$ and $p = 1000$ and compute the associated *i.i.d.* sample $Z_1, \dots, Z_p$.

   (b) Draw the histogram of $Z_1, \dots, Z_p$ along with the density of $\mathcal{N}(0, 1/3)$. What do you observed ?

(c) Draw the Quantile-Quantile plot in order to compare the distribution of $Z_1, \ldots, Z_p$ with the Normal distribution $\mathcal{N}(0, 1/3)$.

(d) Do the same study for $n = 1000$

2. Let now assume that X_{ij} are *i.i.d.* random variables distributed according to the Student distribution with 2 degrees of freedom.

(a) Generate a realisation of $\mathbf{X}$ with $n = 1000$ and $p = 1000$ and compute the associated *i.i.d.* sample $Z_1, \ldots, Z_p$.

(b) Draw the histogram of $Z_1, \ldots, Z_p$ along with the density of $\mathcal{N}(0, \sigma^2)$, where $\sigma^2 = \texttt{sd}(Z_1, \ldots, Z_p)$. What do you observed ?

(c) Draw the Quantile-Quantile plot in order to compare the distribution of $Z_1, \ldots, Z_p$ with the Normal distribution $\mathcal{N}(0, \sigma^2)$. What do you observed ?