

Practical n°2

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Objectives This sheet illustrates convergence theorems useful in statistics.

Exercise 1 (*Law of Large Numbers*).

If $X_1, \dots, X_n$ are *i.i.d.* random variables with well-defined expectation $\mathbb{E}[X_1]$, then

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow[n \rightarrow +\infty]{} \mathbb{E}[X_1] \quad a.s.$$

1. Write a function `cum_mean` which returns the cumulative average of a sample.
2. Using a scatterplot, illustrate the Law of Large Numbers on a sequence of 1000 *i.i.d.* random variables distributed according to a Poisson distribution with mean 0.5.
3. What do you observe if you consider a sequence of 1000 *i.i.d.* random variables distributed according to the standard Cauchy distribution?

Exercise 2 (*Central Limit Theorem*).

If $X_1, \dots, X_n$ are *i.i.d.* random variables with well-defined expectation $\mathbb{E}[X_1]$ and a finite variance $\text{Var}[X_1] = \sigma^2$, then

$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X_1] \right) \xrightarrow[n \rightarrow +\infty]{d} \mathcal{N}(0, \sigma^2).$$

1. Consider a matrix $\mathbf{X} = (X_{ij}) \in \mathcal{M}_{n,p}(\mathbb{R})$ such that X_{ij} are *i.i.d.* random variables distributed according to the uniform $\mathcal{U}([0, 2])$. The theorem states that, for all $j \in \llbracket 1, p \rrbracket$, the random variables

$$Z_j = \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n X_{ij} - \mathbb{E}[X_{11}] \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_{ij} - \sqrt{n} \xrightarrow[n \rightarrow +\infty]{d} \mathcal{N}(0, 1/3). \quad (1)$$

- (a) Generate a realisation of $\mathbf{X}$ with $n = 50$ and $p = 1000$ and compute the associated *i.i.d.* sample $Z_1, \dots, Z_p$.
 - (b) Draw the histogram of $Z_1, \dots, Z_p$ along with the density of $\mathcal{N}(0, 1/3)$. What do you observed ?
 - (c) Draw the Quantile-Quantile plot in order to compare the distribution of $Z_1, \dots, Z_p$ with the Normal distribution $\mathcal{N}(0, 1/3)$.
 - (d) Do the same study for $n = 1000$
2. Let now assume that X_{ij} are *i.i.d.* random variables distributed according to the Student dis-

tribution with 2 degrees of freedom.

- (a) Generate a realisation of $\mathbf{X}$ with $n = 1000$ and $p = 1000$ and compute the associated *i.i.d.* sample $Z_1, \dots, Z_p$.
- (b) Draw the histogram of $Z_1, \dots, Z_p$ along with the density of $\mathcal{N}(0, \sigma^2)$, where $\sigma^2 = \text{sd}(Z_1, \dots, Z_p)$. What do you observed ?
- (c) Draw the Quantile-Quantile plot in order to compare the distribution of $Z_1, \dots, Z_p$ with the Normal distribution $\mathcal{N}(0, \sigma^2)$. What do you observed ?