

TD n°1

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Exercise 1. A company is developing a new type of sensor designed to operate continuously in industrial facilities. Before launching it on the market, the reliability department wishes to estimate its failure rate and deduce its mean lifetime.

Part I. In a first experimental campaign, each sensor is monitored until it fails. For every $\theta \in \mathbb{R}_+^*$, let $\mathbb{P}_\theta$ denote the exponential distribution with parameter θ . We observe a sample $X_1, \dots, X_n$ from $\mathbb{P}_\theta$, where θ is unknown.

1. Write the statistical model associated with one observation, and then with the sample.
2. Give a measure that dominates the model and write the likelihood.
3. Determine the maximum likelihood estimator of θ .
4. Show that it is also a Z -estimator associated with a function Ψ to be specified.
5. Show that this estimator is strongly consistent.
6. Establish its asymptotic normality and then construct an asymptotic confidence interval for θ with confidence level $1 - \alpha$, where $\alpha \in]0, 1[$.

Part II. The previous procedure may keep the laboratory occupied for a long time. To shorten a second experimental campaign, the study is stopped at a fixed time $c \in \mathbb{R}_+^*$, even if some sensors are still operating. For every $i \in \llbracket 1, n \rrbracket$, we observe $Y_i = \min(X_i, c)$. The likelihood is then given by

$$\theta \mapsto L_n(\theta) = \theta^{D_n} \exp(-\theta T_n), \quad D_n = \sum_{i=1}^n \mathbb{1}_{\{Y_i < c\}}, \quad T_n = \sum_{i=1}^n Y_i.$$

7. Study the existence and uniqueness of the maximum likelihood estimator $\hat{\theta}_n^{\text{MLE}}$ of θ . If it does not always exist, compute the probability of the event on which it fails to exist and study the behaviour of this probability as n tends to infinity.
8. Derive the method-of-moments estimator based on Y_1 . Specify the event on which it exists.
9. Extend both estimators arbitrarily by assigning them the same fixed value $\theta_\star > 0$ on the event $\{D_n = 0\}$. Show that they are strongly consistent.

The estimator $\hat{\theta}_n^{\text{MLE}}$ can be interpreted as a Z -estimator associated with $(u, y) \mapsto \mathbb{1}_{\{y < c\}}/u - y$. The asymptotic normality theorem for Z -estimators then gives

$$\sqrt{n} \left(\hat{\theta}_n^{\text{MLE}} - \theta \right) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N} \left(0, \frac{\theta^2}{1 - e^{-c\theta}} \right).$$

10. Comment on the effect of censoring on the asymptotic variance of the maximum likelihood estimator. What result is recovered as c tends to $+\infty$?

Exercise 2. Let $(\mathcal{Y}, \mathcal{A}, \{\mathbb{P}_\theta; \theta \in \Theta\})$ be a statistical model dominated by a σ -finite measure ν . Let $g : \Theta \rightarrow \tilde{\Theta}$ be a measurable bijection. For $\tau = g(\theta)$, consider the reparametrized model $\{\tilde{\mathbb{P}}_\tau = \mathbb{P}_{g^{-1}(\tau)}; \tau \in \tilde{\Theta}\}$.

1. Express the likelihood $\tilde{L}_n(\tau)$ of the reparametrized model in terms of the likelihood L_n associated with $\mathbb{P}_\theta$.
2. Show that

$$\arg \max_{\tau \in \tilde{\Theta}} \tilde{L}_n(\tau) = g \left(\arg \max_{u \in \Theta} L_n(u) \right).$$

Deduce the invariance of the maximum likelihood estimator under reparametrization.

3. In the exponential model of Exercise 1, set $\tau = 1/\theta$. Determine the maximum likelihood estimator of τ without maximizing a new likelihood.

Exercise 3. Let Θ be a compact subset of $\mathbb{R}^d$. For $\theta \in \Theta$, consider a continuous function $\mathbf{Z}_\theta : \Theta \rightarrow \mathbb{R}^d$ whose unique zero is θ . For every $n \geq 1$, let $\mathbf{Z}_n : \Theta \rightarrow \mathbb{R}^d$ be a random function, and let $\hat{\theta}_n^Z$ be a random variable taking values in Θ and satisfying

$$\mathbf{Z}_n(\hat{\theta}_n^Z) = \mathbf{0}.$$

Assume finally that

$$\sup_{u \in \Theta} \|\mathbf{Z}_n(u) - \mathbf{Z}_\theta(u)\| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}_\theta} 0.$$

1. Let $\varepsilon > 0$ be such that the set $K_\varepsilon = \{u \in \Theta; \|u - \theta\| \geq \varepsilon\}$ is nonempty. Show that K_ε is compact and that

$$c_\varepsilon = \inf_{u \in K_\varepsilon} \|\mathbf{Z}_\theta(u)\| > 0.$$

2. Show that

$$\|\mathbf{Z}_\theta(\hat{\theta}_n^Z)\| \leq \sup_{u \in \Theta} \|\mathbf{Z}_n(u) - \mathbf{Z}_\theta(u)\|.$$

3. Show that $\hat{\theta}_n^Z$ is a consistent estimator of θ . What happens when K_ε is empty?

◇ Further exercises ◇

Exercise 4 (*Extending Exercise 1 – Part II*). For the censored model, establish the asymptotic normality of the maximum likelihood estimator, i.e.,

$$\sqrt{n}(\hat{\theta}_n^{\text{MLE}} - \theta) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{\theta^2}{1 - e^{-c\theta}}\right),$$

by interpreting it as a Z -estimator associated with the function

$$(u, y) \mapsto \frac{\mathbb{1}_{\{y < c\}}}{u} - y.$$

Exercise 5. Let $(\mathcal{Y}, \mathcal{A}, \{\mathbb{P}_\theta; \theta \in \Theta\})$ be a statistical model dominated by two σ -finite measures ν

and μ . For every $u \in \Theta$, let p_u and q_u denote the densities of $\mathbb{P}_u$ with respect to ν and μ , respectively. For a sample $Y_1, \dots, Y_n$ from $\mathbb{P}_\theta$, let L_n^ν and L_n^μ denote the likelihoods associated with these two dominating measures.

1. Show that $\eta = \nu + \mu$ is a σ -finite measure that dominates ν , μ , and every distribution in the model.
2. For every $u \in \Theta$, set

$$a = \frac{d\nu}{d\eta}, \quad b = \frac{d\mu}{d\eta}, \quad f_u = \frac{d\mathbb{P}_u}{d\eta}.$$

Show that, for every $\theta \in \Theta$,

$$\mathbb{P}_\theta(S) = 1, \quad S = \{y \in \mathcal{Y}; a(y) > 0 \text{ and } b(y) > 0\}.$$

3. Deduce that, for every $u \in \Theta$,

$$q_u(y) = \frac{a(y)}{b(y)} p_u(y), \quad \eta\text{-almost everywhere on } S.$$

4. Deduce that, for every $\theta \in \Theta$,

$$\arg \max_{u \in \Theta} L_n^\mu(u) = \arg \max_{u \in \Theta} L_n^\nu(u), \quad \mathbb{P}_\theta^{\otimes n}\text{-almost surely}.$$

Exercise 6 (*Extending Exercise 3 – Is uniqueness sufficient without compactness?*). Let $\Theta = \mathbb{R}$ and $\theta = 0$. Consider the continuous function

$$Z_\theta : u \in \Theta \mapsto \frac{u}{1+u^2} \in \mathbb{R}.$$

1. Show that θ is the unique zero of Z_θ , but that this zero is not well separated.
2. For every $n \geq 1$, define Z_n for every $u \in \Theta$ by

$$Z_n(u) = Z_\theta(u) - Z_\theta(n).$$

Show that Z_n converges uniformly to Z_θ on Θ .

3. Verify that $\hat{\theta}_n^Z = n$ is a zero of Z_n . Is this sequence of zeros consistent? Conclude.