On the convergence of monotone schemes for path-dependent PDE *

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April 8, 2015

Abstract

We propose a reformulation of the convergence theorem of monotone numerical schemes introduced by Zhang and Zhuo [32] for viscosity solutions of path-dependent PDEs (PPDE), which extends the seminal work of Barles and Souganidis [1] on the viscosity solution of PDE. We prove the convergence theorem under conditions similar to those of the classical theorem in [1]. These conditions are satisfied, to the best of our knowledge, by all classical monotone numerical schemes in the context of stochastic control theory. In particular, the paper provides a unified approach to prove the convergence of numerical schemes for non-Markovian stochastic control problems, second order BSDEs, stochastic differential games etc.

Key words. Numerical analysis, monotone schemes, viscosity solution, path-dependent PDE

1 Introduction

In their seminal work [1], Barles and Souganidis proved a convergence theorem for monotone numerical schemes for viscosity solutions of fully nonlinear PDEs. Assuming that a strong comparison principle holds true for viscosity solutions of a PDE, they show that for all numerical schemes satisfying the three properties, "monotonicity", "consistency" and "stability", the numerical solutions converge locally uniformly to the unique viscosity solution of the PDE as the discretization parameters converge to zero. They mainly use the stability of viscosity solutions of PDEs and the local compactness of the state space. Due to their result, one only needs to check some local properties of a numerical scheme in order to get a global convergence result. Also, their

*The authors gratefully acknowledge the financial support of the ERC 321111 Rofirm, the ANR Isotace, the Chairs Financial Risks (Risk Foundation, sponsored by Société Générale), Finance and Sustainable Development (IEF sponsored by EDF and CA), and the region of l'île de France. The authors are also grateful to Ibrahim Ekren and Nizar Touzi for their helpful remarks and suggestions.

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result and method are widely used in the numerical analysis of viscosity solutions to PDEs.

It is well known that, by the Feynmann-Kac formula, the conditional expectation of a random variable can be characterized by a viscosity solution of the corresponding parabolic linear PDE. This relationship has also been generalized in the theory of BSDE (corresponding to semi-linear PDEs), 2BSDEs (corresponding to fully nonlinear PDEs) etc. However, the probabilistic tools have their PDE counterparts only in the Markovian case. Recently, a theory of path-dependent PDE (PPDE) has been developed by [9, 11, 12] etc., which permits to study non-Markovian problems. In particular, it provides a unified approach for many Markovian, or non-Markovian stochastic dynamic problems, e.g. BSDEs, second order BSDEs, stochastic control problems and stochastic differential games, etc.

It would be interesting to extend the convergence theorem of Barles and Souganidis [1] in the context of PPDE. The main obstacle for a direct extension of their arguments is that the state space is no longer locally compact. Zhang and Zhuo [32] provided recently a formulation of the convergence theorem of monotone schemes for PPDEs. They mainly use the stability of the viscosity solution of PPDE, and overcome the difficulty of non-local compactness by an optimal stopping argument as in the well-posedness theory of PPDE. They also provide an illustrative numerical scheme which satisfies all the conditions of their convergence theorem. However, this illustrative numerical scheme is not applicable in the general case. Moreover, most of the monotone numerical schemes in the sense of Barles and Souganidis [1], for example the finite difference scheme, do not satisfy their conditions.

Our main objective is to provide a new formulation of the convergence theorem for numerical schemes of PPDE. Our conditions are slightly stronger than the classical conditions of Barles and Souganidis [1], as PPDEs degenerate to be PDEs. Nevertheless, to the best of our knowledge these conditions are satisfied by all classical monotone numerical schemes in the optimal control context, including the classical finite difference scheme, the Monte-Carlo scheme of Fahim, Touzi and Warin [13], the semi-Lagragian scheme, the trinomial tree scheme of Guo, Zhang and Zhuo [15], the switching system scheme of Kharroubi, Langrené and Pham [19], etc. Therefore, our result extends all these numerical schemes to the path-dependent case. In particular, it provides numerical schemes for non-Markovian second order BSDEs, and stochastic differential games, which is new in the literature, see also Possamaï and Tan [24].

Similar to [32], we use an optimal stopping argument to overcome the difficulty of non-local compactness. Instead of looking into an optimal stopping problem of a controlled diffusion as in [32], we consider a discrete time optimal stopping problem of a controlled process. Therefore, our argument is quite different from that in [32].

The paper is organized as follows. In Section 2 we provide some preliminary notations used in the paper. In Section 3 we recall the definition of viscosity solution to the path-dependent PDE, and present our main result, that is, a convergence theorem of monotone schemes for PPDEs. Further we compare with the result of Guo, Zhang and Zhuo [15] and that of Barles and Souganidis [1]. In Section 4 we review some classical monotone schemes for PDEs, and verify that they satisfy the technical conditions of our main convergence theorem, and thus can be applied in the PPDE context. Finally, we complete the proof of the main theorem in Section 6.

2 Preliminaries

Throughout this paper let $T > 0$ be a given finite maturity, $\Omega := \{\omega \in C([0,T];\mathbb{R}^d) : \omega_0 = 0\}$ the set of continuous paths starting from the origin, and $\Theta := [0,T] \times \Omega$. We denote by B the canonical process on Ω, $\mathbb{F} = \{\mathcal{F}_t, 0 \le t \le T\}$ the canonical filtration, $\mathcal{T}$ the set of all $\mathbb{F}$-stopping times taking values in $[0,T]$, and $\mathbb{P}_0$ the Wiener measure on Ω. Moreover, let $\mathcal{T}^+$ denote the subset of $\tau \in \mathcal{T}$ taking values in $(0,T]$, and for $\textsc{h} \in \mathcal{T}$, let $\mathcal{T}_{\textsc{h}}$ and $\mathcal{T}^+_{\textsc{h}}$ be the subset of $\tau \in \mathcal{T}$ taking values in $[0,\textsc{h}]$ and $(0,\textsc{h}]$, respectively.

Following Dupire [8], we introduce the following pseudo-distance on Θ: for all $(t,\omega),(t',\omega') \in \Theta$,

$$\|\omega\|_t := \sup_{0\le s\le t} |\omega_s|, \qquad d\big((t,\omega),(t',\omega')\big) := |t-t'| + \|\omega_{t\wedge\cdot} - \omega'_{t'\wedge\cdot}\|_T.$$

Let E be a metric space, we say a process $X : \Theta \to E$ is in $C^0(\Theta, E)$ whenever $t \mapsto X_t(\omega)$ is continuous for all $\omega \in \Omega$. Similarly, $\mathbb{L}^0(\mathcal{F}, E)$ and $\mathbb{L}^0(\mathbb{F}, E)$ denote the set of all $\mathcal{F}$-measurable random variables and $\mathbb{F}$-progressively measurable processes, respectively. We remark that $C^0(\Theta, E) \subset \mathbb{L}^0(\mathbb{F}, E)$, and when $E = \mathbb{R}$, we shall omit it in these notations. We also denote by $\mathrm{BUC}(\Theta)$ the set of all functions bounded and uniformly continuous with respect to d.

For any $A \in \mathcal{F}_T$, $\xi \in \mathbb{L}^0(\mathcal{F}_T, E)$, $X \in \mathbb{L}^0(\mathbb{F}, E)$, and $(t,\omega) \in \Theta$, define respectively the shifted set, the shifted random variable and the shifted process by

$$A^{t,\omega} := \{\omega' \in \Omega : \omega \otimes_t \omega' \in A\}, \quad \xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \quad X^{t,\omega}_s(\omega') := X(t+s, \omega \otimes_t \omega')$$

where $\omega \otimes_t \omega'$ is the concatenated path defined as

$$(\omega \otimes_t \omega')_s := \omega_s \mathbf{1}_{[0,t]}(s) + (\omega_t + \omega'_{s-t})\mathbf{1}_{(t,T]}(s), \quad 0 \le s \le T.$$

Following the standard arguments with monotone class theorem, we have the following results.

Lemma 2.1. *Let $(t,\omega) \in \Theta$ and $s \in [t,T]$. Then $A^{t,\omega} \in \mathcal{F}_{s-t}$ for all $A \in \mathcal{F}_s$, $\xi^{t,\omega} \in \mathbb{L}^0(\mathcal{F}_{s-t}, E)$ for all $\xi \in \mathbb{L}^0(\mathcal{F}_s, E)$, $X^{t,\omega} \in \mathbb{L}^0(\mathbb{F}, E)$ for all $X \in \mathbb{L}^0(\mathbb{F}, E)$, and $\tau^{t,\omega} - t \in \mathcal{T}_{s-t}$ for all $\tau \in \mathcal{T}_s$.*

Next, let us introduce the nonlinear expectation. As in [11], we fix a constant $L > 0$ throughout the paper, and denote by $\mathcal{P}$ the collection of all continuous semimartingale measures $\mathbb{P}$ on Ω whose drift and diffusion coefficients are bounded by L. More precisely, a probability measure $\mathbb{P} \in \mathcal{P}$ if under $\mathbb{P}$, the canonical process B is a semimartingale with natural decomposition $B = A^{\mathbb{P}} + M^{\mathbb{P}}$, where $A^{\mathbb{P}}$ is a process of finite variation, $M^{\mathbb{P}}$ is a continuous martingale with quadratic variation $\langle M^{\mathbb{P}}\rangle$, such that $A^{\mathbb{P}}$ and $\langle M^{\mathbb{P}}\rangle$ are absolutely continuous in t, and

$$\|\mu^{\mathbb{P}}\|_\infty, \|a^{\mathbb{P}}\|_\infty \le L, \quad \text{where } \mu^{\mathbb{P}}_t := \frac{dA^{\mathbb{P}}_t}{dt},\ a^{\mathbb{P}}_t := \frac{d\langle M^{\mathbb{P}}\rangle_t}{dt}, \quad \mathbb{P}\text{-a.s.} \tag{2.1}$$

We then define the nonlinear expectations:

$$\overline{\mathcal{E}}[\cdot] := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\cdot] \quad \text{and} \quad \underline{\mathcal{E}}[\cdot] := \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\cdot]. \tag{2.2}$$

3 Convergence of monotone schemes for PPDE

We consider the following PPDE

$$-\partial_t u(t,\omega) \; - \; G\big(\cdot, u, \partial_\omega u, \partial^2_{\omega\omega} u\big)(t,\omega) \; = \; 0, \quad \text{for all} \;\; (t,\omega) \in [0,T) \times \Omega, \tag{3.1}$$

with the terminal condition $u(T,\cdot) = \xi$.

3.1 Definition of PPDE

As in the survey of Ren, Touzi and Zhang [25], one may define viscosity solution of path dependent PDE by using the jets. For $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^d, \gamma \in \mathbb{S}^d$, denote:

$$\phi^{\alpha,\beta,\gamma}(t,x) := \alpha t + \beta \cdot x + \frac{1}{2}\gamma : (xx^T) \quad \text{for all} \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

where $A_1 : A_2 := \mathrm{Tr}[A_1 A_2]$. Then define the semijets of a function $u \in \mathrm{BUC}(\Theta)$ at $(t,\omega) \in [0,T) \times \Omega$:

$$\underline{\mathcal{J}}u(t,\omega) := \Big\{(\alpha,\beta,\gamma) : u(t,\omega) = \max\nolimits_{\tau\in\mathcal{T}_{\mathrm{H}_\delta}} \overline{\mathcal{E}}[u^{t,\omega}_\tau - \phi^{\alpha,\beta,\gamma}_\tau], \text{ for some } \delta > 0\Big\},$$
$$\overline{\mathcal{J}}u(t,\omega) := \Big\{(\alpha,\beta,\gamma) : u(t,\omega) = \min\nolimits_{\tau\in\mathcal{T}_{\mathrm{H}_\delta}} \underline{\mathcal{E}}[u^{t,\omega}_\tau - \phi^{\alpha,\beta,\gamma}_\tau], \text{ for some } \delta > 0\Big\},$$

where $\mathrm{H}_\delta(\omega') := \delta \wedge \inf\{s \geq 0 : |\omega'_s| \geq \delta\} \in \mathcal{T}^+$ and $\phi^{\alpha,\beta,\gamma}_t := \phi^{\alpha,\beta,\gamma}(t, B_t)$.

Definition 3.1. *Let $u \in \mathrm{BUC}(\Theta)$.*
(i) *u is a $\mathcal{P}$-viscosity subsolution (resp. supersolution) of the path dependent PDE* (3.1)*, if at any point $(t,\omega) \in [0,T) \times \Omega$ it holds for all $(\alpha,\beta,\gamma) \in \underline{\mathcal{J}}u(t,\omega)$ (resp. $\overline{\mathcal{J}}u(t,\omega)$) that*

$$-\alpha - G(t,\omega,u(t,\omega),\beta,\gamma) \;\leq \textit{(resp. } \geq\textit{)}\; 0.$$

(ii) *u is a $\mathcal{P}$-viscosity solution of the path dependent PDE* (3.1)*, if u is both a $\mathcal{P}$-viscosity subsolution and a $\mathcal{P}$-viscosity supersolution of* (3.1)*.*

There are equivalent definitions of viscosity solution of path dependent PDE, for example in [25] we may find the definition in which one uses smooth test functions in the time-path space Θ. Here we are going to introduce another equivalent definition using constant localization and test functions in $C^{1,2}_0(\mathbb{R}^+ \times \mathbb{R}^d)$, i.e. the class of all $C^{1,2}$ scalar functions φ of which the partial derivatives $\partial_t\varphi, \partial_x\varphi, \partial^2_{xx}\varphi$ are of compact support. Consider the set of test functions:

$$\underline{\mathcal{A}}u(t,\omega) := \Big\{\varphi \in C^{1,2}_0(\mathbb{R}^+ \times \mathbb{R}^d) : (u^{t,\omega} - \varphi)_0 = \max_{\tau\in\mathcal{T}_\delta} \overline{\mathcal{E}}\big[(u^{t,\omega} - \varphi)_\tau\big], \text{ for some } \delta > 0\Big\},$$

$$\overline{\mathcal{A}}u(t,\omega) := \Big\{\varphi \in C^{1,2}_0(\mathbb{R}^+ \times \mathbb{R}^d) : (u^{t,\omega} - \varphi)_0 = \min_{\tau\in\mathcal{T}_\delta} \underline{\mathcal{E}}\big[(u^{t,\omega} - \varphi)_\tau\big], \text{ for some } \delta > 0\Big\},$$

where $\varphi_t = \varphi(t, B_t)$.

Proposition 3.2. *Assume that $G(t,\omega,y,z,\gamma)$ is continuous in (t,ω). A function u is a $\mathcal{P}$-viscosity subsolution (resp. supersolution) of Equation (3.1), if and only if at any point $(t,\omega)\in[0,T)\times\Omega$ it holds for all $\varphi\in\underline{\mathcal{A}}u(t,\omega)$ (resp. $\overline{\mathcal{A}}u(t,\omega)$) that*

$$\mathcal{L}^{t,\omega}\varphi_0 := -\partial_t\varphi_0 - G(t,\omega,u(t,\omega),\partial_x\varphi_0,\partial^2_{xx}\varphi_0) \ \le \text{(resp. } \ge\text{) } 0. \tag{3.2}$$

We will report the proof of the above proposition in Section 6. The next lemma is proved in Section 4.4 of [12].

Lemma 3.3. *Let $u\in\mathrm{BUC}(\Theta)$ be a $\mathcal{P}$-viscosity subsolution of PPDE (3.1). Then for the constant $L\in\mathbb{R}$, $\tilde{u}(t,\omega):=e^{-Lt}u(t,\omega)$ is a $\mathcal{P}$-viscosity subsolution of the PPDE*

$$-\partial_t\tilde{u}(t,\omega) - L\tilde{u}(t,\omega) - e^{-Lt}G\big(\cdot,e^{Lt}\tilde{u},e^{Lt}\partial_\omega\tilde{u},e^{Lt}\partial^2_{\omega\omega}\tilde{u}\big)(t,\omega)=0.$$

The similar result holds for supersolutions.

Since we only consider nonlinearity $G(t,\omega,y,z,\gamma)$ uniformly Lipschtiz in y, it follows from the previous lemma that without loss of generality we may assume that G is non-decreasing in y.

Remark 3.4 (Examples of path-dependent PDE)**.** One of the motivations of the PPDE theory is to characterize the value functions of non-Markovian stochastic control problems.

In particular, let consider a 2BSDE (Cheridito, Soner, Touzi and Victoir [4], Soner, Touzi and Zhang [28]) with generator $F:\Theta\times\mathbb{R}\times\mathbb{R}^d\times K\to\mathbb{R}$ and the controlled generating process with diffusion coefficient $\sigma:\Theta\times K\to S_d$, where K is some set in which the control processes take values. Then the solution of the 2BSDE corresponds to a PPDE with the nonlinearity:

$$G(t,\omega,y,z,\gamma) \ := \ \sup_{k\in K}\Big[\frac{1}{2}\sigma^2(t,\omega,k):\gamma + F(t,\omega,y,\sigma(t,\omega,k)z,k)\Big]. \tag{3.3}$$

Another example is the application of PPDEs in the stochastic differential games (see e.g. Pham and Zhang [23]), where the nonlinearity of PPDE turns to be of the form:

$$G(t,\omega,y,z,\gamma) := \sup_{k_1\in K_1}\inf_{k_2\in K_2}\Big[\frac{1}{2}\sigma^2(t,k_1,k_2):\gamma + F(t,\omega,y,\sigma(t,k_1,k_2)z,k_1,k_2)\Big]. \tag{3.4}$$

We refer to Section 4 of Ekren, Touzi and Zhang [11] for more details.

3.2 Main results

Definition 3.5. *Let $\{U_i, i\ge 1\}$ be a sequence of independent random variables defined on a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$. Every U_i follows the uniform distribution on $[0,1]$. Let $h>0$, K be a subset of a metric space, $\Phi_h: K\times[0,1]\to\mathbb{R}$ be a Borel measurable function such that for all $\nu\in K$ we have*

$$|\tilde{\mathbb{E}}\big[\Phi_h(v,U)\big]| \ \le \ Lh, \quad \mathrm{Var}\big[\Phi_h(\nu,U)\big]\le Lh \ \text{ and } \ \tilde{\mathbb{E}}\big[\Phi_h(\nu,U)^3\big]\le Lh^{3/2}. \tag{3.5}$$

Denote the filtration $\tilde{\mathbb{F}} := \{\tilde{\mathcal{F}}_i, i \in \mathbb{N}\}$*, where* $\tilde{\mathcal{F}}_n := \sigma\{U_i, i \le n\}$*. Let* $\mathcal{K} = \mathbb{L}^0(\tilde{\mathbb{F}}, K)$ *denote the collection of all* $\tilde{\mathbb{F}}$*-adapted control processes taking values in* K*. For all* $\nu \in \mathcal{K}$*, we define*

$$X^{h,\nu}_{(i+1)h} \;=\; X^{h,\nu}_{ih} \;+\; \Phi_h(\nu_{ih}, U_i). \tag{3.6}$$

Further, we denote by $\widehat{X}^{h,\nu} : [0,T] \times \tilde{\Omega} \to \Omega$ *the linear interpolation of the discrete process* $\{X^{h,\nu}_{ih}, i \in \mathbb{N}\}$ *such that* $\widehat{X}^{h,\nu}_{ih} = X^{h,\nu}_{ih}$ *for all* i*. Finally, for any function* $\varphi \in \mathbb{L}^0(\mathcal{F})$*, we define the nonlinear expectation:*

$$\underline{\mathcal{E}}_h[\varphi] \;:=\; \inf\nolimits_{\nu\in\mathcal{U}} \tilde{\mathbb{E}}\Big[\varphi\big(\widehat{X}^{h,\nu}\big)\Big] \quad \textit{and} \quad \overline{\mathcal{E}}_h[\varphi] \;:=\; \sup\nolimits_{\nu\in\mathcal{U}} \tilde{\mathbb{E}}\Big[\varphi\big(\widehat{X}^{h,\nu}\big)\Big]. \tag{3.7}$$

We next introduce the numerical schemes $\mathbb{T}$. Let $(t,\omega) \in [0,T) \times \Omega$ and $0 < h \le T-t$, $\mathbb{T}^{t,\omega}_h$ be a family of functions from $\mathbb{L}^0(\mathcal{F}_{t+h})$ to $\mathbb{R}$. We then define

$$u^h(t,\omega) \;:=\; \mathbb{T}^{t,\omega}_h u^h_{t+h},$$

and assume that $\mathbb{T}$ satisfies the following conditions.

Assumption 3.6. (i) *Consistency: for every* $(t,\omega) \in [0,T) \times \Omega$ *and* $\varphi \in C^{1,2}_0(\mathbb{R}^+ \times \mathbb{R}^d)$*,*

$$\lim_{(t',\omega',h)\to(t,0,0)} \frac{\varphi(t',\omega\otimes_t\omega') - \mathbb{T}^{t',\omega\otimes_t\omega'}_h\big[\varphi(t'+h,\cdot)\big]}{h} \;=\; \mathcal{L}^{t,\omega}\varphi_0.$$

(ii) *Monotonicity: there exists a nonlinear expectation* $\underline{\mathcal{E}}_h$ *as in Definition 3.5 such that, for any* $\varphi, \psi \in \mathbb{L}^0(\mathcal{F}_{t+h})$*, it holds that*

$$\mathbb{T}^{t,\omega}_h[\varphi] \;-\; \mathbb{T}^{t,\omega}_h[\psi] \;\ge\; \inf_{0\le\alpha\le L} \underline{\mathcal{E}}_h\big[e^{\alpha h}(\varphi-\psi)^{t,\omega}\big] \;-\; h\rho(h). \tag{3.8}$$

(iii) *Stability:* u^h *is uniformly bounded and uniformly continuous in* (t,ω)*, uniformly on* h*.*

Our main theorem is the following convergence result of the monotone scheme for PPDE (3.1).

Theorem 3.7. *Assume that*

- *PPDE* (3.1) *is parabolic, i.e.* $G(t,\omega,y,z,\gamma)$ *is nondecreasing in* γ*,*
- *the nonlinearity* G *of PPDE* (3.1) *and the terminal condition* ξ *are continuous in all arguments, and* $G(t,\omega,y,z,\gamma)$ *is uniformly Lipschitz in* y*,*
- *the comparison principle of viscosity solutions of* (3.1) *holds, i.e. if* $u, v \in \mathrm{BUC}(\Theta)$ *are* $\mathcal{P}$*-viscosity subsolution and supersolution of PPDE* (3.1)*, respectively, and* $u(T,\cdot) \le v(T,\cdot)$*, then* $u \le v$ *on* Θ*.*

If the numerical scheme $\mathbb{T}$ *satisfies Assumption 3.6, then PPDE* (3.1) *admits a unique bounded viscosity solution* u*, and*

$$u^h \;\to\; u \quad \textit{locally uniformly}, \quad \textit{as} \quad h \to 0. \tag{3.9}$$

Remark 3.8. A comparison result of viscosity solutions of fully nonlinear PPDEs is proved in Ekren, Touzi and Zhang [12] for PPDE (3.1) under certain conditions. Further, in the case of semilinear PPDEs, a comparison result is proved in Ren, Touzi and Zhang [26] under very general assumptions.

Remark 3.9 (Comparison with Zhang and Zhuo [32])**.** Let us compare our Assumption 3.6 with that in [32]. Our condition (i) is weaker and thus easier to verify comparing to that in [32]. The essential difference is between our condition (ii) and theirs. Our condition (ii), although stated in a complicated way, is satisfied by all (to the best of our knowledge) classical monotone scheme in PDE context. Moreover, by the interpretation of the finite difference scheme for stochastic control problem as controlled Markov chains (see Kushner and Dupuis [20]), this condition is consistent with the classical one in [1].

Comparison with Barles and Souganidis's theorem

When a PPDE degenerates to be a PDE:

$$\mathsf{L}u(t,x) \;:=\; -\partial_t u(t,x) - G_0(\cdot,u,\partial_x u,\partial^2_{xx}u)(t,x) \;=\; 0, \quad \text{on} \quad [0,T)\times\mathbb{R}^d, \tag{3.10}$$

with the terminal condition $u(T,\cdot)=g$. Note that the definition of viscosity solution of PDE is slightly different from that of PPDE recalled in Section 3.1, but under general conditions a viscosity solution of PDE (3.10) is a viscosity solution of the corresponding PPDE.

Assumption 3.10. (i) *The terminal condition g is bounded continuous.*
(ii) *The function G_0 is continuous and $G_0(t,x,y,z,\gamma)$ is nondecreasing in γ.*
(iii) *PDE* (3.10) *admits a comparison principle for bounded viscosity solution, i.e. if u,v are bounded viscosity subsolution and supersolution of PDE* (3.10)*, respectively, and $u(T,\cdot)\le v(T,\cdot)$, then $u\le v$ on $[0,T]\times\mathbb{R}^d$.*

For any $t\in[t,T)$ and $h\in(0,T-t]$, let $\mathbb{T}^{t,x}_h$ be an operator on the set of bounded functions defined on $\mathbb{R}^d$. For $n\ge 1$, denote $h:=\frac{T}{n}<T-t$, $t_i=ih$, $i=0,1,\cdots,n$, let the numerical solution be define by

$$u^h(T,x):=g(x),\quad u^h(t,x):=\mathbb{T}^{t,x}_h[u^h(t+h,\cdot)],\quad t\in[0,T),\quad i=n,\cdots,1.$$

Assumption 3.11. (i) *Consistency: for any $(t,x)\in[0,T)\times\mathbb{R}^d$ and any smooth function $\varphi\in C^{1,2}([0,T)\times\mathbb{R}^d)$,*

$$\lim_{(t',x',h,c)\to(t,x,0,0)}\frac{(c+\varphi)(t',x')-\mathbb{T}^{t',x'}_h\big[(c+\varphi)(t'+h,\cdot)\big]}{h} \;=\; \mathsf{L}\varphi(t,x).$$

(ii) *Monotonicity: $\mathbb{T}^{t,x}_h[\varphi]\le\mathbb{T}^{t,x}_h[\psi]$ whenever $\varphi\le\psi$.*
(iii) *Stability: u^h is bounded uniformly in h whenever g is bounded.*
(iv) *Boundary condition: $\lim_{(t',x',h)\to(T,x,0)}u^h(t',x')=g(x)$ for any $x\in\mathbb{R}^d$.*

We now recall the convergence theorem of the monotone scheme, deduced from Barles and Souganidis [1] in this context of the parabolic PDE (3.10).

Theorem 3.12. *Let the generator function G_0 in* (3.10) *and the terminal condition g satisfy Assumption 3.10, and the numerical scheme $\mathbb{T}_h^{t,x}$ satisfy Assumption 3.11. Then the parabolic PDE* (3.10) *has a unique bounded viscosity solution and u^h converges to u locally uniformly as $h \to 0$.*

Remark 3.13 (Comparison with Assumption 3.6)**.** (i) First, Assumption 3.6 (i) reduces exactly to be Assumption 3.11 (i) when the PPDE degenerates to a PDE.

(ii) Assumption 3.6 (ii) is stronger than Assumption 3.11 (ii). It is clear that the bound (3.8) of the difference of numerical solutions, provided by the discrete sublinear expectation, implies the monotonicity condition in Assumption 3.11 in the PDE case. In their book of numerical methods for stochastic control problem, Kushner and Dupuis [20] studied the classical finite-difference scheme, and highlighted that the monotonicity condition is in fact equivalent to a controlled Markov chain interpretation, where the increments of the Markov chain satisfy (3.5). Our formulation of the monotonicity in Assumption 3.6 (ii) is consistent with this spirit. In particular, for concrete numerical schemes, the two monotonicity formulations demand exactly the same conditions on the coefficients. Moreover, it is satisfied by all classical monotone scheme, to the best of our knowledge, in the context of stochastic control theory. See also our review in Section 4.

(iii) The stability condition in Assumption 3.6 (iii) is also stronger than Assumption 3.11 (iii). Nevertheless, in the classical numerical analysis for parabolic PDE (3.10), in order to check Assumption 3.11 (iv), one needs (explicitly or implicitly) to prove a uniform continuity property of numerical solutions uniformly on the discretization parameter, which leads to the same condition as in Assumption 3.6 (iii). See also our review in Section 4.

4 Examples of monotone schemes

We discuss here some classical monotone numerical schemes in the stochastic control context, and provide some sufficient conditions Assumption 3.6 to hold true. Let us first add some assumptions on the functions G and ξ for PPDE (3.1).

Assumption 4.1. *The terminal condition ξ is Lipschitz in ω, G is increasing in γ, and G is Lipschitz in (y, z, γ): i.e. there is some constant C such that for all $(t, \omega) \in \Theta$ and $(y, z, \gamma), (y', z', \gamma') \in \mathbb{R} \times \mathbb{R}^d \times S_d$,*

$$\Big| G(t, \omega, y, z, \gamma) - G(t, \omega, y, z', \gamma') \Big| \;\; \leq \;\; C\Big(\big|y - y'\big| \; + \; \big|z - z'\big| \; + \; \big|\gamma - \gamma'\big| \Big).$$

In this section, we denote $t_k := hk$ for $h = \Delta t > 0$. Given $\mathbf{x} = (\mathbf{x}_{t_0}, \mathbf{x}_{t_1}, \cdots, \mathbf{x}_{t_k})$ a sequence of points in $\mathbb{R}^d$, we denote by $\widehat{\mathbf{x}} \in \Omega$ the linear interpolation of $\mathbf{x}$ such that $\widehat{\mathbf{x}}_{t_i} = \mathbf{x}_{t_i}$ for all i. Further, for $(t, \omega) \in \Theta$, $h > 0$ and $z \in \mathbb{R}^d$, we define a path

$$(\omega \otimes_t^h z) \;:=\; \omega \otimes_t z^h, \quad \text{where} \;\; z_s^h \;:=\; \begin{cases} \dfrac{s}{h} z, & \text{for} \;\; 0 \leq s \leq h; \\ z, & \text{for} \;\; s > h. \end{cases}$$

Let E be some normed vector space, then for maps $\psi : \Theta \to E$, we introduce the norm $|\psi|_0$ and $|\psi|_1$ by

$$|\psi|_0 \ := \ \sup_{(t,\omega)\in\Theta} |\psi(t,\omega)| \quad \text{and} \quad |\psi|_1 \ := \ \sup_{(t,\omega)\neq(t',\omega')} \frac{|\psi(t,\omega)-\psi(t',\omega')|}{(|\omega_{t\wedge\cdot}-\omega'_{t'\wedge\cdot}|+|t-t'|^{1/2}}.$$

4.1 Finite difference scheme

For simplicity, we assume that the state space is of dimension one ($d=1$). Let $\Delta x > 0$ be the space discretization size. For every $(t,\omega)\in\Theta$, $h>0$ and $\mathcal{F}_{t+h}$-measurable random variable $\psi : \Omega \to \mathbb{R}$, we define the discrete derivatives

$$\mathcal{D}_h\psi(t,\omega) := \big(\mathcal{D}_h^0\psi, \mathcal{D}_h^1\psi, \mathcal{D}_h^2\psi\big)(t,\omega),$$

where

$$\mathcal{D}_h^0\psi(t,\omega) \ := \ \psi(\omega_{t\wedge\cdot}), \quad \mathcal{D}_h^1\psi(t,\omega) \ := \ \tfrac{\psi(\omega\otimes_t^h\Delta x)-\psi(\omega_{t\wedge\cdot})}{\Delta x},$$
$$\text{and} \quad \mathcal{D}_h^2\psi(t,\omega) \ := \ \tfrac{\psi\left(\omega\otimes_t^h\Delta x\right)-2\psi(\omega_{t\wedge\cdot})+\psi\left(\omega\otimes_t^h(-\Delta x)\right)}{\Delta x^2}.$$

Then an explicit finite difference scheme is given by

$$\mathbb{T}_h^{t,\omega}[u_{t+h}^h] \ := \ u^h(t+h,\omega_{t\wedge\cdot}) + hG\big(t,\omega,\mathcal{D}_h u_{t+h}^h(t,\omega)\big). \tag{4.1}$$

Proposition 4.2. *Suppose that Assumption 4.1 holds true and G is Lipschitz in ω, i.e. there is a constant C such that for all $\omega,\omega'\in\Omega$ and all $(t,y,z,\gamma)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times S_d$,*

$$\Big|G(t,\omega,y,z,\gamma)-G(t,\omega',y,z,\gamma)\Big| \ \le \ C\big\|\omega_{t\wedge\cdot}-\omega'_{t\wedge\cdot}\big\|.$$

Assume in addition the CFL (Courant-Friedrichs-Lewy) condition, i.e.

$$\varepsilon \ \le \ \frac{\Delta t|\nabla_\gamma G|_0}{\Delta x^2} \ \le \ \frac{1}{2}-\varepsilon, \tag{4.2}$$

and that $\nabla_\gamma G \ge \varepsilon$ for some small constant $\varepsilon > 0$. Then Assumption 3.6 holds true for finite difference scheme (4.1)*. In particular, the numerical solution u^h is $\frac{1}{2}$–Hölder in t and Lipschitz in ω, uniformly on h.*

Proof. We will check each condition in Assumption 3.6. For the simplicity of presentation, we assume that G is independent of y. Clearly, the argument still works if G is Lipschitz in y.

(i) The consistency condition (Assumption 3.6 (i)) is obviously satisfied by (4.1) as in the no path-dependent case.

(ii) For the monotonicity in Assumption 3.6 (ii), let us consider two different bounded functions φ and ψ. Denote $\phi := \varphi - \psi$, then by direct computation,

$$\mathbb{T}_h^{t,\omega}[\varphi]-\mathbb{T}_h^{t,\omega}[\psi] \ = \ \phi(\omega^0) \ + \ h\Big(G_y\mathcal{D}_h^{0,t}\phi + G_z\mathcal{D}_h^{1,t}\phi + G_\gamma\mathcal{D}_h^{2,t}\phi\Big),$$

where G_y, G_z and G_γ is some function depending on (t,ω) and (φ,ψ), but uniformly bounded by the Lipschitz constant L of G. Let $b\in[-L,L]$ and $\varepsilon \le a \le |\nabla_\gamma G|_0$ be two constants, and $\zeta^{a,b}$ be a random variable defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that

$$\tilde{\mathbb{P}}(\zeta^{a,b} = 0) \;=\; 1 - b\tfrac{\Delta t}{\Delta x} - 2a\tfrac{\Delta t}{\Delta x^2},$$
$$\tilde{\mathbb{P}}(\zeta^{a,b} = \Delta x) \;=\; b\tfrac{\Delta t}{\Delta x} + a\tfrac{\Delta t}{\Delta x^2} \quad \text{and} \quad \tilde{\mathbb{P}}(\zeta^{a,b} = -\Delta x) \;=\; a\tfrac{\Delta t}{\Delta x^2}.$$

The law of $\zeta^{a,b}$ is well defined for $\Delta t = h$ small enough, because every term above is positive and the sum of all terms equals to 1 under condition (4.2). Further, we have

$$\tilde{\mathbb{E}}\big[\zeta^{a,b}\big] = bh, \quad \mathrm{Var}\big[\zeta^{a,b}\big] = ah, \quad \text{and} \quad \tilde{\mathbb{E}}\big[|\zeta^{a,b}|^3\big] \;\le\; |\Delta x|^3 \;\le\; Ch^{3/2}, \tag{4.3}$$

where the last terms follows by $\Delta x \approx h^{\frac{1}{2}}$.

Then let $F_h(a,b,\cdot) : \mathbb{R} \to [0,1]$ be the distribution function of $\zeta^{a,b}$ and $\Phi_h(a,b,\cdot) : [0,1] \to \mathbb{R}$ be the generalized inverse function of $F_h(a,b,\cdot)$, i.e.

$$\Phi_h(a,b,x) \;\;:=\;\; \inf\{y : F_h(a,b,y) > x\}. \tag{4.4}$$

In view of (4.3), the monotonicity condition of Assumption 3.6 (ii) holds true.

(iii) To prove Assumption 3.6 (iii), we will prove that there is a constant C independent of h such that

$$\big|u^h(t,\omega) - u^h(t',\omega')\big| \;\;\le\;\; C\Big(\|\omega_{t\wedge\cdot} - \omega'_{t'\wedge\cdot}\| + \sqrt{|t'-t|}\Big), \quad \forall (t,\omega),(t',\omega') \in \Theta. \tag{4.5}$$

Let us first prove that u^h is Lipschtiz in ω. Denote

$$L^h_t \;\;:=\;\; \sup_{(t,\omega),(t',\omega')\in\Theta} \frac{u^h(t,\omega) - u^h(t,\omega')}{\|\omega_{t\wedge\cdot} - \omega'_{t\wedge\cdot}\|} 1_{\{\|\omega_{t\wedge\cdot} - \omega'_{t\wedge\cdot}\|>0\}}.$$

By direct computation, we have

$$u^h(t,\omega) \;-\; u^h(t,\omega') \;\;=\;\; hG_\omega\|\omega_{t\wedge\cdot} \;-\; \omega'_{t\wedge\cdot}\| \;+\; \tilde{D}^G_h u^h_{t+h}(t,\omega) \;-\; \tilde{D}^G_h u^h_{t+h}(t,\omega'), \tag{4.6}$$

where

$$\tilde{D}^G_h u^h_{t+h} \;:=\; \big((1+hG_y)\mathcal{D}^0_h + hG_z\mathcal{D}^1_h + hG_\gamma\mathcal{D}^2_h\big)u^h_{t+h},$$

with G_y, G_z and G_γ uniformly bounded by L. Then there is a constant C independent of h such that

$$L^h_t \;\;\le\;\; (1+Ch)L^h_{t+h} + Ch.$$

Notice that the terminal condition ξ is Lipschitz, it follows by the discrete Gronwall inequality, we have $L^h_t \le Ce^{CT}$ for a constant C independent of h. Hence, there is a constant C' independent of h such that

$$\big|u^h(t,\omega) - u^h(t,\omega')\big| \;\;\le\;\; C'\|\omega_{t\wedge\cdot} - \omega'_{t\wedge\cdot}\|, \quad \forall t \in [0,T],\; \omega,\omega' \in \Omega. \tag{4.7}$$

We next consider the regularity of u^h in t. Let $t := ih$ and $t' := jh > t$. Note that

$$u^h(t,\omega) = u^h(t+h,\omega_{t\wedge\cdot}) + hG(t,\omega,0,0,0) + h\big(G(t,\omega,\mathcal{D}_h u^h_{t+h}(t,\omega)) - G(t,\omega,0,0,0)\big).$$

By a direct computation, we have

$$u^h(t,\omega) \;=\; \tilde{\mathbb{E}}\Big[\sum_{k=i}^{j-1} G(t_k,\omega\otimes_t \widehat{X}^h,0,0,0)\;h \;+\; u^h(t',\omega\otimes_t \widehat{X}^h)\Big], \tag{4.8}$$

where X^h is a discrete process defined as $X^h_0 := 0$,

$$X^h_{t_{k+1}} \;:=\; X^h_{t_k} \;+\; \Phi_h(\nabla_\gamma G, \nabla_z G, U_{k+1}),$$

with Φ_h be given by (4.4), and $\widehat{X}^h$ is the linear interpolation of X^h. Define

$$A^h_0 := 0, \quad A^h_{t_k} := \sum_{i=0}^{k-1} \tilde{\mathbb{E}}\big[\Phi_h(\nabla_\gamma G, \nabla_z G, U_{i+1})\big|\tilde{\mathcal{F}}_i\big], \quad \text{and} \quad M^h := X^h - A^h.$$

Clearly, M^h is a martingale and A^h is a predictable process. Further, it follows from the property of Φ_h in (4.3) that

$$\tilde{\mathbb{E}}\Big|A^h_{t_{k+1}} - A^h_{t_k}\Big| \le Lh \quad \text{and} \quad \mathrm{Var}\Big[M^h_{t_{k+1}} - M^h_{t_k}\Big] \le Lh.$$

Then by (4.8), we have

$$|u(t,\omega) - u(t',\omega_{t\wedge\cdot})| \;\le\; C(t'-t) \;+\; C\tilde{\mathbb{E}}\Big[\sup_{i\le k\le j}\big|M_{t_k}\big|\Big]. \tag{4.9}$$

Further, by Doob's inequality, it follows that

$$\tilde{\mathbb{E}}\Big[\sup_{i\le k\le j}\big|M^h_{t_k}\big|\Big] \;\le\; \sqrt{\tilde{\mathbb{E}}\Big[\sup_{i\le k\le j}\big|M^h_{t_k}\big|^2\Big]} \;\le\; 2\Big(\sqrt{\tilde{\mathbb{E}}\big[(M^h_{t_j})^2\big]}\Big) \;\le\; C\sqrt{t_j - t_i}.$$

Finally, combining the above estimation with (4.7) and (4.9), we obtain (4.5). ∎

Remark 4.3. We here assume that the PPDE is non-degenerate ($\nabla_\gamma G \ge \varepsilon > 0$). When $\nabla_\gamma G = 0$ and $\nabla_z G \ge 0$, the scheme is still monotone. When $\nabla_\gamma G = 0$ and $\nabla_z G \le 0$, it is possible to redefine the first order discrete derivative by

$$\mathcal{D}^1_h\psi(t,\omega) \;:=\; \frac{\psi(\omega_{t\wedge\cdot}) - \psi(\omega\otimes^h_t(-\Delta x))}{\Delta x}$$

to obtain a monotone scheme.

Remark 4.4. In the multidimensional case, $\nabla_\gamma G$ is a matrix. If $\nabla_\gamma G$ is diagonal dominated, then following Kushner and Dupuis [20], it is easy to construct a monotone scheme under similar CFL condition (4.2). When $\nabla_\gamma G$ is not diagonal dominated, it is possible to use the generalized finite difference scheme proposed by Bonnans, Ottenwaelter and Zidani [2].

4.2 The trinomial tree scheme of Guo-Zhang-Zhuo [15]

We consider the PPDE of the form (3.1). Let σ_0 be some symmetric $d\times d$ matrix, denote

$$F(t,\omega,y,z,\gamma) \;:=\; G(t,\omega,y,z,\gamma)-\frac{1}{2}\sigma_0^2:\gamma,\quad \tilde G_\gamma \;:=\; \sigma_0^{-1}G_\gamma\sigma_0^{-1}.$$

Let $\zeta=(\zeta_1,\cdots,\zeta_d)$ a random vector defined on a probability space $(\tilde\Omega,\tilde{\mathcal F},\tilde{\mathbb P})$ such that $\zeta_i, i=1,\cdots,d$ are i.i.d and

$$\tilde{\mathbb P}(\zeta_i=\frac{1}{\sqrt p})=\frac{p}{2},\quad \tilde{\mathbb P}(\zeta_i=-\frac{1}{\sqrt p})=\frac{p}{2},\quad \tilde{\mathbb P}(\zeta_i=0)=1-p,\quad \text{with } p\in(0,1).$$

For every $\mathcal F_{t+h}$-measurable function $\psi:\Omega\to\mathbb R$, let us define $\mathcal D_h^i\psi(t,\omega):=\tilde{\mathbb E}\Big[\psi\big(\omega\otimes_t^h(\sqrt h\sigma_0\zeta)\big)K_i(\zeta)\Big]$ with

$$K_0:=1,\quad K_1:=\frac{\sigma_0^{-1}\zeta}{\sqrt h},\quad K_2:=\frac{\sigma_0^{-1}[(1-p)\zeta\zeta^T-(1-3p)\mathrm{Diag}[\zeta\zeta^T]-2pI_d]\sigma_0^{-1}}{(1-p)h},$$

where for any matrix $\gamma=[\gamma_{i,j}]_{1\le i,j\le d}\in S_d$, $\mathrm{Diag}[\gamma]$ denotes the diagonal matrix whose (i,i)-th component is γ_{ii}. Then the numerical scheme is defined as

$$\mathbb T_h^{t,\omega}[u^h(t+h,\cdot)] \;\;:=\;\; \mathcal D_h^0u^h(t,\omega)\;+\;hF\big(\cdot,\mathcal D_hu_{t+h}^h\big)(t,\omega). \tag{4.10}$$

Proposition 4.5. *Let Assumptions 4.1 hold true and G is Lipschitz in ω. Suppose in addition that Assumption 3.3 in Guo, Zhang and Zhuo [15] holds true (where we replace their notation $\tilde G_\gamma$ by $\nabla_\gamma\tilde G$ in our context). Then the trinomial tree scheme* (4.10) *satisfies Assumption 3.6.*

Proof. The consistency and monotonicity condition in Assumption 3.6 (i) and (ii) can be justified by almost the same argument as in [15]. Similarly to the finite difference scheme, the monotonicity in sense of Barles and Souganidis [1] implies the interpretation of the controlled discrete processes of the numerical scheme, which implies the monotonicity condition (3.8) in our context. Further, using the same argument as in Proposition 4.2, it is easy to show that

$$\big|u^h(t,\omega)-u^h(t',\omega')\big| \;\;\le\;\; C\Big(\|\omega_{t\wedge\cdot}-\omega'_{t'\wedge\cdot}\|+\sqrt{|t'-t|}\Big),\quad \forall(t,\omega),(t',\omega')\in\Theta,$$

for some constant C independent of h, which implies in particular (iii) of Assumption 3.6. ■

Remark 4.6. As a PPDE degenerates to be a classical PDE, the conditions in Proposition 4.5 turns to be exactly the same conditions in Theorem 3.10 of [15].

4.3 The probabilistic scheme of Fahim-Touzi-Warin [13]

We consider PPDE (3.1) in which G is in the form of

$$G(t,\omega,y,z,\gamma) \;\;=\;\; \mu(t,\omega)\cdot z\;-\;\frac{1}{2}\sigma\sigma^T(t,\omega):\gamma\;-\;F(t,\omega,y,z,\gamma).$$

Before introducing the numerical scheme, we first define a random vector

$$X_h^{(t,\omega)} \;\; := \;\; \mu(t,\omega)h \; + \; \sigma(t,\omega)W_h,$$

where $W_h \sim N(0, hI_d)$ is a Gaussian vector. For every bounded function $\psi \in \mathbb{L}^0(\mathcal{F}_{t+h})$, we define

$$\mathcal{D}_h\psi(t,\omega) \;\; := \;\; \mathbb{E}\Big[\psi(\omega \otimes_t \widehat{X}_{\cdot}^{(t,\omega)})H_h(t,\omega)\Big],$$

where $H_h(t,\omega) = (H_0^h, H_1^h, H_2^h)^T$ with

$$H_0^h := 1, \quad H_1^h := (\sigma^T(t,\omega))^{-1}\tfrac{W_h}{h}, \quad H_2^h := (\sigma^T(t,\omega))^{-1}\frac{W_h W_h^T - hI_d}{h^2}\sigma^{-1}(t,\omega).$$

Then the probabilistic scheme is given by

$$\mathbb{T}_h^{t,\omega}[u^h(t+h,\cdot)] \;:=\; \mathbb{E}\Big[u^h(t+h, \widehat{X}^{(t,\omega)})\Big] + hF\big(\cdot, \mathcal{D}_h u_{t+h}^h\big)(t,\omega). \tag{4.11}$$

Remark 4.7. The probabilistic scheme in [13] is inspired by the second order BSDE theory of Cheridito, Soner, Touzi and Victoir [4], and extends the classical numerical scheme of BSDE (see e.g. Bouchard and Touzi [3], Zhang [31]). In practice, one can use the simulation-regression method to estimate the conditional expectation in the above scheme (see e.g. Gobet, Lemor and Warin [14]). We refer to Guyon and Henry-Labordère [16] for more details on the use of the scheme, to Tan [29] for an extension to a degenerate case, and to Tan [30] for an extension to path-dependent control problems.

Assumption 4.8. (i) *The nonlinearity F is Lipschtiz w.r.t. (ω, u, z, γ) uniformly in t and $|F(\cdot,\cdot,0,0,0)|_0 < \infty$.*
(ii) *F is elliptic and dominated by the diffusion term of X, that is,*

$$\nabla_\gamma F \le \sigma\sigma^T, \quad on \quad \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d. \tag{4.12}$$

(iii) *$\nabla_p F \in Image(\nabla_\gamma F)$ and $\big|(\nabla_p F)^T(\nabla_\gamma F)^{-1}\nabla_p F\big|_0 < \infty$.*
(iv) *$|\mu|_1, |\sigma|_1 < \infty$ and σ is invertible and ξ is bounded Lipschitz.*

Proposition 4.9. *Suppose that Assumption 4.8 holds true. Then the probabilistic numerical scheme* (4.11) *satisfies Assumption 3.6.*

Proof. (i) Assumption 3.6 (i) is obviously satisfied in view of Lemma 3.11 of [13].
(ii) Further, using probabilistic interpretation of this scheme in Tan [30, Section 3.2], we may verify (ii) of Assumption 3.6. See also the estimation given by Lemma 3.1 of [30].
(iii)For (iii)of Assumption 3.6, we shall prove that the numerical solution u^h is Lipschitz in ω and 1/2-Hölder in t. In [13], the authors proved this property in the case of PDEs. Their arguments for the Lipschitz continuity in ω can be easily adapted to this path-dependent case. For the regularity of u^h in t, they used a regularization technique, which seems impossible to be adapted to the path-dependent case. However, we can still use similar arguments as in Proposition 4.2, i.e. use the discrete-time controlled semimartingale interpretation, to prove the Hölder property of u^h in t. ∎

Remark 4.10. As a PPDE degenerates to be a PDE, the conditions in Assumption 4.8 reduce exactly the same conditions as in [13] (see their Theorem 3.6).

4.4 The semi-Lagrangian scheme

For the semi-Lagrangian scheme, we shall consider the PPDE (3.1) of the Bellman-Issac type, i.e. the function G is in the form of

$$G(t,\omega,y,z,\gamma) \;=\; \inf_{k_1\in K_1}\sup_{k_2\in K_2}\Big(\frac{1}{2}a^{k_1,k_2}(\cdot):\gamma + b^{k_1,k_2}(\cdot)\cdot z + c^{k_1,k_2}(\cdot)y + f^{k_1,k_2}(\cdot)\Big)(t,\omega),$$

where K_1 and K_2 are some sets, $(a^{k_1,k_2}, b^{k_1,k_2}, c^{k_1,k_2}, f^{k_1,k_2})$ are functionals defined on Θ.

Let ζ be a random vector satisfying

$$\mathbb{E}\big[\zeta\big]=0, \quad \mathrm{Var}\big[\zeta\big]=I_d \quad \text{and} \quad \mathbb{E}\big[|\zeta|^3\big]<\infty. \tag{4.13}$$

Then the semi-Lagragian scheme is defined as

$$\begin{aligned}\mathbb{T}_h^{t,\omega}[u^h(t+h,\cdot)] \;:=\; \inf_{k_1\in K_1}\sup_{k_2\in K_2}\Big\{& u^h\big(t+h,\omega\otimes_t\big(\sigma^{k_1,k_2}(t,\omega)\zeta\sqrt{h}+b^{k_1,k_2}(t,\omega)h\big)\big)\\ &+u^h\big(t+h,\omega\big)c^{k_1,k_2}(t,\omega)h \;+\; f^{k_1,k_2}(t,\omega)h\Big\}. \end{aligned}\tag{4.14}$$

Proposition 4.11. *Suppose that $|a|_1+|b|_1+|c|_1+|f|_1<\infty$, and* (4.13) *holds true. Then the semi-Lagrangian scheme* (4.14) *for the Bellman-Issac path-dependent equation satisfies Assumption 3.6.*

Proof. (i) The consistency condition (Assumption 3.6 (i)) is easy to check.
(ii) Let E be a set, $e: K_1\times K_2\to E$ be an arbitrary mapping, and $\psi,\varphi: E\to\mathbb{R}$ be two bounded functions. Note that

$$\inf_{k_1\in K_1}\sup_{k_2\in K_2}\psi(e(k_1,k_2)) - \inf_{k_1\in K_1}\sup_{k_2\in K_2}\varphi(e(k_1,k_2)) \le \sup_{k_1\in K_1,k_2\in K_2}(\psi-\varphi)(e(k_1,k_2)).\tag{4.15}$$

Notice that $\mathbb{R}^d$ is isomorphic to $\mathbb{R}$, we can always consider the random vector $\sigma^{k_1,k_2}\zeta\sqrt{h}+b^{k_1,k_2}h$ as a one-dimensional random variable. By consider the inverse function of its distribution function, then there is a family $\Phi_h(k_1,k_2,\cdot)$ such that $\Phi^h(k_1,k_2,U)\sim\sigma^{k_1,k_2}\zeta\sqrt{h}+b^{k_1,k_2}h$ in law with $U\sim\mathcal{U}([0,1])$, for all $(k_1,k_2)\in K_1\times K_2$. Then it follows from (4.15) that the monotonicity condition in Assumption 3.6 (ii) holds true with $\Phi_h(k_1,k_2,\cdot)$ and $K=K_1\times K_2$.
(iii) Finally, by the same arguments as in Proposition 4.2, we can easily deduce that u^h is Lipschitz in ω and $1/2$-Hölder in t, uniformly on h, and hence complete the proof for the stability condition in Assumption 3.6. ∎

Remark 4.12. Solutions of path dependent Bellman-Issac equations can characterize value functions of stochastic differential games (see e.g. Pham and Zhang [23]).

Remark 4.13. (i) For Bellman-Issac PDE, Debrabant and Jakobsen [5] studied the semi-Lagrangian scheme with a random variable ζ following a discrete distribution, together with an interpolation technique for the implementation.

(ii) For Bellman equation (PDE), Kharroubi, Langrené and Pham [19] propose a semi-Lagrangian type numerical scheme with $\zeta\sim N(0,1)$, and provide a simulation-regression technique for the implementation. It is worth of mentioning that [19] provides a convergence rate for the scheme, while we only prove in this paper a general convergence theorem as in Barles and Souganidis [1].

5 Numerical examples

In this section, we provide two toy examples of numerical implementation in low-dimensional case. For more numerical examples (in high-dimensional case), we would like to refer to [13, 15, 16, 19, 29], etc.

A first numerical example For a first numerical example, we consider the PPDE

$$-\partial_t u - \min_{\mu\in[\underline{\mu},\overline{\mu}]} \mu\partial_\omega u - \max_{a\in[\underline{a},\overline{a}]} \frac{a}{2}\partial^2_{\omega\omega} u = f(t,\omega,\bar{\omega}), \quad u(T,\omega) = g(\omega_T,\bar{\omega}_T). \tag{5.1}$$

where $d = 1$, $\bar{\omega}_t := \int_0^t \omega_s ds$, $f : [0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are two functions.

The above PPDE (5.1) is motivated by a stochastic differential game:

$$u_0 = \inf_{\underline{\mu}\le\mu_t\le\overline{\mu}} \sup_{\underline{a}\le a_t\le\overline{a}} \mathbb{E}\Big[\int_0^T f(t, X^{\mu,a}_t, \overline{X}^{\mu,a})dt + g(X^{\mu,a}_T, \overline{X}^{\mu,a}_T)\Big],$$

where $X^{\mu,\sigma}$ is controlled diffusion such that

$$X^{\mu,a}_t = \int_0^t \mu_s ds + \int_0^t \sqrt{a_s}dW_s, \quad \text{with } W \text{ a Brownian motion},$$

and $\overline{X}^{\mu,a}_t = \int_0^t X^{\mu,a}_s ds$ (see e.g. Pham and Zhang [23] for more details).

We choose the terminal condition $g(x,y) = \cos(x+y)$ and the function

$$\begin{aligned} f(t,x,y) \;\; = \;\; & -(x-\underline{\mu})\big(\sin(x-y)\big)^- + (x+\overline{\mu})\big(\sin(x-y)\big)^+ \\ & + \frac{\underline{a}}{2}\big(\cos(x-y)\big)^+ - \frac{\overline{a}}{2}\big(\cos(x-y)\big)^-, \end{aligned}$$

so that the solution of PPDE (5.1) is given explicitly by $u(t,\omega) = \cos(\omega_t + \bar{\omega}_t)$, which serves as a reference value for the numerical examples. This idea is borrowed from Guo, Zhang and Zhuo [15]. For numerical test, we implemented the finite difference scheme in Section 4.1 and the probabilistic scheme (of Fahim, Touzi and Warin [13]) in Section 4.3. The results are reported in Figure 1.

A second numerical example The second example of PPDE we considered is given by

$$-\partial_t u - \max_{\underline{a}\le a\le\overline{a}} \Big(\tfrac{1}{2}a\partial^2_{\omega\omega}u - f(t,u,\partial_\omega u, a)\Big) \;=\; 0, \tag{5.2}$$

$$\text{where} \quad f(t,y,z,a) = \tfrac{1}{2}\big((\sqrt{a}z + b/\sqrt{a})^-\big)^2 - zb - b^2/2a,$$

which is taken from Matoussi, Possamai and Zhou [21]. The above equation is motivated by solving a robust utility maximization problem using 2BSDE, which can be instead characterized by a PPDE (see e.g. (3.3)).

We consider the terminal condition

$$u(T,\omega) = K_1 + (\bar{\omega}_T - K_1)^+ - (\bar{\omega}_T - K_2)^+, \quad \overline{\omega}_T := \int_0^T \omega_s ds.$$

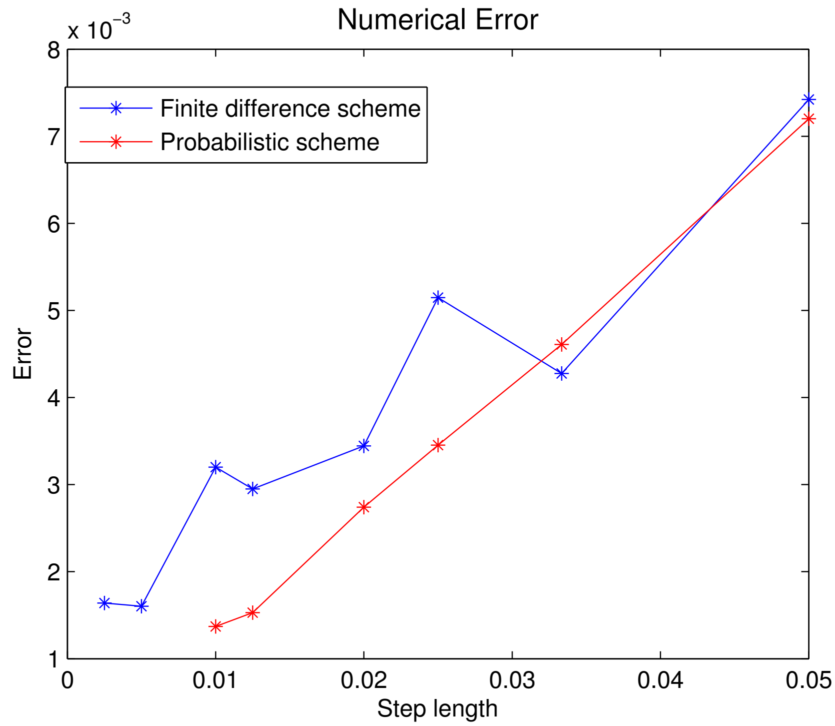

Figure 1: For PPDE (5.1), we choose $\underline{\mu} = -0.2$, $\overline{\mu} = 0.2$, $\underline{a} = 0.04$, $\overline{a} = 0.09$, $T = 1$ and $\omega_0 = \overline{\omega}_0 = 0$. Then the reference solution is given by $u(0,0) = \cos(0) = 1$. We compute the error between the reference solution and the numerical solutions, w.r.t. difference time step length Δt.

Then the solution of PPDE (5.2) can also be characterized by the PDE, by adding an associated variable y,

$$
\begin{aligned}
-\partial_t v - x\partial_y v - \max_{\underline{a}\le a\le \overline{a}}\Big(\tfrac{1}{2}a\partial^2_{xx}v - f(t,v,\partial_x v,a)\Big) = 0, \\
v(T,x,y) = K_1 + (y-K_1)^+ - (y-K_2)^+.
\end{aligned}
\tag{5.3}
$$

We implemented the finite difference scheme (Section 4.1) and the probabilistic scheme (Section 4.3) for PPDE (5.2). For reference, we implemented the classical finite difference scheme of PDE (5.3). We also notice that the generator in PPDE (5.2) is in fact not Lipschitz but quadratic in z, however, the convergence of the numerical solutions can be still observed, see Figure 2.

6 Proofs

6.1 Preliminary results

In preparation of the proof of Theorem 3.7, we prove the following lemmas.

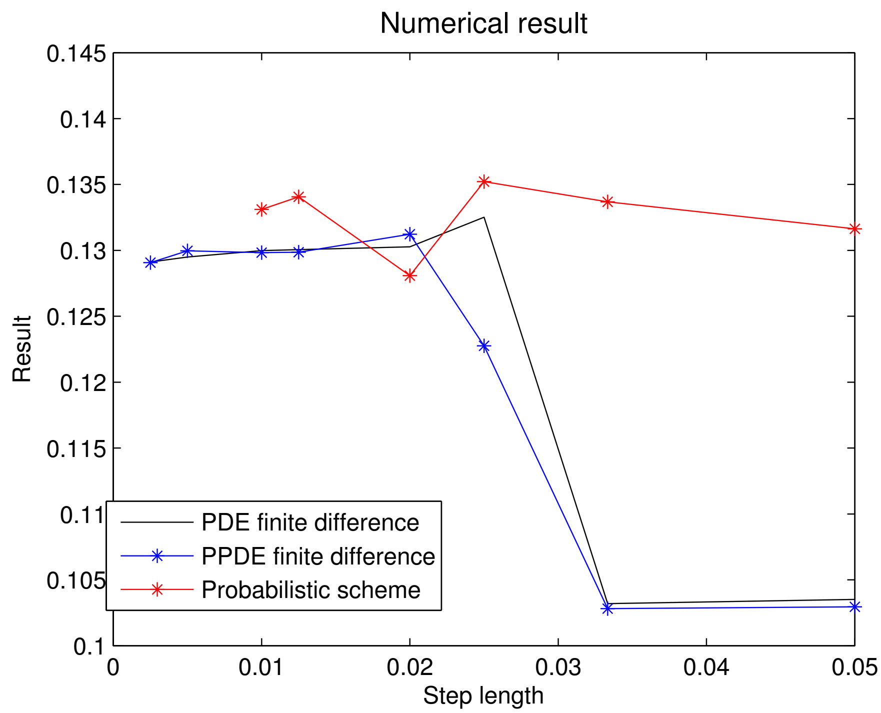

Figure 2: For PPDE (5.1), we choose $K_1 = -0.2$, $K_2 = 0.2$, $\underline{a} = 0.04$, $\overline{a} = 0.09$, $b = 0.05$ and $T = 1$. We provide all the numerical solutions w.r.t. difference time step length Δt. It seems that the faire value is closed to 0.129. For finite-difference scheme, when Δt is greater than 0.025, we need to use a coarser space-discretization to ensure the monotonicity (similar to the classical CFL condition), which makes a big difference to the numerical solutions for the case $\Delta t < 0.25$. However, the convergence as $\Delta t \to 0$ is still obvious.

Lemma 6.1 (Fatou's Lemma). *Assume that the random variables $X^n \in C^0(\mathcal{F})$ are bounded. Then we have*

$$\varliminf_{n\to\infty} \underline{\mathcal{E}}[X^n] \geq \underline{\mathcal{E}}\big[\varliminf_{n\to\infty} X^n\big]$$

Proof. In order to prove the Fatou lemma, it is enough to show the monotone convergence theorem, i.e. given a sequence $\{X^n : n \in \mathbb{N}\}$ of increasing random variables, we have

$$\lim_{n\to\infty} \underline{\mathcal{E}}[X^n] = \underline{\mathcal{E}}[\lim_{n\to\infty} X^n]. \tag{6.1}$$

Since $X^n \in C^0(\mathcal{F})$ for each n, it follows from Theorem 31 in [6] that (6.1) holds true.

Recall the nonlinear expectation $\underline{\mathcal{E}}_h$ defined in (3.7).

Lemma 6.2. *Let $\varphi : \Omega \to \mathbb{R}$ be bounded uniformly continuous. Then there exists a modulus continuity $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ which depends only on the continuity modulus of φ and $|\varphi|_0$, such that*

$$\underline{\mathcal{E}}[\varphi] \quad \leq \quad \underline{\mathcal{E}}_h[\varphi] + \rho(h).$$

Proof. Denote $\rho' : \mathbb{R}_+ \to \mathbb{R}_+$ as a continuity modulus of φ. Let $\nu \in \mathcal{K}$ and $X^{h,\nu}$ be defined by (3.6) and $\widehat{X}^{h,\nu}$ its linear interpolation on $[0,T]$. Then under the condition (3.5), it follows from Lemma 4.8 of Tan [30] (see also Dolinsky [7]) that we can construct a process $\widehat{X}^{h,\nu}$ and another process $\overline{X}$ in the same probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that the image measure of $\overline{X}$ lies in $\mathcal{P}$, and for some constant C independent of h,

$$\tilde{\mathbb{P}}\Big(\big|\widehat{X}^{h,\nu} - \overline{X}\big| \geq h^{1/8}\Big) \;\;\leq\;\; Ch^{1/8}.$$

Let $\rho(h) := \rho'(h^{1/8}) + 2\|\varphi\|_\infty h^{1/8}$, then it follows that

$$\underline{\mathcal{E}}[\varphi] \leq \tilde{\mathbb{E}}[\varphi(\overline{X})] \leq \tilde{\mathbb{E}}\big[\varphi(\widehat{X}^{h,\nu})\big] + \rho(h),$$

which concludes the proof by the arbitrariness of $\nu \in \mathcal{K}$. ∎

Lemma 6.3. *Let $\varphi : \Omega \to \mathbb{R}$ be lower semicontinuous and bounded from below, then it holds for all $(t,\omega) \in \Theta$ that*

$$\underline{\lim}_{h\to 0} \underline{\mathcal{E}}_h[\varphi] \;\;\geq\;\; \underline{\mathcal{E}}[\varphi].$$

In particular, by defining $\mathrm{H} := \inf\{t \geq 0 : |B_t| \geq x\}$ *for some $x > 0$, we have*

$$\overline{\lim}_{h\to 0} \overline{\mathcal{E}}_h[1_{\{\mathrm{H}\leq\delta\}}] \;\;\leq\;\; \overline{\mathcal{E}}[1_{\{\mathrm{H}\leq\delta\}}] \quad \textit{for any} \;\; \delta > 0.$$

Proof. Define the approximation for the function φ:

$$\varphi^n(\omega) := \inf_{\omega' \in \Omega} \big\{\varphi(\omega') + n\|\omega - \omega'\|\big\}.$$

Clearly, for each $n \in \mathbb{N}$, function φ^n is Lipschitz continuous, and $\varphi^n \uparrow \varphi$. By Lemma 6.2, we obtain that

$$\underline{\lim}_{h\to 0} \underline{\mathcal{E}}_h[\varphi] \;\;\geq\;\; \overline{\lim}_{h\to 0} \underline{\mathcal{E}}_h[\varphi^n] \;\;\geq\;\; \underline{\mathcal{E}}[\varphi^n], \quad \text{for all} \;\; n \in \mathbb{N}.$$

Since $\varphi^n \uparrow \varphi$, by Fatou's lemma we have

$$\underline{\lim}_{n\to\infty} \underline{\mathcal{E}}[\varphi^n] \;\;\geq\;\; \underline{\mathcal{E}}[\varphi].$$

Therefore

$$\underline{\lim}_{h\to 0} \underline{\mathcal{E}}_h[\varphi] \;\;\geq\;\; \underline{\mathcal{E}}[\varphi]. \tag{6.2}$$

Then we easily get the symmetric result for upper semicontinuous function ψ, i.e.

$$\overline{\lim}_{h\to 0} \overline{\mathcal{E}}_h[\psi] \;\;\leq\;\; \overline{\mathcal{E}}[\varphi].$$

To conclude, it remains to prove that the function $\omega \longmapsto 1_{\{\mathrm{H}(\omega)\leq\delta\}}$ is upper semicontinuous. Note that

$$\{\mathrm{H} \leq \delta\} = \{\max_{t\in[0,\delta]} |B_t| \geq x\}$$

Since the function $\varphi : \omega \mapsto \max_{t\in[0,\delta]} |B_t(\omega)|$ is continuous, the set $\{\mathrm{H} \leq \delta\}$ is closed. Consequently, the function $1_{\{\mathrm{H}\leq\delta\}}$ is upper semicontinuous. ∎

Lemma 6.4. *For any $\delta > 0$ and $\varepsilon > 0$, define $x(\delta) = Ld\sqrt{\delta}\big(\sqrt{\delta} + \sqrt{-2\ln\frac{\varepsilon\delta}{4d}}\big)$ and $\mathrm{H}^{\delta,x} = \inf\{t \ge 0 : |B_t| \ge x\}$. Then, for δ small enough we have*

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[\mathrm{H}^\delta \le \delta] \le \varepsilon\delta. \tag{6.3}$$

Proof. Note that

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[\mathrm{H}^\delta \le \delta] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\big[\max_{t\in[0,\delta]} |B_t| \ge x\big] \le d \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\big[\max_{t\in[0,\delta]} |B^1_t| \ge \frac{x}{d}\big]$$

By the definition of $\mathcal{P}$ above (2.1), for all $\mathbb{P} \in \mathcal{P}$, the canonical process B admits the canonical decomposition $B = A^{\mathbb{P}} + M^{\mathbb{P}}$, where $A^{\mathbb{P}} = (A^1, \cdots, A^d)$ is a finite variation process and $M = (M^1, \cdots, M^d)$ is a $\mathbb{P}$-martingale. Moreover, for each $i = 1, \cdots, d$,

$$\mathbb{P}\big[\max_{t\in[0,\delta]} |B^i_t| \ge \frac{x}{d}\big] = \mathbb{Q}\big[\max_{t\in[0,\delta]} |A^i_t + M^i_t| \ge \frac{x}{d}\big] \le \mathbb{Q}\big[\max_{t\in[0,\delta]} |M^i_t| \ge \frac{x}{d} - L\delta\big].$$

Further, by the time-change for martingales (see e.g. Theorem 4.6 on page 174 of [18]), there is a scalar Brownian motion W defined on a probability space $(\Omega, \mathsf{F}, \mathsf{P})$ such that

$$\begin{aligned}
\mathbb{P}\big[\max_{t\in[0,\delta]} |M^i_t| \ge \frac{x}{d} - L\delta\big] &= \mathsf{P}\big[\max_{t\in[0,\delta]} |W_{<M^1>_t}| \ge \frac{x}{d} - L\delta\big] \\
&\le \mathsf{P}\big[\max_{t\in[0,L^2\delta]} |W_t| \ge \frac{x}{d} - L\delta\big] \\
&= 4\mathsf{P}\big[W_1 \ge \frac{x/d - L\delta}{L\sqrt{\delta}}\big]
\end{aligned}$$

Since $\eta := \frac{x/d - L\delta}{L\sqrt{\delta}} = \sqrt{-2\ln\frac{\varepsilon\delta}{4d}} > 1$ when δ is small enough, we have

$$4\mathsf{P}\big[W_1 \ge \eta\big] \le 4e^{-\frac{\eta^2}{2}} = \frac{\varepsilon\delta}{d}.$$

We then conclude that $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[\mathrm{H}^\delta \le \delta] \le \varepsilon\delta$. ■

6.2 Proof of Proposition 3.2

We only discuss the case of subsolution. The result about the supersolution follows similarly.

1. We first prove the only if part. Let $(t,\omega) \in [0,T) \times \Omega$ and $(\alpha,\beta,\gamma) \in \underline{\mathcal{J}}u(t,\omega)$ with a localizing time H_δ. Clearly, there is a function $\varphi \in C^{1,2}_0(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\varphi = \phi^{\alpha,\beta,\gamma}$ on the set $[0,\delta] \times \{x \in \mathbb{R}^d : |x| \le x(\delta)\}$, where $x(\cdot)$ is defined as in Lemma 6.4. Thus,

$$(\varphi - u)_0 = \max_{\tau\in\mathcal{T}_{\bar{\mathrm{H}}^{\delta,x}}} \overline{\mathcal{E}}[(\varphi - u)_\tau],$$

where $\bar{\mathrm{H}}^{\delta,x} := \delta \wedge \mathrm{H}^{\delta,x}$ with $\mathrm{H}^{\delta,x}$ be defined as in Lemma 6.4. We have

$$(\varphi - u)_0 \quad \ge \quad \overline{\mathcal{E}}[(\varphi - u)_\delta] - \overline{\mathcal{E}}[(\varphi - u)_\delta - (\varphi - u)_{\bar{\mathrm{H}}^{\delta,x}}]. \tag{6.4}$$

For the second term on the right hand side of (6.4), we have

$$\begin{aligned}\overline{\mathcal{E}}[(\varphi-u)_\delta-(\varphi-u)_{\bar{\mathrm{H}}^{\delta,x}}] &\le \overline{\mathcal{E}}\big[|(\varphi-u)_\delta-(\varphi-u)_{\bar{\mathrm{H}}^{\delta,x}}|;\mathrm{H}^{\delta,x}\le\delta\big]\\ &\le C\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}[\mathrm{H}^{\delta,x}\le\delta].\end{aligned}$$

Take $\varepsilon>0$. By Lemma 6.4, there is a constant $C(\varepsilon)>0$ such that for all $\delta<C(\varepsilon)$ we have $\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}[\mathrm{H}^{\delta,x}\le\delta]<\frac{\varepsilon\delta}{2C}$. Then it follows from (6.4) that

$$(\varphi-u)_0>\overline{\mathcal{E}}[(\varphi-u)_\delta]-\frac{\varepsilon\delta}{2}.$$

We next consider the optimal stopping problem:

$$Y_t(\omega)=\sup_{\tau\in\mathcal{T}_{\delta-t}}\overline{\mathcal{E}}[(\varphi-u)^{t,\omega}_\tau-\varepsilon\tau].$$

According to Ekren, Touzi and Zhang [10], $\tau^*:=\inf\{t:Y_t=\varphi_t-u_t-\varepsilon t\}$ is an optimal stopping rule. Suppose that we always have $\bar{\mathrm{H}}^{\delta,x}\le\tau^*\le\delta$. Then we obtain that

$$\begin{aligned}\overline{\mathcal{E}}[(\varphi-u)_{\tau^*}-\varepsilon\tau^*] &\le \overline{\mathcal{E}}[(\varphi-u)_\delta-\varepsilon\delta]+\overline{\mathcal{E}}[(\varphi-u)_{\tau^*}-(\varphi-u)_\delta-\varepsilon(\tau^*-\delta)]\\ &\le \overline{\mathcal{E}}[(\varphi-u)_\delta-\varepsilon\delta]+\overline{\mathcal{E}}[|(\varphi-u)_{\tau^*}-(\varphi-u)_\delta-\varepsilon(\tau^*-\delta)|;\mathrm{H}^{\delta,x}\le\delta]\\ &\le \overline{\mathcal{E}}[(\varphi-u)_\delta-\varepsilon\delta]+C\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}[\mathrm{H}^{\delta,x}\le\delta]\\ &\le \overline{\mathcal{E}}[(\varphi-u)_\delta]-\frac{\varepsilon\delta}{2}\ <\ (\varphi-u)_0.\end{aligned}$$

However, this is in contradiction with the optimality of τ^*. Therefore, there is ω^* such that $t^*:=\tau^*(\omega^*)<\bar{\mathrm{H}}^{\delta,x}(\omega^*)$ and

$$(\varphi-u)_{t^*}(\omega^*)\quad=\quad\max_{\tau\in\mathcal{T}_{\delta-t^*}}\overline{\mathcal{E}}[(\varphi-u)^{t^*,\omega^*}_\tau-\varepsilon\tau].$$

So we have

$$\big(-\partial_t\varphi+\varepsilon-G(\cdot,u,\partial_x\varphi,\partial^2_{xx}\varphi)\big)(t^*,\omega^*)\quad\le\quad 0.$$

By letting $\delta\to0$ and then $\varepsilon\to0$, we obtain

$$\big(-\partial_t\varphi-G(\cdot,u,\partial_x\varphi,\partial^2_{xx}\varphi)\big)(0,0)\quad\le\quad 0.$$

Finally, since $\alpha=\partial_t\varphi_0,\beta=\partial_x\varphi_0,\gamma=\partial^2_{xx}\varphi_0$, this provides that $-\alpha-G(0,u_0,\beta,\gamma)\le0$.

2. For the if part, one may apply the same argument as in Proposition 3.11 in [25]. For completeness, we provide the full argument. Let $(t,\omega)\in[0,T)\times\Omega$ and $\varphi\in\underline{\mathcal{A}}u(t,\omega)$ with a localizing time $\delta\in\mathbb{R}^+$. Without loss of generality, we assume that $(t,\omega)=(0,0)$ and $(\varphi-u)_0=0$. Denote

$$\alpha:=\partial_t\varphi_0,\quad\beta:=\partial_x\varphi_0,\quad\text{and}\quad\gamma:=\partial^2_{xx}\varphi_0.\tag{6.5}$$

For any $\varepsilon>0$, since φ is smooth, by otherwise choosing a stopping time $\mathrm{H}_{\delta'}<\delta$ we may assume

$$|\partial_t\varphi_t-\alpha|\le\varepsilon,\quad|\partial_x\varphi_t-\beta|\le\varepsilon,\quad|\partial^2_{xx}\varphi_t-\gamma|\le2\varepsilon,\quad 0\le t\le\mathrm{H}_{\delta'}.$$

Denote $\alpha_\varepsilon := \alpha + [1+2L]\varepsilon$. Then, for all $\tau \in \mathcal{T}_{\mathrm{H}_{\delta'}}$,

$$
\begin{aligned}
& \overline{\mathcal{E}}\big[(u-\phi^{\alpha_\varepsilon,\beta,\gamma})_\tau\big] - u_0 = \overline{\mathcal{E}}\big[(u-u_0-\phi^{\alpha_\varepsilon,\beta,\gamma})_\tau\big] \\
\le\ & \overline{\mathcal{E}}\big[(u-\varphi)_\tau\big] + \overline{\mathcal{E}}\big[(\varphi-\varphi_0-\phi^{\alpha_\varepsilon,\beta,\gamma})_\tau\big] \\
\le\ & \overline{\mathcal{E}}\Big[\int_0^\tau (\partial_t\varphi_s - \alpha_\varepsilon)ds + (\partial_x\varphi_s-\beta)\cdot dB_s + \frac{1}{2}(\partial^2_{xx}\varphi_s-\gamma) : d\langle B\rangle_s\Big].
\end{aligned}
$$

where the last inequality is due to the Itô's formula. Note that, for any $\|\mu\|_\infty, \|a\|_\infty \le L$, we have

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}_{\mu,\sigma}}\Big[\int_0^\tau (\partial_t\varphi_s - \alpha_\varepsilon)ds + (\partial_x\varphi_s-\beta)\cdot dB_s + \frac{1}{2}(\partial^2_{xx}\varphi_s-\gamma) : d\langle B\rangle_s\Big] \\
= & \mathbb{E}^{\mathbb{Q}_{\mu,\sigma}}\Big[\int_0^\tau \Big(\partial_t\varphi_s - \alpha + (\partial_x\varphi_s-\beta)\cdot\mu_s + \frac{1}{2}(\partial^2_{xx}\varphi_s-\gamma) : a_s\Big)ds - [1+2L]\varepsilon\tau\Big] \le 0.
\end{aligned}
$$

By the arbitrariness of μ, σ, we see that

$$
\overline{\mathcal{E}}\big[(u-\phi^{\alpha_\varepsilon,\beta,\gamma})_\tau\big] - u_0 \quad \le \quad 0.
$$

That is, $(\alpha_\varepsilon, \beta) \in \underline{\mathcal{J}} u_0$. Since u is a $\mathcal{P}$-viscosity subsolution, it follows that

$$
-\alpha_\varepsilon - G(0,0,u_0,\beta,\gamma) \quad \le \quad 0.
$$

Let $\varepsilon \to 0$, then the desired result follows. ∎

6.3 Proof of Theorem 3.7

We first introduce two functions:

$$
\underline{u}(t,\omega) = \varliminf_{h\to 0} u^h(t,\omega) \quad \text{and} \quad \overline{u}(t,\omega) = \varlimsup_{h\to 0} u^h(t,\omega). \tag{6.6}
$$

Note that $\underline{u}, \overline{u}$ inherit the uniform modulus of continuity of u^h, so $\underline{u}, \overline{u} \in \mathrm{BUC}(\Theta)$. It is also clear that $\underline{u} \le \overline{u}$ and $\underline{u}_T = \overline{u}_T$. Then it is enough to prove that $\underline{u}$ is a $\mathcal{P}$-viscosity supersolution and $\overline{u}$ is a $\mathcal{P}$-viscosity subsolution, so that by the comparison principle we may obtain $\overline{u} \le \underline{u}$, to conclude the proof of Theorem 3.7.

Proposition 6.5. *The functions $\underline{u}$ and $\overline{u}$ defined in (6.6) are $\mathcal{P}$-viscosity supersolution and subsolution, respectively.*

Proof. We only prove the result for $\underline{u}$. The corresponding result for $\overline{u}$ can be proved similarly.

1. Without loss of generality, we only verify the viscosity supersolution property at the point $(0,0)$. Let function $\varphi \in \overline{\mathcal{A}}\underline{u}(0,0)$, and by adding a constant to φ, we assume that $\underline{u}(0,0) > \varphi(0,0)$, so that

$$
0 < \eta := (\underline{u}-\varphi)_0 = \min_{\tau\in\mathcal{T}_\delta} \underline{\mathcal{E}}[(\underline{u}-\varphi)_\tau], \quad \text{for some } \delta > 0. \tag{6.7}
$$

Assume that $\underline{u}$ and φ are both bounded by a constant $M \ge 0$. Take a subsequence still named as u^h such that $\underline{u}_0 = \lim_{h\to 0} u^h_0$. Now fix a constant $\varepsilon > 0$, and denote

$\varphi^\varepsilon(t,x) = \varphi(t,x) - \varepsilon t$. By Lemma 6.4, there is a constant $C(\varepsilon) \in (0, 1/L)$ such that for all $0 < \delta < C(\varepsilon)$, we have

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[\mathrm{H}^{\delta,x} \leq \delta] \leq \frac{\varepsilon}{32(2M+\varepsilon)}\delta. \tag{6.8}$$

Since u^h is uniformly continuous uniformly in h, by considering δ small enough we may assume that $u^h - \varphi^\varepsilon > 0$ on $[0, \bar{\mathrm{H}}^{\delta,x}]$, where $\bar{\mathrm{H}}^{\delta,x} := \delta \wedge \mathrm{H}^{\delta,x}$. It follows from (6.7) that

$$(\underline{u} - \varphi^\varepsilon)_0 \;\leq\; \underline{\mathcal{E}}[(\underline{u} - \varphi)_\delta] \;=\; \underline{\mathcal{E}}[(\underline{u} - \varphi^\varepsilon)_\delta] - \varepsilon\delta. \tag{6.9}$$

In Step **2** we will show that

$$\underline{\mathcal{E}}[(\underline{u} - \varphi^\varepsilon)_\delta] \leq \underline{\lim}_{h\to 0} \underline{\mathcal{E}}_h[(u^h - \varphi^\varepsilon)_\delta]. \tag{6.10}$$

It follows that for h sufficiently small

$$(u^h - \varphi^\varepsilon)_0 \quad\leq\quad \underline{\mathcal{E}}_h\big[(u^h - \varphi^\varepsilon)_\delta\big] - \frac{3\varepsilon\delta}{4}. \tag{6.11}$$

Then by the optimal stopping argument in Step **3**, we may find $(t^*, \omega^*) \in \Theta$ such that $\bar{\mathrm{H}}^{\delta,x}(\omega^*) \wedge (\delta - h) > t^* \in \Delta_h$ and

$$(u^h - \varphi^\varepsilon)_0^{t^*,\omega^*} \quad=\quad \min_{\tau\in\mathcal{T}^h_{\delta-t^*}, \beta\in\mathcal{B}^h} \underline{\mathcal{E}}_h[\beta_\tau (u^h - \varphi^\varepsilon)_\tau^{t^*,\omega^*}], \tag{6.12}$$

where $\Delta_h := \{kh : k \in \mathbb{N}\}$, $\mathcal{T}^h_{\delta-t^*} := \{\tau \in \mathcal{T}_{\delta-t^*} : \tau \text{ takes values in } \Delta_h\}$ and $\mathcal{B}^h$ is the collection of all processes β defined by $\beta_t := e^{\sum_{i=0}^{[t/h]-1}\alpha_i h}$ for some $\mathcal{F}_{ih}$-measurable α_i taking value in $[0, L]$. In particular, (6.12) implies that

$$(u^h - \varphi^\varepsilon)(t^*, \omega^*) \leq \inf_{0\leq\alpha\leq L} \underline{\mathcal{E}}_h[e^{\alpha h}(u^h - \varphi^\varepsilon)_h^{t^*,\omega^*}]$$

By (ii) of Assumption 3.6, we obtain

$$(u^h - \varphi^\varepsilon)(t^*, \omega^*) \leq \mathbb{T}_h^{t^*,\omega^*}[u^h] - \mathbb{T}_h^{t^*,\omega^*}[\varphi^\varepsilon] + h\rho(h).$$

Since $u^h(t^*, \omega^*) = \mathbb{T}_h^{t^*,\omega^*}[u^h]$, it follows that

$$\frac{\mathbb{T}_h^{t^*,\omega^*}[\varphi^\varepsilon] - \varphi^\varepsilon(t^*, \omega^*)}{h} \leq \rho(h).$$

Further, by (i) of Assumption 3.6, letting $h \to 0$, we obtain

$$-\partial_t\varphi(t^*, \omega^*_{t^*}) + \varepsilon - G(\cdot, \varphi, \partial_x\varphi, \partial^2_{xx}\varphi)(t^*, \omega^*_{t^*}) \geq 0. \tag{6.13}$$

We next let $\delta \to 0$. Since $t^* < \bar{\mathrm{H}}^{\delta,x}(\omega^*)$, we have $(t^*, \omega^*) \to 0$ as $\delta \to 0$. Therefore, it follows from (6.13) that

$$-\partial_t\varphi_0 + \varepsilon - G(0, \underline{u}_0 - \eta, \partial_x\varphi_0, \partial^2_{xx}\varphi_0) \geq 0.$$

Finally, we can conclude the proof by letting $\varepsilon \to 0$ and then $\eta \to 0$.

2. For the simplification of notation, we denote $X := (\underline{u}-\varphi^\varepsilon)_\delta$ and $X^h := (u^h-\varphi^\varepsilon)_\delta$. It follows from (iii) of Assumption 3.6 that $\{X^h : h > 0\}$ is uniformly bounded and uniformly continuous uniformly in h, and note that $X = \underline{\lim}_{h\to 0} X^h$. By Lemma 6.1 and 6.2, we obtain that

$$\begin{aligned}\underline{\lim}_{h\to 0}\underline{\mathcal{E}}_h[X^h] &\geq \underline{\lim}_{h\to 0}\underline{\mathcal{E}}[X^h] + \underline{\lim}_{h\to 0}\big(\underline{\mathcal{E}}_h[X^h]-\underline{\mathcal{E}}[X^h]\big)\\ &\geq \underline{\mathcal{E}}[X] + \underline{\lim}_{h\to 0}\inf_{\ell>0}\big(\underline{\mathcal{E}}_h[X^\ell]-\underline{\mathcal{E}}[X^\ell]\big) \;\geq\; \underline{\mathcal{E}}[X] + \underline{\lim}_{h\to 0}\rho(h) \;=\; \underline{\mathcal{E}}[X].\end{aligned}$$

3. We consider the mixed control and optimal stopping problem in finite discrete-time:

$$Y_t^h(\omega) := \inf_{\tau\in\mathcal{T}^h_{\delta-t},\beta\in\mathcal{B}} \underline{\mathcal{E}}_h[\beta_\tau (Z^h)^{t,\omega}_\tau], \quad \text{where} \;\; Z_t^h := (u^h-\varphi^\varepsilon)_t,\; t\in\Delta_h. \tag{6.14}$$

By standard argument, we have

$$Y_0^h = \inf_{\beta\in\mathcal{B}}\underline{\mathcal{E}}_h[\beta_{\tau^*}Z^h_{\tau^*}], \quad \text{where} \;\; \tau^* := \inf\{t\in\Delta_h : Y_t^h = Z_t^h\}.$$

Recall that $Z^h > 0$ on $[0, \mathrm{H}^{\delta,x}]$ for h small enough. Then since $\tau^*\leq\delta$, we have

$$\begin{aligned}\underline{\mathcal{E}}_h[Z^h_{\tau^*}] &\leq \underline{\mathcal{E}}_h[Z^h_{\tau^*};\mathrm{H}^{\delta,x}>\delta] + \overline{\mathcal{E}}_h[Z^h_{\tau^*};\mathrm{H}^{\delta,x}\leq\delta]\\ &= \inf_{\beta\in\mathcal{B}}\underline{\mathcal{E}}_h[\beta_{\tau^*}Z^h_{\tau^*};\mathrm{H}^{\delta,x}>\delta] + \overline{\mathcal{E}}_h[Z^h_{\tau^*};\mathrm{H}^{\delta,x}\leq\delta]\\ &\leq \inf_{\beta\in\mathcal{B}}\underline{\mathcal{E}}_h[\beta_{\tau^*}Z^h_{\tau^*}] + \sup_{\beta\in\mathcal{B}}\overline{\mathcal{E}}[\beta_{\tau^*}|Z^h_{\tau^*}|;\mathrm{H}^{\delta,x}\leq\delta] + \overline{\mathcal{E}}_h[|Z^h_{\tau^*}|;\mathrm{H}^{\delta,x}\leq\delta]\\ &\leq Y_0^h + (1+e^{L\delta})\overline{\mathcal{E}}_h[|Z^h_{\tau^*}|;\mathrm{H}^{\delta,x}\leq\delta]\\ &\leq Y_0^h + (1+e^{L\delta})(2M+\varepsilon)\overline{\mathcal{E}}_h[1_{\{\mathrm{H}^{\delta,x}\leq\delta\}}].\end{aligned}$$

Further, we obtain from Lemma 6.3 that for h small enough it holds

$$\overline{\mathcal{E}}_h[1_{\{\mathrm{H}^{\delta,x}\leq d\}}] < \overline{\mathcal{E}}[1_{\{\mathrm{H}^{\delta,x}\leq d\}}] + \frac{\varepsilon\delta}{8(4M+2\varepsilon)}, \tag{6.15}$$

So we get

$$\underline{\mathcal{E}}_h[Z^h_{\tau^*}] \leq Y_0^h + (1+e^{L\delta})(2M+\varepsilon)\overline{\mathcal{E}}[1_{\{\mathrm{H}^{\delta,x}\leq\delta\}}] + \frac{\varepsilon\delta}{8}.$$

Further, it follows from (6.8) that $(1+e^{L\delta})(2M+\varepsilon)\overline{\mathcal{E}}[1_{\{\mathrm{H}^{\delta,x}\leq\delta\}}] \leq \frac{\varepsilon\delta}{8}$. Therefore,

$$Y_0^h \;\geq\; \underline{\mathcal{E}}_h[Z^h_{\tau^*}] - \frac{\varepsilon\delta}{4}.$$

Suppose that

$$\bar{\mathrm{H}}^{\delta,x}(\omega)\wedge(\delta-h) \leq \tau^*(\omega) \leq \delta, \quad \text{for all} \;\; \omega. \tag{6.16}$$

Note that

$$\underline{\mathcal{E}}_h[Z^h_{\tau^*}] \geq \underline{\mathcal{E}}_h[Z^h_\delta] + \underline{\mathcal{E}}_h[Z^h_{\tau^*} - Z^h_{\tau^*\vee(\delta-h)}] + \underline{\mathcal{E}}_h[Z^h_{\tau^*\vee(\delta-h)} - Z^h_\delta]. \tag{6.17}$$

Further it follows from (6.8) and (6.15) that

$$\begin{aligned} I_1 &:= \underline{\mathcal{E}}_h[Z^h_{\tau^*} - Z^h_{\tau^*\vee(\delta-h)}] \;=\; \underline{\mathcal{E}}_h[Z^h_{\tau^*} - Z^h_{\delta-h}; \tau^* < \delta - h] \\ &\geq -(4M+2\varepsilon)\overline{\mathcal{E}}_h[1_{\{\mathrm{H}^{\delta,x}\leq\delta\}}] \;>\; -(4M+2\varepsilon)\overline{\mathcal{E}}[1_{\{\mathrm{H}^{\delta,x}\leq\delta\}}] - \frac{\varepsilon\delta}{8} \;\geq\; -\frac{\varepsilon\delta}{4}. \end{aligned}$$

On the other hand, we have

$$I_2 := \underline{\mathcal{E}}_h\big[Z^h_{\tau^*\vee(\delta-h)} - Z^h_\delta\big] \geq -\overline{\mathcal{E}}_h\big[(\rho_u+\rho_\varphi)(h + 2\|B_{(\delta-h)\wedge\cdot} - B_{\delta\wedge\cdot}\|)\big] - \varepsilon h,$$

where ρ_u, ρ_φ are module of continuity of function u, φ, and are chosen to be bounded and continuous. Again by Lemma 6.3, we have for h sufficiently small that

$$\begin{aligned} &\overline{\mathcal{E}}_h\big[(\rho_u+\rho_\varphi)(h+2\|B_{(\delta-h)\wedge\cdot} - B_{\delta\wedge\cdot}\|)\big] < \overline{\mathcal{E}}\big[(\rho_u+\rho_\varphi)(h+ \\ &\qquad 2\|B_{(\delta-h)\wedge\cdot} - B_{\delta\wedge\cdot}\|)\big] + \frac{\varepsilon\delta}{8} = \overline{\mathcal{E}}\big[(\rho_u+\rho_\varphi)(h+2\|B_{h\wedge\cdot}\|\big] + \frac{\varepsilon\delta}{8}, \end{aligned}$$

It follows that $\lim_{h\to 0}\overline{\mathcal{E}}\big[(\rho_u+\rho_\varphi)(h+2\|B_{h\wedge\cdot}\|\big] = 0$ and therefore

$$I_2 \;>\; -\frac{\varepsilon\delta}{4}, \quad \text{for } h \text{ sufficiently small.}$$

Finally, by (6.17) and (6.11) we have

$$Y^h_0 \;\geq\; \underline{\mathcal{E}}_h[Z^h_\delta] - \frac{\varepsilon\delta}{4} + I_1 + I_2 \;>\; \underline{\mathcal{E}}_h[Z^h_\delta] - \frac{3\varepsilon\delta}{4} \;\geq\; Z^h_0,$$

which contradicts the definition of Y in (6.14). Therefore, (6.16) is wrong, i.e. there is ω^* such that $t^* := \tau^*(\omega^*) < \bar{\mathrm{H}}^{\delta,x}(\omega^*) \wedge (\delta - h)$. Further, since $Y_{t^*}(\omega^*) = Z^h_{t^*}(\omega^*)$, we obtain (6.12). ■

7 Conclusion

We provide a convergence theorem of monotone numerical schemes for a class of parabolic PPDE, which generalizes the classical convergence theorem of Barles and Souganidis [1]. In contrast to the formulation of [32], our conditions are satisfied by all classical monotone numerical scheme for PDEs, to the best of our knowledge. Moreover, our results permit to deduce some numerical schemes for path-dependent stochastic differential game problems and the second order BSDEs whose generator depends on z (see (3.3), (3.4)), which are new in literatures.

Other numerical schemes, such as the branching process scheme of Henry-Labordère, Tan and Touzi [17], are possible for some PPDE, but it is not analyzed by the monotone scheme arguments.

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