About a theorem of Pemantle et Volkov

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Abstract: Vertex-Reinforced Random Walk (VRRW), defined by Pemantle (1988,[5]), is a random process taking values in the vertex set of a graph G, which is more likely to visit vertices it has visited before. Pemantle and Volkov (1997,[7]) proved that, when the underlying graph is the one-dimensional integer lattice $\mathbb{Z}$, the random walk eventually gets stuck in a finite set containing at least five points and, with positive probability, in exactly five points. We give here a short proof of these results.

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let G be a locally finite graph, $\sim$ be its neighbor relation, and $V(G)$ be its vertex set.

Let $(X_n)_{n\in\mathbb{N}}$ be a process taking values in $V(G)$. Let $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ denote the filtration generated by the process, i.e $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ for all $n \in \mathbb{N}$.

For any $v \in V(G)$, let $Z_n(v)$ be the number of times plus one that the process visits site v up through time $n \in \mathbb{N} \cup \{\infty\}$, i.e

$$Z_n(v) = 1 + \sum_{i=0}^{n} \mathbb{1}_{\{X_i = v\}}.$$

Then $(X_n)_{n\in\mathbb{N}}$ is called Vertex-Reinforced Random Walk (VRRW) with starting point $v_0 \in V(G)$ if $X_0 = v_0$ and, for all $n \in \mathbb{N}$,

$$\mathbb{P}(X_{n+1} = x \mid \mathcal{F}_n) = \mathbb{1}_{\{x \sim X_n\}} \frac{Z_n(x)}{\sum_{w \sim X_n} Z_n(w)}.$$

These non-Markovian random walks were introduced in 1988 by Pemantle ([5]), in the spirit of the model of Edge-Reinforced Random Walks introduced by Coppersmith and Diaconis in 1987 in an unpublished manuscript ([3], the weights being accumulated on edges rather than vertices).

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Similar processes are useful in models involving self-organization or learning behavior, and in economics. For more details, we refer the reader to the articles of Pemantle ([6]), Pemantle and Volkov ([7]) and Skyrms and Pemantle ([9]).

Vertex-reinforced random walks on finite graphs, with reinforcements weighted by factors associated to each edge of the graph, have been studied by Pemantle (1992,[6]) and Benaïm (1997,[1]).

Pemantle and Volkov (1997,[7]) have obtained, when the underlying graph is $\mathbb{Z}$, very precise results on the asymptotic behavior of the random walk, recalled hereafter.

Define the two random sets

$$R = \{v \in \mathbb{Z} / \ \exists n \in \mathbb{N}, \ X_n = v\},$$

$$R' = \{v \in \mathbb{Z} / \ X_n = v \text{ infinitely often}\}$$

and, given $v \in \mathbb{Z}$ and $\alpha \in (0,1)$, the six events:

(i) $R' = \{v-2, v-1, v, v+1, v+2\}$;

(ii) $\ln Z_n(v-2)/\ln n \to \alpha$;

(iii) $\ln Z_n(v+2)/\ln n \to 1-\alpha$;

(iv) $Z_n(v-1)/n \to \alpha/2$;

(v) $Z_n(v+1)/n \to (1-\alpha)/2$;

(vi) $Z_n(v)/n \to 1/2$.

Theorem 1.1 *([7])* $\mathbb{P}(|R| < \infty) = 1$.

Theorem 1.2 *([7])* $\mathbb{P}(|R'| \leqslant 4) = 0$.

Theorem 1.3 *([7])* $\mathbb{P}(|R| = 5) > 0$. *Further, for any open set $I \subset [0,1]$ and any integer $v \in \mathbb{Z}$ there exists, with positive probability, $\alpha \in I$ such that events* **(i)** *to* **(vi)** *occur.*

Conjecture 1 *([7]) There exist almost surely $v \in \mathbb{Z}$ and $\alpha \in (0,1)$ such that events* **(i)** *to* **(vi)** *occur.*

We prove this conjecture in [10]. The purpose of this paper is to give a short proof of theorems 1.1, 1.2 and 1.3.

We partly use the heuristics of the article of Pemantle and Volkov ([7]). However, we provide here a new method (introduced in [10]) to describe the local behavior of the random walk, based on a symmetry property observed on a logarithmic scale (claimed in lemma 2.2). This result enables us to prove some local a.s. properties of the VRRW (corollary 2.1), and to yield a condition ensuring that, with large probability, the random walk remains right-hand or left-hand from a point (corollary 2.2). We deduce theorems 1.1, 1.2 and 1.3 from these corollaries.

NOTATIONS: Let $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Let $\mathsf{Cst}(a_1, \ldots, a_p)$ denote a positive constant depending only on a_1, a_2 ... a_p. Let ξ denote a random positive constant.
We write $u_n \equiv v_n$ iff $\lim(u_n - v_n)$ exists a.s. and is finite, and $u_n \asymp v_n$ iff $\lim u_n/v_n = 1$.

2 Proof of the results

Given $x \in \mathbb{Z}$, $n \in \mathbb{N} \cup \{\infty\}$, and j, $k \in \mathbb{N}$ such that $k \geqslant j$, denote

$$\begin{aligned}
Y_n(x) &:= \sum_{i=1}^{n} \mathbb{1}_{\{X_{i-1}=x\}} \frac{1}{Z_{i-1}(x-1) + Z_{i-1}(x+1)} \\
Y_n^{\pm}(x) &:= \sum_{i=1}^{n} \mathbb{1}_{\{X_{i-1}=x, X_i=x\pm 1\}} \frac{1}{Z_{i-1}(x \pm 1)} \\
\alpha_n^{\pm}(x) &:= \frac{Z_n(x \pm 1)}{Z_n(x-1) + Z_n(x+1)}, \ \beta_n^{\pm}(x) := \frac{Z_n(x \pm 1)}{Z_n(x)}, \\
Y_{j,k}(x) &:= Y_k(x) - Y_j(x), \ Y_{j,k}^{\pm}(x) := Y_k^{\pm}(x) - Y_j^{\pm}(x).
\end{aligned}$$

Let us define, given $x \in \mathbb{Z}$, the event

$$\Upsilon(x) = \{Y_\infty(x) < \infty\}$$

on which x is seldom visited in comparison with its neighbors $x-1$ and $x+1$.

Let us define, given $x \in \mathbb{Z}$, $n \in \mathbb{N}^*$ and $t > 0$, the probability events

$$\Lambda_n^{\pm}(x,t) := \left\{ \sup_{k \geqslant n} \left| Y_{n,k}^{\pm}(x) - Y_{n,k}(x) \right| < t \right\}, \ \Lambda^{\pm}(x,t) := \bigcup_{n \in \mathbb{N}} \Lambda_n^{\pm}(x,t).$$

$Y_n^{\pm}(x)$ estimates the contribution -on a logarithmic scale- of the visits starting from x to the number of visits to $x \pm 1$. This contribution $Y_n^{\pm}(x)$ is almost independent of the sign of $\pm$: the probability of visit to $x \pm 1$ starting from x is $Z_n(x \pm 1)/(Z_n(x-1) + Z_n(x+1))$, so that $Y_n^{\pm}(x)$ increases in average of approximately $1/(Z_n(x-1) + Z_n(x+1))$, like $Y_n(x)$. Lemmas 2.1 and 2.2 rely on these two remarks.

2.1 Preliminary results

Lemma 2.1

$$\ln Z_n(x) \equiv Y_n^{+}(x-1) + Y_n^{-}(x+1).$$

PROOF:

$$\begin{aligned}
\ln Z_n(x) \equiv \sum_{i=1}^{n} \frac{\mathbb{1}_{\{X_i=x\}}}{Z_{i-1}(x)} &= \sum_{i=1}^{n} \frac{\mathbb{1}_{\{X_{i-1}=x-1, X_i=x\}}}{Z_{i-1}(x)} + \frac{\mathbb{1}_{\{X_{i-1}=x+1, X_i=x\}}}{Z_{i-1}(x)} \\
&= Y_n^{+}(x-1) + Y_n^{-}(x+1). \qquad \square
\end{aligned}$$

Lemma 2.2

$$\textbf{(a)}\quad Y_n^{\pm}(x) \equiv Y_n(x)$$
$$\textbf{(b)}\quad \mathbb{P}(\Lambda_n^{\pm}(x,t) \mid \mathcal{F}_n) \geqslant 1 - \frac{8}{Z_n(x\pm 1)t^2}.$$

PROOF: Let us first prove the following inequality

$$\mathbb{P}(\Lambda_n^{\pm}(x,t)^c \mid \mathcal{F}_n) \leqslant \frac{8\mathbb{P}(Z_\infty(x\pm 1) \neq Z_n(x\pm 1) \mid \mathcal{F}_n)}{Z_n(x\pm 1)t^2}, \tag{1}$$

which will in particular imply statement **(b)**.

Observe that $M_n = Y_n^{\pm}(x) - Y_n(x)$ is a martingale; using Doob's inequality, for all $k \geqslant n$,

$$\mathbb{E}(\sup_{k\geqslant n}(Y_{n,k}^{\pm}(x) - Y_{n,k}(x))^2 \mid \mathcal{F}_n) \leqslant 4\mathbb{E}\left(\sum_{i=n}^{\infty}\mathbb{E}((M_{i+1}-M_i)^2 \mid \mathcal{F}_i) \mid \mathcal{F}_n\right)$$
$$\leqslant 4\mathbb{E}\left(\sum_{i=n}^{\infty}\frac{\mathbb{1}_{\{X_i=x\}}}{Z_i(x\pm 1)^2}\alpha_i^{\pm}(x) \mid \mathcal{F}_n\right) = 4\mathbb{E}\left(\sum_{i=n}^{\infty}\frac{\mathbb{1}_{\{X_i=x, X_{i+1}=x\pm 1\}}}{Z_i(x\pm 1)^2} \mid \mathcal{F}_n\right)$$
$$\leqslant 4\mathbb{P}(Z_\infty(x\pm 1) \neq Z_n(x\pm 1) \mid \mathcal{F}_n)\left(\sum_{p=Z_n(x\pm 1)}^{\infty}\frac{1}{p^2}\right) \leqslant \frac{8\mathbb{P}(Z_\infty(x\pm 1) \neq Z_n(x\pm 1) \mid \mathcal{F}_n)}{Z_n(x\pm 1)},$$

which implies inequality (1) by Bienaymé-Tchebychev inequality.

Let us now prove that inequality (1) implies statement **(a)**. It suffices to prove that, for all $t > 0$, we a.s. belong to $\Lambda^{\pm}(x,t)$. Using the fact that $\mathbb{P}(\Lambda^{\pm}(x,t) \mid \mathcal{F}_n)$ converges a.s. to $\mathbb{1}_{\Lambda^{\pm}(x,t)}$, it is sufficient to prove that $\mathbb{P}(\Lambda_n^{\pm}(x,t) \mid \mathcal{F}_n)$ converges a.s. to 1. This last statement follows from (1) on $\{Z_\infty(x\pm 1) = \infty\}$, and remains true on $\{Z_\infty(x\pm 1) < \infty\}$, since

$$\begin{aligned}\{Z_\infty(x\pm 1) = \infty\} &\subset \bigcup_{k\in\mathbb{N}}\{Z_\infty(x\pm 1) = Z_k(x\pm 1)\}\\ &\subset \{\lim_{n\to\infty}\mathbb{P}(Z_\infty(x\pm 1) \neq Z_n(x\pm 1) \mid \mathcal{F}_n) = 0\}.\end{aligned}$$ □

Corollary 2.1

(a) $\Upsilon(x\pm 2) \subset \{\exists\alpha_\infty^{\pm}(x) := \lim_{n\to\infty}\alpha_n^{\pm}(x) \in [0,1)\}$

(b) $\Upsilon(x-3)\cap\Upsilon(x-2)\cap\Upsilon(x+2)\cap\Upsilon(x+3)\cap\{Z_\infty(x+2) = Z_\infty(x-2) = \infty\}$
$$\subset \left\{\frac{\ln Z_n(x\pm 2)}{\ln Z_n(x\pm 1)} \longrightarrow \alpha_\infty^{\pm}(x) = \lim_{n\to\infty}\beta_n^{-}(x)\right\}$$

(c) *Conditionally to event* **(i)**, *events* **(ii)** *to* **(vi)** *hold.*

A. Bienvenüe proved part **(c)** of this corollary in his PhD dissertation (1999,[2]) using different ideas related to the theory of continuous vertex-reinforced random walks (see Sellke,[8]).

Let us explain parts **(a)** and **(b)** of corollary 2.1, in the case $\pm := -$.

In part **(a)**, the quantity $\ln Z_n(x-1)/Z_n(x+1)$ does not change significantly over the visits from $x-2$ (we belong to $\Upsilon(x-2)$), and its contribution over the visits from x is $Y_n^+(x) - Y_n^-(x)$, which remains stable by lemma 2.2; lastly, this quantity decreases over the visits from $x+2$. In summary, $\ln Z_n(x-1)/Z_n(x+1)$ decreases in average, and therefore converges towards a value $\lambda \in \mathbb{R} \cup \{-\infty\}$.

In part **(b)**, the term $\ln Z_n(x-2)$ remains stable over the visits from $x-3$ (we belong to $\Upsilon(x-3)$); starting from $x-1$, the probability of visit to $x-2$ is $\alpha_n^-(x-1) \asymp Z_n(x-2) \times \beta_n^-(x)/Z_n(x-1)$, so that the average variation is $\beta_\infty^-(x)/Z_n(x-1)$ (in the concerned case, $\alpha_n^-(x-2) \to 0$, and $\beta_n^-(x)$ converges). Therefore, for large n, $\ln Z_n(x-2)$ behaves like $\beta_\infty^-(x) \ln Z_n(x-1)$.

PROOF: We will consider the case $\pm := -$, the study of the other case being similar. Let us assume we belong to $\Upsilon(x-2)$, and prove **(a)**: using lemmas 2.1 and 2.2,

$$\ln Z_n(x-1) \equiv Y_n^+(x-2) + Y_n^-(x) \equiv Y_n^-(x) \equiv Y_n(x) \equiv Y_n^+(x). \tag{2}$$

Equation (2) and lemma 2.1 imply

$$\ln \frac{Z_n(x-1)}{Z_n(x+1)} \equiv Y_n^+(x) - \ln Z_n(x+1) \equiv -Y_n^-(x+2), \tag{3}$$

which completes the proof, since $Y_n^-(x+2)$ is nondecreasing in n.

Let us assume we belong to the left-hand side of **(b)**. First, this implies that we belong to $\Upsilon(x+1)^c$, since otherwise we would belong to

$$\Upsilon(x+1) \cap \Upsilon(x+3) = \{Y_\infty^+(x+1) < \infty\} \cap \{Y_\infty^-(x+3) < \infty\} \subset \{\ln Z_\infty(x+2) < \infty\}$$

by lemmata 2.1 and 2.2. Therefore, using equation 3 for $x := x-1$, $\alpha_n^-(x-1)$ tends to 0 (using $Y_\infty^-(x+1) = \infty$). Similarly, $\alpha_\infty^+(x+1) = 0$. Hence, using equation (2) for $x := x-1$,

$$\ln Z_n(x-2) \equiv Y_n(x-1) = \sum_{i=0}^{n-1} \frac{\mathbb{1}_{\{X_i = x-1\}}}{Z_i(x-2) + Z_i(x)} \asymp \sum_{i=0}^{n-1} \frac{\mathbb{1}_{\{X_i = x-1\}}}{Z_i(x-1)} \beta_i^-(x) \tag{4}$$

$$\asymp \alpha_\infty^-(x) \ln Z_n(x-1). \tag{5}$$

Let us now prove **(c)**: by lemma 2.2 **(a)**, we belong to $\{Y_\infty^-(v-2) < \infty\} = \Upsilon(v-2)$, and similarly to $\Upsilon(v+2)$, and to $\Upsilon(v-3) \cap \Upsilon(v+3)$. We can conclude by **(a)** and **(b)**. □

Given $x \in \mathbb{Z}$, let $t_n(x)$ be the n-th visit time to site x. Given $a \in \mathbb{N}^*$ and $\zeta < 1$, let $\mathcal{E}_n^\pm(x, a, \zeta)$ denote the following property

$$\mathcal{E}_n^\pm(x, a, \zeta) := \left\{X_n = x,\ \beta_n^\pm(x) \leqslant \zeta,\ Z_n(x \pm 2) \leqslant Z_n(x \pm 1)^\zeta,\ Z_n(x \pm 3) \leqslant a\right\}.$$

Corollary 2.2 *Fix $a \in \mathbb{N}^*$ and $\zeta < 1$. Then, for all $c \in (0,1)$, if $Z_n(x \pm 1) \geqslant \mathsf{Cst}(a, c, \zeta)$,*

$$\mathbb{P}(\{Z_\infty(x \pm 3) = Z_n(x \pm 3)\} \mid \mathcal{F}_n) \geqslant c \ \mathbb{I}_{\mathcal{E}_n^\pm(x,a,\zeta)}.$$

The proof of corollary 2.2 is given in subsection 2.3.

2.2 Proof of theorems 1.1, 1.2 and 1.3

2.2.1 Proof of theorem 1.1

Let us prove there exists a constant $\delta > 0$ such that, for all $v \in \mathbb{Z}$ such that $v \leqslant v_0$,

$$\mathbb{P}(v - 3 \in R \mid v \in R) \leqslant 1 - \delta. \tag{6}$$

This completes the proof of the theorem: (6) implies $\mathbb{P}(v_0 - 3n \in R) \leqslant (1-\delta)^n$, which goes to zero as $n \to \infty$. Hence $\inf R' > -\infty$ a.s.; similarly, $\sup R' < \infty$ a.s.

Observe that $\mathbb{P}(\mathcal{E}^-_{t_n(v-1)+1}(v, 1, 1/2) \mid v \in R) \geqslant \mathsf{Cst}(n)$, with the convention that $\mathcal{E}^-_{t_n+1}$ holds whenever $t_n = \infty$. Therefore, using corollary 2.2, there exists a universal constant $n \in \mathbb{N}^*$ such that

$$\begin{aligned}\mathbb{P}(v - 3 \notin R \mid v \in R) &\geqslant \mathbb{P}(\{Z_\infty(v-3) = 1\} \cap \mathcal{E}^-_{t_n(v-1)+1}(v, 1, 1/2) \mid v \in R) \\ &\geqslant \mathbb{E}(\mathbb{P}(\{Z_\infty(v-3) = 1\} \mid \mathcal{F}_{t_n(v-1)+1}) \ \mathbb{I}_{\mathcal{E}_{t_n(v-1)+1}(v,1,1/2)} \mid v \in R) \geqslant \mathsf{Cst}(n)/2.\end{aligned}$$

2.2.2 Proof of theorem 1.2

Theorem 1.1 implies that there exists a.s. $x \in \mathbb{Z}$ such that $\inf R' = x > -\infty$. Now, for all $x \in \mathbb{Z}$, using successively lemma 2.2 **(a)**, corollary 2.1 **(a)** and conditional Borel-Cantelli lemma (see for instance [4], t.2, th.2.7.33, p.76),

$$\begin{aligned}&\{\inf R' = x\} \subset \{Y^-_\infty(x) < \infty\} = \Upsilon(x) \cap \{Z_\infty(x+1) = \infty\} \\ &\subset \{\exists \alpha^-_\infty(x+2) := \lim_{n\to\infty} \alpha^-_n(x+2) \in [0,1)\} \cap \{Z_\infty(x+3) = \infty\} \\ &\subset \left\{\sum_{k=1}^\infty \mathbb{I}_{\{X_k = x+3\}} \mathbb{P}(X_{k+1} = x+4 \mid \mathcal{F}_k) \geqslant \sum_{k=1}^\infty \frac{\mathbb{I}_{\{X_k=x+3\}}}{1 + Z_k(x+2)} \geqslant \xi \sum_{k=1}^\infty \frac{\mathbb{I}_{\{X_k=x+3\}}}{Z_k(x+3)} = \infty\right\} \\ &\subset \left\{\sum_{k=1}^\infty \mathbb{I}_{\{X_k=x+3, X_{k+1}=x+4\}} = \infty\right\} \subset \{\sup R' \geqslant x+4\}.\end{aligned}$$

2.2.3 Proof of theorem 1.3

Assume without loss of generality $I = (\overline{\alpha} - 2\epsilon, \overline{\alpha} + 2\epsilon) \subset (0,1)$, with $\overline{\alpha} \in (0,1)$ and $\epsilon > 0$. Let $t_n := t_n(v)$ for simplicity.

There exists $n_0 \in \mathbb{N}^*$ such that, for all $n \geqslant n_0$,

$$\mathbb{P}(\mathcal{E}^-_{t_n}(v, 2, \overline{\alpha} + \epsilon) \cap \mathcal{E}^+_{t_n}(v, 2, 1 - \overline{\alpha} + \epsilon)) > 0,$$

therefore, using corollary 2.2 successively for $\pm := +$ and $-$, if $n \geqslant \mathsf{Cst}(a,\zeta)$, letting

$$\Lambda = \{Z_\infty(v-3) = Z_{t_n}(v-3),\ Z_\infty(v+3) = Z_{t_n}(v+3)\},$$

$$\mathbb{P}(\Lambda) \geqslant \mathbb{E}(\mathbb{P}(\Lambda \mid \mathcal{F}_{t_n})\ \mathbb{1}_{\mathcal{E}^-_{t_n}(v,2,\overline{\alpha}+\epsilon)\cap\mathcal{E}^+_{t_n}(v,2,1-\overline{\alpha}+\epsilon)}) \geqslant \frac{1}{2}\mathbb{P}(\mathcal{E}^-_{t_n}(v,2,\overline{\alpha}+\epsilon)\cap\mathcal{E}^+_{t_n}(v,2,1-\overline{\alpha}+\epsilon)).$$

On the other hand, $\Lambda \subset \{R' = \{v-2, v-1, v, v+1, v+2\}\}$ by lemma 2.2. Corollary 2.1 **(c)** completes the proof. The case $v = v_0$ gives $\mathbb{P}(|R| = 5) > 0$.

2.3 Proof of corollary 2.2

We give the proof for $\pm := -$, and let $t_k := t_k(x-1)$ for simplicity. Assume we belong to $\mathcal{E}^-_n(x,a,\zeta)$. Set $\mu = (1+1/\zeta)/2 \in (1, 1/\zeta)$; let us define the stopping times

$$T_1 = \inf\{k \geqslant n/\ Z_k(x-3) \neq Z_n(x-3)\},$$
$$T_2 = \inf\{k \geqslant n/\ \beta^-_k(x) > \mu\zeta \text{ or } Z_k(x-2) > Z_k(x-1)^{\mu\zeta}\},$$

and consider the two events

$$\Omega_1 = \{T_1 = T_1 \wedge T_2 < \infty\},\ \Omega_2 = \{T_2 = T_1 \wedge T_2 < \infty\}.$$

It suffices to prove that $\mathbb{P}(\Omega_1^c \cap \Omega_2^c \mid \mathcal{F}_n) \geqslant c$.

First, at each time $t_k \geqslant n$ such that $t_k < T_1 \wedge T_2$, the probability to visit $x-3$ at time $t_k + 2$ is

$$\mathbb{P}(X_{t_k+2} = x-3 \mid \mathcal{F}_{t_k}) \leqslant \frac{Z_{t_k}(x-2)}{Z_{t_k}(x-1)}\frac{Z_{t_k}(x-3)}{Z_{t_k}(x-1)} \leqslant \frac{a}{Z_{t_k}(x-1)^{2-\mu\zeta}}.$$

Therefore, if $Z_n(x-1) \geqslant \mathsf{Cst}(a,c,\zeta)$,

$$\mathbb{P}(\Omega_1 \mid \mathcal{F}_n) \leqslant \sum_{i\in\mathbb{N}^*} \frac{a}{(Z_n(x-1)+i)^{2-\mu\zeta}} \leqslant \frac{a}{1-\mu\zeta}\frac{1}{Z_n(x-1)^{1-\mu\zeta}} \leqslant \frac{1-c}{2}. \quad (7)$$

We will now prove that, if $t \leqslant \mathsf{Cst}(\zeta)$,

$$\Omega_2^c \supset \Gamma_n(x,t) = \Lambda^-_n(x,t) \cap \Lambda^-_n(x-1, 8(1-c)^{-1/2}) \cap \Lambda^+_n(x-2,t), \quad (8)$$

which will complete the proof of the corollary since, by lemma 2.2 **(b)**, for all $t > 0$, if $Z_n(x-1) \geqslant \mathsf{Cst}(c,t)$, $\mathbb{P}(\Gamma_n(x,t) \mid \mathcal{F}_n) \geqslant 1 - (1-c)/2$.

Assume we belong to $\Gamma_n(x,t)$, and let $k \geqslant n$ such that $k \leqslant T_1 \wedge T_2$; then, if $Z_n(x-1) \geqslant \mathsf{Cst}(\mu\zeta, t) = \mathsf{Cst}(\zeta,t)$,

$$Y_{n,k}(x-2) \leqslant \sum_{i=n}^{k-1} \frac{\mathbb{1}_{\{X_i = x-2\}}}{Z_i(x-1)} \leqslant \sum_{i\leqslant k-1} \frac{\mathbb{1}_{\{X_i=x-2, Z_i(x-2)\geqslant Z_n(x-1)^{\mu\zeta}\}}}{Z_i(x-2)^{\mu^{-1}\zeta^{-1}}} + \frac{Z_n(x-1)^{\mu\zeta}}{Z_n(x-1)} \leqslant t$$

and, similarly as in lemmas 2.1 and 2.2, since we belong to $\Gamma_n(x,t)$,

$$\ln\frac{Z_k(x-1)}{Z_n(x-1)} \leqslant t+Y^+_{n,k}(x-2)+Y^-_{n,k}(x) \leqslant 3t+Y_{n,k}(x-2)+Y_{n,k}(x) \leqslant 5t+\ln\frac{Z_k(x)}{Z_n(x)}; \quad (9)$$

hence

$$\ln\beta^-_k(x) = \ln\frac{Z_k(x-1)}{Z_k(x)} \leqslant 5t+\ln\frac{Z_n(x-1)}{Z_n(x)} \leqslant 5t+\ln\zeta \leqslant \ln\mu\zeta$$

if $t \leqslant \mathsf{Cst}(\mu) = \mathsf{Cst}(\zeta)$.

Now, an equation similar to (4) occurs: letting $t' = t+8(1-c)^{-1/2}$,

$$\ln\frac{Z_k(x-2)}{Z_n(x-2)} \leqslant t+Y^-_{n,k}(x-1) \leqslant t'+Y_{n,k}(x-1) = t'+\sum_{i=n}^{k-1}\frac{1\!\!1_{\{X_i=x-1\}}}{Z_k(x-1)}\beta^-_k(x)$$
$$\leqslant t'+\mu\zeta\left(t+\ln\frac{Z_k(x-1)}{Z_n(x-1)}\right).$$

Hence

$$\ln Z_k(x-2) \leqslant \zeta\ln Z_n(x-1)+t'+\mu\zeta t+\mu\zeta\ln\frac{Z_k(x-1)}{Z_n(x-1)} \leqslant \mu\zeta\ln Z_k(x-1)$$

since $\zeta\ln Z_n(x-1)+t'+\mu\zeta t \leqslant \mu\zeta\ln Z_n(x-1)$ if $Z_n(x-1) \geqslant \mathsf{Cst}(c,t,\zeta)$. Therefore $k < T_2$; this completes the proof of (8).

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