Habilitation à diriger des recherches

UNIVERSITÉ PAUL SABATIER TOULOUSE III

Domaine : Mathématiques

par

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Auto-interaction et apprentissage dans les structures aléatoires

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À Telma

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Contents

Chapter 1

Introduction

Self-interacting random processes evolve in an environment constantly modified by their own behaviour. Depending on the nature of this self-interaction, they can be self-repelling or self-attracting, in other words more likely to stay away from or to come back to the places already visited before, or strategies that have been played before. These non-Markov processes "learn" from their past behaviour, either localizing on particular subsets or strategies or on the contrary scattering, as a consequence of the interaction feature.

The first part on self-interacting random walks starts with a description of almost-sure localization results I could obtain, namely the conjecture of Pemantle and Volkov that the vertex-reinforced random walk (VRRW) on $\mathbb{Z}$ eventually gets stuck at five (random) sites [A1], and Sellke's conjecture (jointly with Limic [A3]) that the strongly edge-reinforced random walk on general graphs eventually localizes in a single edge. These proofs are based on the same technique combining carefully chosen "local" martingales to describe a global behaviour.

Then I describe a joint result with Benaïm [A5] on localization with positive probability of the VRRW on general graphs, generalizing one of Volkov [90]: our proof is based on a different technique, using an approximation of the occupation density by the *replicator* dynamics invented by Fisher, Wright et Haldane in the 1920s to model the evolution of populations submitted to natural selection [41, 80].

Our analysis shows that the set of dominating species in this *replicator* dynamics has to be a complete d-partite subset for some $d \geqslant 2$ in the general case, and hence that there is a strong "clustering" effect; we deduce localization results on our walk.

We finish that first part by a recent result with Sabot [A6], which represents the edge-reinforced random walk (ERRW) introduced by Coppersmith-Diaconis [20], as a vertex-reinforced jump process (VRJP, invented by Werner [25]) with independent gamma conductances. We calculate the limit measure of the latter and show that it can be interpreted as a supersymmetric hyperbolic sigma model introduced by Disertori, Spencer and Zirnbauer [32] in quantum field theory, which enables us to deduce that VRJP and ERRW with large reinforcement are strongly recurrent in any dimension. The question of recurrence/transience of edge-reinforced random walks was initially raised by Diaconis in 1986.

The second part is on Brownian polymers, which are a certain type of self-interacting diffusions. It is dedicated to the proofs of two conjectures of Durrett-Rogers in 1992, the first with Mountford [A7] on heavy-tailed interaction, and the second with Tóth and Valkó [A8] on local interaction. The second result shows that a smoothed version of the local time seen from the particle is a Markov process with *Gaussian invariant measure,* which introduces a new tool for the analysis of these polymers.

The last part describes results on adaptive learning algorithms. The first one is theoretical and concerns criteria for nonconvergence towards "traps", i.e. unstable subsets of the associated deterministic dynamics. It was shown during my PhD [A9,A10], but had a great impact on the rest of this work, in particular the study of self-interacting walks, so I recall it for convenience.

The second one studies the Narendra two-armed bandit algorithm, and shows that there is a phase transition in its behaviour, with respect to the "speed" of learning: if it learns too fast it can get trapped into a bad strategy for ever. We describe under which assumptions this behaviour can occur, first with Lamberton and Pagès [A11] in the i.i.d. case, second with Vandekerkhove [A12] under ergodic assumptions.

The third one shows with Yao [A13] that inference from examples in statistical learning can be achieved by "online" algorithms at the same speed as for classical "batch" algorithms, through a careful choice of step sequences, using an analysis relying on techniques from stochastic algorithms: the advantage of "online" procedures lies in their flexibility and computational complexity.

The last one, with Hu and Skyrms [A14], analyzes a model describing how different agents can learn to signal without direct communication, by reinforcing themselves on state-signal connections that have worked out, originally introduced by Argiento, Pemantle, Skyrms and Volkov, who solved the 2 agents 2 signals case[1]. We show in general that, on the bipartite graph of states and signals, the state-signals choices eventually concentrate with positive probability on any set of edges that have the property that no signal can simultaneously be synonymic and polysemic.

Publications

by section and [citation] number

2 Self-interacting random walks

2.1.2 First proof of the conjecture of Pemantle-Volkov
[A1] Vertex-reinforced random walk on $\mathbb{Z}$ eventually gets stuck on five points, Annals of Probability 32 (2004), no.3B, 2650-2701.

2.1.4 New proof of the conjecture of Pemantle-Volkov
[A2] Localization of Reinforced Random Walks.
Available on http://arxiv.org/abs/1103.5536.

2.1.5 Proof of Sellke's conjecture
[A3] Attracting edge and strongly edge reinforced random walks, with V. Limic, Annals of Probability (2007), Vol. 35, No. 5, 1783-1806.
[A4] What is the difference between a square and a triangle? with V. Limic, In and out of equilibrium 2. Series: Progress in Probability (2008), Vol. 60, 481–496, Birkhäuser.

2.2 Localization: positive probability results
[A5] Dynamics of vertex-reinforced random walks, with M. Benaïm, Annals of Probability (2011), Vol. 39, No. 6, 2178–2223.

2.3 Edge-reinforced random walk, vertex-reinforced jump process and a conjecture of Diaconis
[A6] Edge-reinforced random walk, Vertex-Reinforced Jump Process and the supersymmetric hyperbolic sigma model, with Christophe Sabot.
Available on http://arxiv.org/abs/1111.3991.

3 Brownian polymers

3.1 Conjecture 3 of Durrett-Rogers in [36]
[A7] An asymptotic result for Brownian polymers, with T. Mountford, Annales de l'Institut Henri Poincaré, Probabilités and Statistiques (2008), Vol. 44, No.1, 29-46.

3.2 Conjecture 2 of Durrett-Rogers in [36]
[A8] Diffusivity bounds for 1d Brownian polymers, with B. Tóth and B. Valkó, to appear in Annals of Probability (2011).
Available on http://www.imstat.org/aop/future_papers.htm.

4 Stochastic algorithms

4.1 Unstable traps
[A9] Pièges répulsifs, Comptes rendus de l'Académie des Sciences 330 (2000), Série I, p.125-130.
[A10] Pièges répulsifs, additif, Chapter 3 of PhD thesis.

4.2 Narendra two-armed bandit algorithm
[A11] When can the two-arm bandit algorithm be trusted?, with D. Lamberton and G. Pagès, Annals of Applied Probability 14 (2004), no.3, 1424-1454.
[A12] On ergodic two-armed bandits, with P. Vandekerkhove, to appear in Annals of Applied Probability (2012).
Available on http://www.imstat.org/aap/future_papers.htm

4.3 Online learning algorithms
[A13] Online learning as stochastic approximation of the regularization paths, with Y. Yao. Available on http://arxiv.org/abs/1103.5538.

4.4 Reinforcement learning in signaling game
[A14] Reinforcement learning in signaling game, with Y. Hu and B. Skyrms. Available on http://arxiv.org/abs/1103.5818.

The following paper is not discussed in these notes, since I did not work on it or develop related techniques after my PhD.

[A15] Generalized urn models of evolutionary processes, with M. Benaïm and S. Schreiber, Annals of Applied Probability 14 (2004), no.3, 1455-1478

Chapter 2

Self-interacting random walks

Let $(\Omega, \mathcal{F}, \mathbb{P})$ probability space. Let $(G; \sim)$ nonoriented locally finite graph; let $V = V(G)$ (resp. $E = E(G)$) be its set of vertices (resp. nonoriented edges). For any $e \in E(G)$, let $W_e : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$ be a weight function. Let $(X_n)_{n \in \mathbb{N}}$ be a random process that takes values in $V(G)$, and let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ be the filtration of its past. For all $v \in G$, $n \in \mathbb{N} \cup \{\infty\}$, let

$$Z_n(v) = \sum_{k=0}^{n} \mathbb{1}_{\{X_k = v\}} + 1 \tag{2.1}$$

be the number of visits to v up to time n plus one.

Then $(X_n)_{n \in \mathbb{N}}$ is a Vertex Self-Interacting Random Walk (VSIRW) with starting point $v_0 \in G$ and weight functions W_e, $e \in E(G)$, if $X_0 = v_0$ and, for all $n \in \mathbb{N}$, if $X_n = i$ then

$$\mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{i \sim j} \frac{W_{\{i,j\}}(Z_n(j))}{\sum_{k \sim i} W_{\{i,k\}}(Z_n(k))}. \tag{2.2}$$

An Edge Self-Interacting Random Walk (ESIRW) is defined similarly, replacing in (2.2) the numbers of visits to vertices $l \sim i$ by those to the corresponding nonoriented edges $\{i, l\}$:

$$Z_n(\{i,l\}) := \sum_{k=1}^{n} (\mathbb{1}_{\{X_{k-1}=i, X_k=l\}} + \mathbb{1}_{\{X_{k-1}=l, X_k=i\}}); \tag{2.3}$$

for notational reasons, we do not add one to that number, contrary to (2.1).

When the weight functions W_e are identical equal to W for all $e \in E(G)$, the process will simply be called VSIRW or ESIRW with weight function W.

We will define the Edge (resp. Vertex) Reinforced Random Walk (ERRW, resp. VRRW) as an ESIRW (resp. VSIRW) with affine weight function $W(n) = n + \Delta$, $\Delta > 0$ (resp. $\Delta > -1$); we will in later sections also consider the case of affine weight functions depending on edges. These processes were introduced by Coppersmith and Diaconis in 1986 [20].

My interest in these walks was triggered by the fact that they can eventually only visit a finite consecutive subset of the graph, which we call here *localization* behaviour. Note that the standard recurrence/transience dichotomy does not apply anymore, since the self-interaction makes the process non-Markov.

This localization occurs both gradually and erratically, in the sense that the walk first concentrates on several disconnected clusters -separated by seldom visited sites- so that the relative numbers of visits follow a rather unpredictable dynamics, before it finally settles in a small subset. This makes almost sure localization particularly interesting to analyse.

Let us define the two following subsets of the graph, respectively called range and asymptotic range of the process $(X_n)_{n\geqslant 0}$:

$$R := \{v \in G \text{ s.t. } Z_\infty(v) \neq Z_0(v)\} \tag{2.4}$$

$$R' := \{v \in G \text{ s.t. } Z_\infty(v) = \infty\}. \tag{2.5}$$

The equalities and inclusions of probability events are understood to hold almost surely.

I will present in Section 2.1 my results of almost sure localization [A1,A2], and [A3] with Limic, whereas Section 2.2 will explain a theorem of localization with positive probability with Benaïm [A5], using different techniques.

Finally, I will discuss in Section 2.3 a recent result with Sabot [A6], which links edge-reinforced random walk (ERRW) to the vertex-reinforced jump process (VRJP) invented by Werner and introduced by Davis and Volkov [25], and to the supersymmetric hyperbolic sigma model in quantum field theory introduced by Disertori, Spencer and Zirnbauer [32].

2.1 Localization: almost sure results

2.1.1 Statement of the theorems

My first work on localization was on the vertex-reinforced random walk (VRRW) on the integers $\mathbb{Z}$. Pemantle and Volkov [70] had shown that the walk a.s. visits only finitely many vertices, and that it localizes on any set of five consecutive sites with positive probability; I proved that the latter is the a.s. behaviour.

Theorem 1 (Pemantle and Volkov, [70], VRRW on $\mathbb{Z}$) $|R'| < \infty$ *a.s. and, for any* $x \in \mathbb{Z}$, $\mathbb{P}(R' = \{x-2, x-1, x, x+1, x+2\}) > 0$.

Theorem 2 ([A1,A2], VRRW on $\mathbb{Z}$) $|R'| = 5$ *a.s.*

This localization of VRRW, i.e. VSIRW with linear weight function, corresponds to the critical case in the scale of polynomial weights, as the following result of Volkov shows.

Theorem 3 (Volkov, [91], VSIRW on $\mathbb{Z}$) *Suppose that $W(k)/k^\alpha$ converges to $\xi \in (0,\infty)$, $\alpha \in \mathbb{R}$. Then*

(a) *If $\alpha < 1$, then $R' \in \{0,\infty\}$*

(b) *If $\alpha > 1$, then $|R'| = 2$.*

Schapira [81] further obtained a $0-1$ law for the VSIRW with weight of order $W(k) = k^\alpha$, $\alpha \in [0, 1/2)$, i.e. that the walk is either almost surely recurrent or almost surely transient.

The first proof of Theorem 2, in my PhD thesis, was particularly long (75 pages); I gradually simplified it until the end of 2003, which led to a shorter but still technical version of 55 pages published in [A1]. The argument was based on the analysis of the repartition of "seldom visited" sites $x \in \mathbb{Z}$ (on which $\Upsilon(x)$ holds, see (2.6)); this led to study several types of unstable patterns of the dynamics, associated to the different possibilities of repartitions of these sites. It is briefly explained in Section 2.1.2.

I later proposed a shorter argument [A2], introducing a new method of analysis of the instability of the dynamics, based on a variant of Rubin construction, which defines a time-continuous equivalent of the walk. This enables one to couple a VSIRW with nondecreasing weights W, with a small modification of the walk which leans more towards the right (see Definition 2.1.1 Section 2.1.4), and thus prove the impossibility of certain unstable limit dynamics.

This latter technique was very recently used by Basdevant, Schapira et Singh [5] to show a.s. localization of the VSIRW with weight $W(n) := n \log\log n$ on 4 or 5 random consecutive sites, these two patterns occuring with strictly positive probability.

Note that the new proof in [A2] interprets the timeline events as a Poisson point process with constant intensity after a deterministic time change, using a result of Kendall [46]; this point of view was further developed in recent work with Sabot [A6].

Let us now discuss edge self-interacting random walks (ESIRW): the next elementary Proposition 2.1.1 implies that we cannot expect localization anymore for linear reinforcement.

Let (H) be the following condition on W:

$$\sum_{k\in\mathbb{N}} \frac{1}{W(k)} < \infty.$$

Proposition 2.1.1 (ESIRW on G connected) *Assume that W is nondecreasing and that (H) does not hold. Then*

$$\{|R'| \neq 0\} = \{R' = G\} \ a.s.$$

PROOF: Let $t_n := t_n(x)$ be the n-th visit time to x, then

$$\sum_{z\sim x} Z_{t_n(x)}(\{x,z\}) = 2n + a,$$

where $a := \mathbb{1}_{\{X_0 \neq x\}}$. Hence, for all $z \sim x$ and $n \in \mathbb{N}$,

$$\mathbb{P}(X_{t_n+1} = z | \mathcal{F}_{t_n}) \mathbb{1}_{\{t_n < \infty\}} \geqslant \frac{\mathbb{1}_{\{t_n < \infty\}} W(0)}{N W(2n+a)} \geqslant \frac{\mathbb{1}_{\{t_n < \infty\}} W(0)}{2N} \left(\frac{1}{W(2n+1)} + \frac{1}{W(2n+2)} \right)$$

where $N := |\{w \in V : z \sim x\}|$, using that W is nondecreasing.

Therefore, using conditional Borel-Cantelli Lemma,

$$\begin{aligned} \{Z_\infty(x) = \infty\} &\subseteq \left\{ \sum_{n \in \mathbb{N}} \mathbb{P}(X_{t_n+1} = z | \mathcal{F}_{t_n}) \mathbb{1}_{\{t_n < \infty\}} = \infty \right\} \\ &= \left\{ \sum_{n \in \mathbb{N}} \mathbb{1}_{\{X_{t_n+1} = z\}} \mathbb{1}_{\{t_n < \infty\}} = \infty \right\} \subseteq \{Z_\infty(z) = \infty\} \text{ a.s.} \end{aligned}$$

□

Remark 2.1.1 The condition that W should be nondecreasing is important in Proposition 2.1.1. The following counterexample was proposed by Sellke [82]: if $\sum W(2k)^{-1} = \infty$ and $\sum W(2k+1)^{-1} < \infty$, $G = \mathbb{R}^d$ and $X_0 = 0$, then

$$\mathbb{P}(\forall n \in \mathbb{N},\, X_{2n} = 0) > 0.$$

This result can be shown easily, using the time-lines construction in Section 2.1.3.

Now it is easy to show that (H) implies localization on a single edge with positive probability. Sellke [82] conjectured in 1994 that this should occur with probability one on any graph of bounded degree, and proved the statement on $\mathbb{Z}^d$, with an argument that easily extends to bipartite graphs. Surprisingly, even the case of a triangle could not be solved by that technique (see [A4]).

We showed the conjecture in the case of nondecreasing W with Limic [A3], who had solved the case $W(k) = (k+1)^\rho$ in 2003.

Theorem 4 (Sellke [82], Theorem 3) *If $(G, \sim)$ is a bipartite graph of bounded degree, then (H) implies $|R'| = 2$ a.s.*

Theorem 5 ([A3], Corollary 3) *If $(G, \sim)$ has bounded degree and W is nondecreasing, then (H) implies $|R'| = 2$ a.s.*

Theorem 4 is proved in Section 2.1.3, and the key steps of the proof of Theorem 5 are explained in Section 2.1.5.

2.1.2 First proof of the conjecture of Pemantle-Volkov [A1]

The aim of this section is to sketch the proof of Theorem 2 in [A1]. We assume for convenience that $\Delta = 0$, i.e. that $W(n) = n$. For all $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}$, denote

$$Z_n^{\pm}(x) := \sum_{k=1}^{n} \mathbb{1}_{\{X_{k-1}=x, X_k=x\pm1\}},$$
$$\alpha_n^{\pm}(x) := \frac{Z_n(x \pm 1)}{Z_n(x-1) + Z_n(x+1)},$$
$$Y_n^{\pm}(x) := \sum_{k=1}^{n} \mathbb{1}_{\{X_{k-1}=x, X_k=x\pm1\}} \frac{1}{Z_{k-1}(x \pm 1)},$$

also

$$Y_n(x) := \sum_{k=1}^{n} \mathbb{1}_{\{X_{k-1}=x\}} \frac{1}{Z_{k-1}(x-1) + Z_{k-1}(x+1)},$$
$$\hat{Y}_n^{\pm}(x) := Y_n^{\pm}(x) - Y_n(x),$$

which are respectively the previsible and martingale part in the Doob decomposition of $Y_n^{\pm}(x)$, and finally

$$Y_\infty^{\pm}(x) := \lim_{n\to\infty} Y_n^{\pm}(x),\ Y_\infty(x) := \lim_{n\to\infty} Y_n(x).$$

Given (a_n), (b_n) random processes on $\mathbb{R}$, we write $a_n \equiv b_n$ iff $a_n - b_n$ converges a.s. Let us define the probability event

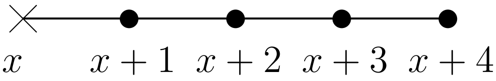
$$\Upsilon(x) := \{Y_\infty(x) < \infty\} \tag{2.6}$$

and, for any finite sequence $(x_i)_{1\leqslant i\leqslant n}$ taking values in $\mathbb{Z}$, the event

$$\Upsilon((x_i)_{1\leqslant i\leqslant n}) = \bigcap_{1\leqslant i\leqslant n} \Upsilon(x_i).$$

On the event $\Upsilon(x)$, x is "seldom" visited, which we represent by a cross on the figure below, hence "neutral" with respect to its neighbours, in the following sense: the respective visits to $x+1$ and $x+3$ starting from $x+2$ evolve similarly as in a Pólya urn model, with the exception that they are perturbed by the visits from x (negligible) and $x+4$ (unknown), which is the heuristics of **(a)-(b)** in the following Corollary 2.1.1 of Proposition 2.1.2.

Another interpretation is that $\Upsilon(x)$ is the event on which the cumulative time of the continuous-time counterpart of the VRRW is finite; see Sections 2.1.3 and 2.1.4.



Proposition 2.1.2 ([A1], Proposition 3.1) *For all $x \in \mathbb{Z}$,*

(a) $\hat{Y}_n^{\pm}(x) = Y_n^{\pm}(x) - Y_n(x)$ *is a martingale, converging a.s. and in* L^2
(b) $Y_n^{\pm}(x) \equiv Y_n(x)$
(c) $\log Z_n(x) \equiv Y_n^{+}(x-1) + Y_n^{-}(x+1)$

PROOF: It follows from its definition that $(\hat{Y}_n^{\pm}(x))_{n \geqslant 0}$ is a martingale. Now

$$\mathsf{Var}(\hat{Y}_{n+1}^{\pm}(x)|\mathcal{F}_n) = \mathsf{Var}(Y_{n+1}^{\pm}(x)|\mathcal{F}_n) \leqslant \mathbb{E}((Y_{n+1}^{\pm}(x) - Y_n^{\pm}(x))^2|\mathcal{F}_n)$$
$$= \mathbb{E}\left(\frac{\mathbb{1}_{\{X_n = x, X_{n+1} = x\pm 1\}}}{Z_n(x \pm 1)^2}|\mathcal{F}_n\right)$$

Hence, for all $n \in \mathbb{N}$,

$$\mathbb{E}((\hat{Y}_n^{\pm}(x))^2) \leqslant \mathbb{E}\left(\sum_{k=0}^{\infty} \frac{\mathbb{1}_{\{X_k = x, X_{k+1} = x\pm 1\}}}{Z_k(x \pm 1)^2}\right) \leqslant \sum_{l = Z_0(x\pm 1)}^{\infty} \frac{1}{l^2} \leqslant \frac{\pi^2}{6}.$$

Hence $\hat{Y}_n^{\pm}(x)$ is bounded in L^2. This implies **(a)-(b)**; **(c)** follows from definitions. □

Corollary 2.1.1 ([A1], Corollary 3.1) *For all $x \in \mathbb{Z}$,*

(a) $\Upsilon(x) \subseteq \{\exists \alpha_\infty^-(x+2) := \lim \alpha_n^-(x+2) \in [0,1)\}$
(b) $\Upsilon(x) \cap \{\alpha_\infty^-(x+2) > 0\} \subseteq \Upsilon(x+4)$
(c) $\Upsilon(x-1, x+1) \subseteq \{Z_\infty(x) < \infty\}$

PROOF: By Proposition 2.1.2 **(b)-(c)**, a.s. on $\Upsilon(x) = \{Y_\infty^+(x) < \infty\}$,

$$\log Z_n(x+1) \equiv Y_n^+(x) + Y_n^-(x+2) \equiv Y_n^+(x+2) \equiv \log Z_n(x+3) - Y_n^-(x+4),$$

so that

$$\log \frac{Z_n(x+1)}{Z_n(x+3)} \equiv -Y_n^-(x+4).$$

Now **(c)** is a direct consequence of Proposition 2.1.2 **(b)-(c)**: a.s. on $\Upsilon(x-1, x+1)$,

$$\log Z_\infty(x) \equiv Y_\infty^+(x-1) + Y_\infty^-(x+1) < \infty.$$

□

Next we can claim a kind of "propagation rule" on seldom visited sites, given by the following

Proposition 2.1.3 ([A1], Proposition 2.1) *For all* $x \in \mathbb{Z}$*,*

$$\Upsilon(x) \subseteq \Upsilon(x+1) \cup \Upsilon(x+4).$$

By Corollary 2.1.1 **(b)**, in order to prove Proposition 2.1.3, it is sufficient to show that, for all $x \in \mathbb{Z}$,

$$\Upsilon_0(x) := \Upsilon(x) \cap \{\alpha_\infty^-(x+2) = 0\} \subseteq \Upsilon(x+1). \tag{2.7}$$

Let us sketch the proof of (2.7). Assume $x := 0$ for simplicity.

The argument relies on a thorough analysis of the following stochastic approximation recursion, see Section 4 [A1]. Let, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} S_n &:= Z_n(4) + Z_n(2) - (Z_n(1) + Z_n(3)) \\ t_n &:= \inf\left\{m \in \mathbb{N}_0 / \ Z_m^+(2) \geqslant n\right\}, \\ z_n &:= \log \frac{Z_{t_n}(3)}{Z_{t_n}(2)}, \ y_n := \frac{S_{t_n}}{Z_{t_n}(2) Z_{t_n}(3)}. \end{aligned}$$

Then we "almost" have

$$z_{n+1} - z_n = y_n + \epsilon_{n+1} + r_n, \tag{2.8}$$

where

$$\mathbb{E}(\epsilon_{n+1} \mid \mathcal{G}_n) = 0, \ \ \mathbb{E}(\epsilon_{n+1}^2 \mid \mathcal{G}_n) \asymp \alpha_{t_n}^-(2)/n^2, \ \ |r_n| = O(1/n^{2-\epsilon}) \text{ for all } \epsilon > 0.$$

First, it is possible to deduce from (2.8), through technical estimates, that, a.s. on $\Upsilon_0(x)$, $\limsup Z_n(4)/Z_n(1) \leqslant e$ (see Lemma 2.4 in [A1]), and therefore $z_n \underset{n\to\infty}{\longrightarrow} 0$.

One the other hand, let us now show that

$$\mathbb{P}\left(\lim_{n\to\infty} z_n = 0\right) = 0, \tag{2.9}$$

by an instability argument, using coupling techniques. First note that we cannot directly use a standard result of nonconvergence towards repulsive traps (for instance Theorem 15 Section 4.1), which would require y_n to be a function of z_n.

Instead, we define a partial order on self-interacting random walks: $\mathcal{M}' \gg \mathcal{M}$ if for each site $j \in \mathbb{Z}$, for all $n \in \mathbb{N}$, at the n-th return time to j, $\mathcal{M}'$ has more visited $j+1$ than $\mathcal{M}$ and less visited $j-1$. Then it easy to show that $\mathcal{M}' \gg \mathcal{M}$ implies

$$S'_{t'_n} \geqslant S_{t_n}.$$

Next, we adequately perturb, from a certain time t_n onwards, the VRRW $\mathcal{M}$ into $\mathcal{M}'$, in order on one hand to add, in the right-hand side of (2.8), a factor equivalent to the standard deviation of $\sum_{k\geqslant t_n} \epsilon_k$ in the recursion (2.8), and on the other hand to have comparable probabilities of paths for $\mathcal{M}$ and $\mathcal{M}'$ (see Proposition 4.1 and Lemma 4.2, [A1]). This argument enables us to show that, conditionally to $\mathcal{F}_{t_n}$, z_n does not converge to 0 with lower bounded probability, and hence to complete the proof of (2.9), (2.7) and Proposition 2.1.3.

We are now ready to describe the repartition of seldom visited sites. Let, for all $x \in \mathbb{Z}$,

$$\Omega(x) = \{x = \inf R'\}.$$

Using Theorem 1, there exists a.s. $x \in \mathbb{Z}$ such that $\Omega(x)$ holds. The following Lemma 2.1.1 shows that, a.s. on $\Omega(x)$, only certain "pavements" of the lattice by events $\Upsilon(x)$ can occur.

Let us define the events

$$\begin{aligned}
&\Omega_0(x) = \Omega(x) \cap \{Z_\infty(x+5) < \infty\}\\
&\Omega_1(x) = \Upsilon(x, x+4, x+8) \cap \{Z_\infty(x+1) = Z_\infty(x+7) = \infty\}\\
&\Omega_2(x) = \Upsilon(x-1, x, x+4, x+5, x+9, x+10\} \cap \{Z_\infty(x+1) = Z_\infty(x+8) = \infty\}
\end{aligned}$$

Lemma 2.1.1 ([A1], Lemma 2.6) *For all $x \in \mathbb{Z}$,*

$$\Omega(x) \subseteq \Omega_0(x) \cup \Omega_1(x) \cup \Omega_1(x+5) \cup \Omega_2(x).$$

PROOF: First, for all $y \in \mathbb{Z}$, using Proposition 2.1.3 and Corollary 2.1.1 **(c)**,

$$\Upsilon(y-1, y) \cap \{Z_\infty(y) = \infty\} \subseteq \Upsilon(y+4). \tag{2.10}$$

Now $\Omega(x) \subseteq \Upsilon(x-1, x)$ by Proposition 2.1.2 **(b)**. Hence

$$\begin{aligned}
&\Omega(x) \cap \Omega_0(x)^c \subseteq \Upsilon(x-1, x) \cap \{Z_\infty(x) = Z_\infty(x+5) = \infty\}\\
&\subseteq \Upsilon(x-1, x, x+4) \cap \{Z_\infty(x) = Z_\infty(x+5) = \infty\}\\
&\subseteq (\Upsilon(x-1, x, x+4, x+5) \cup \Upsilon(x, x+4, x+8)) \cap \{Z_\infty(x+1) = Z_\infty(x+7) = \infty\}\\
&\subseteq \Omega_1(x) \cup (\Upsilon(x-1, x, x+4, x+5) \cap \{Z_\infty(x+1) = Z_\infty(x+8) = \infty\}).
\end{aligned}$$

In the third inclusion, we use (2.10) with $y := x+4$ and, in the fourth one, we note that $Z_\infty(x+7) = \infty$ since $\alpha_n^-(x+6)$ converges on $\Upsilon(x+4)$ by Corollary 2.1.1 **(a)**. Finally, we use in the last one that $Z_\infty(x+8) = \infty$ on $\Omega(x) \cap \Omega_0(x)^c \cap \Omega_1(x)^c \subseteq \Upsilon(x+8)^c$.

Applying again (2.10) with $y := x+5$ and Proposition 2.1.3,

$$\begin{aligned}
&\Omega(x) \cap \Omega_0(x)^c \cap \Omega_1(x)^c\\
&\subseteq \Upsilon(x-1, x, x+4, x+5, x+9) \cap \{Z_\infty(x+1) = Z_\infty(x+8) = \infty\}\\
&\subseteq \Omega_2(x) \cup \Omega_1(x+5),
\end{aligned}$$

where we use in the last inclusion that $Z_\infty(x+10) = Z_\infty(x+12) = \infty$ on $\Upsilon(x+9) \cap \Upsilon(x+10)^c$, since $\alpha_\infty^-(x+11) \in [0, 1)$. □

Now, for all $x \in \mathbb{Z}$, $\Omega_1(x)$ and $\Omega_2(x)$ are of probability 0, as stated in Lemmas 2.1.2 and 2.1.3. These results complete the proof of the conjecture.

Lemma 2.1.2 ([A1], Lemma 2.7) *For all $x \in \mathbb{Z}$, $\mathbb{P}(\Omega_1(x)) = 0$.*

Lemma 2.1.3 ([A1], Lemma 2.8) *For all $x \in \mathbb{Z}$, $\mathbb{P}(\Omega_2(x)) = 0$.*

The proofs of Lemmas 2.1.2 and 2.1.3 rely on an analysis of instability similar to the one carried out for the proof of (2.7). They can be bypassed by the new proof, presented in Section 2.1.4, which shows in particular that, for all $x \in \mathbb{Z}$,

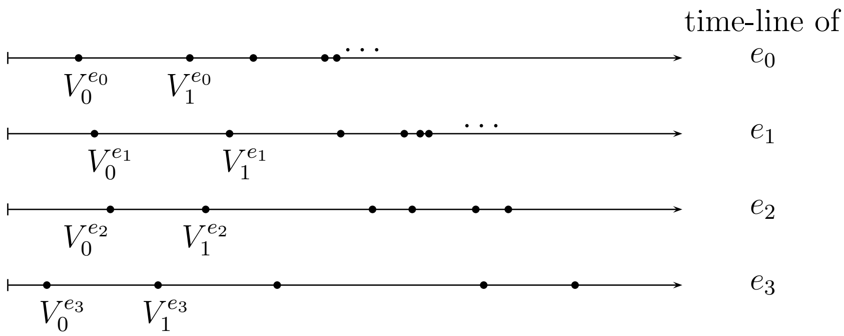
$$\Upsilon(x) \subseteq \{Z_\infty(x-1) < \infty\} \cup \{Z_\infty(x+1) < \infty\}.$$

2.1.3 Rubin continuous time-lines construction

In this section we construct a continuous-time process $(\tilde{X}_t)_{t\in\mathbb{R}_+}$, equal in law to $(X_n)_{n\geqslant 0}$ for ESIRW at times of jumps, initially introduced by Rubin, Davis and Sellke [23, 82], and explain how it can be used to show Theorem 4. We will introduce in Section 2.1.4 an alternative version of that timelines construction for VSRIW.

Let $(\tau_i^e)_{e\in E, i\in\mathbb{N}_0}$ be a collection of i.i.d. exponential random variables of parameter 1. For each edge $e \in E$, we set up a clock with alarms at times

$$V_k^e := \sum_{i=0}^{k} \frac{\tau_i^e}{W_e(i)},\ k \in \mathbb{N}_0 \cup \{\infty\}.$$



The process starts at $\tilde{X}_0 := x_0$ at time 0:

- The clock of an edge e runs when the process $(\tilde{X}_t)_{t\geqslant 0}$ is adjacent to e.
- Each time an edge e sounds an alarm, $\tilde{X}_t$ crosses it instantaneously.

Let ζ_n be the n-th jump time of $(\tilde{X}_t)_{t\geqslant 0}$, with the convention that $\zeta_0 := 0$.

Lemma 2.1.4 (Davis [23], Sellke [82]) *The processes $(\tilde{X}_{\zeta_n})_{n\geqslant 0}$ and $(X_n)_{n\geqslant 0}$ have the same distribution.*

PROOF: The proof is based on the memoryless property of exponentials, and on the observation that, if A and B are two independent random variables of parameters a and b, then $\mathbb{P}[A < B] = a/(a+b)$. □

Let

$$\mathcal{G}_\infty := \{e \in E(G) \text{ s.t. } Z_\infty(e) = \infty\}.$$

An immediate but important consequence of this construction is the following

Proposition 2.1.4 (Davis [23], Sellke [82]) *If, for all $e \in E$, $\sum_{n\in\mathbb{N}} 1/W_e(n) < \infty$, then $\mathcal{G}_\infty$ contains no even cycle.*

PROOF: For all $e \in E$, the assumption $\sum W_e(n)^{-1} < \infty$ implies $\mathbb{E}(V_\infty^e) < \infty$ (hence $V_\infty^e < \infty$ a.s.).

For simplicity, let us denote an even cycle by $\mathbb{Z}/\ell\mathbb{Z}$, ℓ even. Now

$$\{\mathbb{Z}/\ell\mathbb{Z} \subseteq \mathcal{G}_\infty\} \subseteq \left\{ L := \sum_{x\in\mathbb{Z}/\ell\mathbb{Z}} (-1)^x V_\infty^{\{x,x+1\}} = 0 \right\}.$$

Now $L \neq 0$ a.s., as $\tau_0^{\{0,1\}} W_{\{0,1\}}(0)^{-1}$ is independent from $L - \tau_0^{\{0,1\}} W_{\{0,1\}}(0)^{-1}$ and has continuous density, which implies that

$$\mathbb{P}\left(\mathbb{Z}/\ell\mathbb{Z} \subseteq \mathcal{G}_\infty\right) = 0.$$

□

A similar argument can be used to show that, on a graph of bounded degree, under the same assumption of reciprocally summable weight functions W_e, $\mathcal{G}_\infty$ cannot be a tree, unless it consists of a single edge. Then Theorem 4 follows readily, since cycles of a bipartite graph are necessarily even.

Note that, heuristically, the proof of Proposition 2.1.4 exploits the idea that certain limit configurations, such as even limit cycles, are unstable. The martingale technique I contributed to develop in [A9,A10] can also be adapted to yield the same result, as explained Section 2.1.5 (see also Section 3 [A4]); this is one of the building blocks in the proof of Theorem 4 on strongly edge reinforced random walk, since this martingale technique is more adaptable to small perturbations than the timelines technique.

2.1.4 New proof of the conjecture of Pemantle-Volkov [A2]

The aim of this section is to sketch the new proof of Theorem 2, developed in [A2]. The argument relies on the following two propositions.

Proposition 2.1.5 ([A2], Proposition 4.1) *For all $x \in \mathbb{Z}$,*

$$\Upsilon(x) \subseteq \{Z_\infty(x-1) < \infty\} \cup \{Z_\infty(x+1) < \infty\} \text{ a.s.}$$

Proposition 2.1.6 ([A2], Proposition 4.2) *For all $x \in \mathbb{Z}$, $\Omega(x) \subseteq \Upsilon(x+4)$ a.s.*

These will imply Theorem 2, i.e. a.s. localization on the VRRW on five consecutive vertices: a.s. on $\Omega(x)$, $Z_\infty(x+3) < \infty$ or $Z_\infty(x+5) < \infty$ by Propositions 2.1.5 and 2.1.6, and the former cannot occur since $\Upsilon(x)$ holds and $\alpha_\infty^-(x+2) < 1$ by Corollary 2.1.1**(a)**.

Let us now explain the proof of Proposition 2.1.5. Let $\vec{E}$ be the set of directed edges of $\mathbb{Z}$. For all $e = (x,y) \in \vec{E}$, denote $\underline{e} := x$, $\overline{e} := y$, $\sigma(e) := (y,x)$. For all $e = \{j, j+1\} \in E$, set $\underline{e} := j$, $\overline{e} := j+1$.

The argument makes use of a timelines construction for the VRRW but, instead of putting a clock at each vertex as a natural counterpart would do, it sets up one clock at each directed edge $\vec{E}$. Indeed, this will enable us to introduce a coupling, by a simple modification of the collection $(\tau_i^e)_{e\in\vec{E},i\in\mathbb{N}_0}$.

Let us define a continuous time process $(\tilde{X}_t)_{t\in\mathbb{R}_+}$ taking values in $\mathbb{Z}$; for all $x \in V(G)$ and $t \geqslant 0$, by a slight abuse of notation, let also denote by $Z_t(x)$ its number of visits to x plus one at time t:

- Let $(\tau_k^e)_{e\in\vec{E},k\in\mathbb{N}_0}$ be a collection of independent exponential random variables with expectation one.
- Each oriented edge $e \in \vec{E}$ has its own clock, which only runs when the process $(\tilde{X}_t)_{t\geqslant 0}$ is in $\underline{e}$.
- Each time an edge e has just been crossed, the clock of $\sigma(e)$ sets up an alarm at distance $\tau_k^{\sigma(e)}/Z_t(\underline{e})$, if $\sigma(e)$ has been crossed k times so far. At time 0, we set up an initial alarm, at time distance τ_0^e, for the edges $(x, x+1)$, $x \geqslant x_0$, $(x, x-1)$, $x \leqslant x_0$.
- Each time an edge e sounds an alarm, $\tilde{X}_t$ crosses it instantaneously.

Let ζ_n be the n-th jump time of $(\tilde{X}_t)_{t\geqslant 0}$, with the convention that $\zeta_0 := 0$.

Lemma 2.1.5 ([A2], Lemma 4.1) *The processes $(\tilde{X}_{\zeta_n})_{n\geqslant 0}$ and $(X_n)_{n\geqslant 0}$ have the same distribution.*

The proof of Lemma 2.1.5 is similar to the one of Lemma 2.1.4.

Let us denote by $\mathcal{M}$ the function which maps a (deterministic) "collection of alarms" $\mathcal{T} = (\tau_k^e)_{e\in\vec{E},k\in\mathbb{N}_0}$ and an initial site x_0 to the corresponding continuous-time (deterministic) walk $\mathcal{M}(\mathcal{T}, x_0)$ on the vertices of $\mathbb{Z}$, with the convention that, if two alarms ring simultaneously, the walk goes to the right (however this will occur with probability 0 in our setting).

Definition 2.1.1 ([A2], Definition 4.1) *Given $\mathcal{T} = (\tau_k^e)_{e\in\vec{E},k\in\mathbb{N}_0}$ and $\mathcal{T}' = (\tau_k'^e)_{e\in\vec{E},k\in\mathbb{N}_0}$ two collections of random variables on $\mathbb{R}_+$, we say that $\mathcal{T}' \gg \mathcal{T}$ if, for all $k \in \mathbb{N}_0$, $x \in \mathbb{Z}$, $\tau_k'^{(x,x+1)} \leqslant \tau_k^{(x,x+1)}$ and $\tau_k'^{(x,x-1)} \geqslant \tau_k^{(x,x-1)}$ a.s.*

Given $\mathcal{T} = (\tau_k^e)_{e\in\vec{E},k\in\mathbb{N}_0}$ and $\mathcal{T}' = (\tau_k'^e)_{e\in\vec{E},k\in\mathbb{N}_0}$ two collections of random variables on $\mathbb{R}_+$ we let, by a slight abuse of notation, $\mathcal{M} = (\tilde{X}_t)_{t\in\mathbb{R}_+} := \mathcal{M}(\mathcal{T}, x_0)$ and $\mathcal{M}' = (\tilde{X}'_t)_{t\in\mathbb{R}_+} := \mathcal{M}(\mathcal{T}', x_0)$ be the continuous-time random walks starting at x_0 associated to $\mathcal{T}$ and $\mathcal{T}'$. They will satisfy $\mathcal{M}' \gg \mathcal{M}$ with respect to the coupling discussed in Section 2.1.2; however we need here a different property, described in Definition 2.1.2 and Lemma 2.1.6.

For all $i \in \mathbb{N}_0$, $j \in \mathbb{Z}$ and $e \in E$ or $\vec{E}$, let $n_e(i)$ be the i-th visit crossing of e (with the convention $n_e(0) := 0$), let T_j be the total time spent in j by the random walk $\mathcal{M}$; let $n'_e(i)$ and T'_j be the similar notation for $\mathcal{M}'$.

Definition 2.1.2 ([A2], Definition 4.2) *For all $i \in \mathbb{N}_0$ and $e \in E$, let us define the property $E_{i,e}$ as follows:*

$$Z'_{n'_e(i)}(\overline{e}) \geqslant Z_{n_e(i)}(\overline{e}) \text{ and } Z'_{n'_e(i)}(\underline{e}) \leqslant Z_{n_e(i)}(\underline{e}),$$

with the convention that $E_{i,j}$ holds whenever $n_e(i) = \infty$ or $n'_e(i) = \infty$.

Lemma 2.1.6 ([A2], Lemma 4.2) *Assume $\mathcal{T}' \gg \mathcal{T}$; then, for all $i \in \mathbb{N}_0$ and $e \in E$, $E_{i,e}$ holds a.s.*

The proof can be found in [A2] (see also [5] for a more detailed argument).

Fix $x \in \mathbb{Z}$. Let $\mathcal{T} := (\tau_k^e)_{e \in \vec{E}, k \in \mathbb{N}_0}$ be a collection of independent exponential random variables with expectation 1, and let

$$\mathcal{T}'^{(n)} := (\tau_k'^{(n)e})_{e \in \vec{E}, k \in \mathbb{N}_0} = (\tau_k'^{e})_{e \in \vec{E}, k \in \mathbb{N}_0} := (\tau_k^e + \mathbb{1}_{\{e=(x,x-1)\}} \mathbb{1}_{\{k=n\}})_{e \in \vec{E}, k \in \mathbb{N}_0}.$$

Let $\mathcal{M} = (\tilde{X}_t)_{t \in \mathbb{R}_+} := \mathcal{M}(\mathcal{T}, x_0)$ and $\mathcal{M}'^{(n)} = (\tilde{X}'_t)_{t \in \mathbb{R}_+} := \mathcal{M}(\mathcal{T}', x_0)$. Let, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} \mathcal{Q} &:= \{Z_\infty(x+1) = Z_\infty(x-1) = \infty\} \cap \{T_x < \infty\}, \\ \mathcal{Q}'^{(n)} &:= \{Z'_\infty(x+1) = Z'_\infty(x-1) = \infty\} \cap \{T'_x < \infty\}. \end{aligned}$$

The next Lemma implies that $\Upsilon(x) \cap \{Z_\infty(x-1) = Z_\infty(x+1) = \infty\}$ a.s. does not occur simultaneously for $\mathcal{M}$ and $\mathcal{M}'$, which will allow us to conclude, since the probabilities of the same measurable set of paths are comparable for the two random walks $\mathcal{M}$ and $\mathcal{M}'$.

Lemma 2.1.7 ([A2], Lemma 4.3) *For all $n \in \mathbb{N}_0$, $\mathbb{P}(\mathcal{Q} \cap \mathcal{Q}'^{(n)}) = 0$.*

PROOF: If $Z_\infty(x+1) = Z_\infty(x-1) = Z'_\infty(x+1) = Z'_\infty(x-1) = \infty$ and $T_x < \infty$, $T'_x < \infty$, then

$$T_x = \sum_{k=0}^\infty \frac{\tau_k^{(x,x+1)}}{Z_{n_{(x+1,x)}(k)}(x+1)} = \sum_{k=0}^\infty \frac{\tau_k^{(x,x-1)}}{Z_{n_{(x-1,x)}(k)}(x-1)}$$

and, using Lemma 2.1.6,

$$T'_x = \sum_{k=0}^\infty \frac{\tau_k'^{(x,x+1)}}{Z'_{n'_{(x+1,x)}(k)}(x+1)} \leqslant T_x < \sum_{k=0}^\infty \frac{\tau_k'^{(x,x-1)}}{Z'_{n'_{(x-1,x)}(k)}(x-1)} = T'_x \text{ a.s.},$$

which is contradictory. $\square$

PROOF OF PROPOSITION 2.1.5: Let $\mathcal{F}_n := \sigma(\tilde{X}_0, \ldots, \tilde{X}_n) \subseteq \sigma(\tau_k^e, 1 \leqslant k \leqslant n, e \in \vec{E})$. Then Lemma 2.1.7 implies

$$\mathbb{P}(\mathcal{Q}^c \cup (\mathcal{Q}'^{(n+1)})^c | \mathcal{F}_n) = 1.$$

But
$$\mathbb{P}(\mathcal{Q}^c|\mathcal{F}_n) \geqslant e^{-1}\mathbb{P}((\mathcal{Q}'^{(n+1)})^c|\mathcal{F}_n),$$
so that
$$\mathbb{P}(\mathcal{Q}^c|\mathcal{F}_n) \geqslant (1+e)^{-1}.$$
Now $\mathbb{P}(\mathcal{Q}^c|\mathcal{F}_n) \underset{n\to\infty}{\longrightarrow} \mathbb{1}_{\mathcal{Q}^c}$ a.s., so that $\mathcal{Q}^c$ holds almost surely. □

The proof of Proposition 2.1.6 is similar to the proof of (2.7) in Section 2.1.2, but the timelines construction however simplifies the argument. In particular, we use an interpretation of the sequence of "alarms" in the timeline of particular edge as a Poisson point process with constant intensity after a change of time, which helped link ERRW to the vertex-reinforced jump process (VRJP) in recent work with Sabot [A6], see Section 2.3, Theorem 8.

2.1.5 Proof of Sellke's conjecture [A3,A4]

In this section we sketch the proof of Theorem 5 in [A3]. We assume throughout that $\sum W(n)^{-1} < \infty$ and that the graph is of bounded degree.

First, the technique of proof of Proposition 2.1.4 carries over to show [82, 53] that $\mathcal{G}_\infty$ (defined Section 2.1.3) is a.s. either a single edge or an odd cycle. Therefore the aim is to prove that $\mathcal{G}_\infty$ a.s. cannot be a cycle $\mathbb{Z}/\ell\mathbb{Z}$ of length ℓ, for any ℓ odd.

For all $n \in \mathbb{N}$ and $x \in \mathbb{Z}/\ell\mathbb{Z}$, let

$$\begin{aligned} W^*(n) &:= \sum_{k=0}^{n-1} \frac{1}{W(k)} \\ Y_n^{\pm}(x) &:= \sum_{k=1}^{n} \frac{\mathbb{1}_{\{X_{k-1}=x, X_k=x\pm1\}}}{W(Z_{k-1}(\{x,x\pm1\}))} \\ \epsilon_n(x) &:= Y_n^+(x) - Y_n^-(x) \\ \kappa_n(x) &:= W^*(Z_n(\{x,x+1\})) - W^*(Z_n(\{x,x-1\})); \end{aligned}$$

then $Y_n^{\pm}(x)$ is the equivalent of the process with the same name defined for VRRW in Section 2.1.2, in the sense that $(\epsilon_n(x))_{n\in\mathbb{N}}$ is similarly a martingale, and

$$Y_n^+(x) + Y_n^-(x+1) = W^*(Z_n(\{x,x+1\})).$$

Let us first discuss again the case where ℓ is even: obviously

$$u_n = \sum_{x\in\mathbb{Z}/\ell\mathbb{Z},\ x \text{ even}} \kappa_n(x) = \sum_{x\in\mathbb{Z}/\ell\mathbb{Z}} (-1)^x \epsilon_n(x)$$

is a martingale, and

$$R' \supseteq \mathbb{Z}/\ell\mathbb{Z} \implies u_\infty = \sum_{x\in\mathbb{Z}/\ell\mathbb{Z},\ x \text{ even}} \kappa_\infty(x) = 0.$$

We can use an instability argument similar to Theorem 15 Section 4.1 to show that $u_\infty \neq 0$ a.s. (see for instance Section 3 [A4]): then this yields another proof of Proposition 2.1.4.

Now we adapt this heuristics to the case ℓ odd; the process $\kappa_n(x)$ is not a martingale anymore, but approaches one sufficiently closely.

Indeed let us study, given $x \in \mathbb{Z}/\ell\mathbb{Z}$, the evolution of $\kappa_n(x)$. The part of $\kappa_n(x)$ arising from visits originating at x is the martingale $\epsilon_n(x)$, but the behaviour of that process also depends on the difference in probabilities of cycles clockwise and anti-clockwise.

A key step in the proof is an approximate computation of these differences, through a combination of martingales $(\epsilon_n(x))_{n\in\mathbb{N}}$.

More precisely, for all $n \in \mathbb{N}_0$, let

$$\alpha_n := \sum_{k\geqslant n} \frac{1}{W(k)^2},\ \delta_n := \sum_{k=n+1}^{\infty} |\frac{1}{W(k)} - \frac{1}{W(k-1)}|.$$

For all $n \in \mathbb{N}$, let $t_n := \inf\{k \in \mathbb{N}_0 \text{ s.t. } Z_k(0) = n\}$ be the time of n-th visit to site 0 (∞ if 0 is visited less than n times), and let

$$\begin{aligned}\overrightarrow{q_n} &:= \mathbb{P}(X_{t_n+1} = 1 \text{ and } X_{t_{n+1}-1} = -1 \mid \mathcal{F}_{t_n}),\\ \overleftarrow{q_n} &:= \mathbb{P}(X_{t_n+1} = -1 \text{ and } X_{t_{n+1}-1} = 1 \mid \mathcal{F}_{t_n}),\end{aligned}$$

be the probabilities of anti-clockwise and clockwise cycles originating at 0 at time t_n.

Let, for all $x \in \mathbb{Z}/\ell\mathbb{Z}$, $k \leqslant n < \infty$,

$$\begin{aligned}\zeta_n(x) &:= Y_n^+(x) - Y_n^-(x+1)\\ \delta_{k,n}(x) &:= \delta_{Z_k(\{x,x+1\})} - \delta_{Z_n(\{x,x+1\})}.\end{aligned}$$

Then, on one hand, for all $x \in \mathbb{Z}/\ell\mathbb{Z}$, the average increment of $\zeta_\cdot(x)$ between return times t_n and t_{n+1} is of the order of $(\overrightarrow{q_n} - \overleftarrow{q_n})W(Z_{t_n}(\{x,x+1\})^{-1}$, that is,

$$\mathbb{E}(\zeta_{t_{n+1}}(x) - \zeta_{t_n}(x)|\mathcal{F}_{t_n}) = \frac{\overrightarrow{q_n} - \overleftarrow{q_n}}{W(Z_{t_n}(\{x,x+1\})} + O(\mathbb{E}(\delta_{t_n,t_{n+1}}(x) \mid \mathcal{F}_{t_n}));$$

indeed, during this time interval, the process goes back and forth through $\{x, x+1\}$ and, if it goes through a cycle, eventually does one more crossing in the corresponding direction. On the other hand, the sum of processes $\zeta(x)$ over $x \in \mathbb{Z}/l\mathbb{Z}$ is a martingale:

$$\sum_{x\in\mathbb{Z}/\ell\mathbb{Z}} \zeta_n(x) = \sum_{x\in\mathbb{Z}/\ell\mathbb{Z}} \epsilon_n(x).$$

Subsequently,

$$(\overleftarrow{q_n} - \overrightarrow{q_n}) \sum_{x\in\mathbb{Z}/\ell\mathbb{Z}} \frac{1}{W(Z_{t_n}(\{x,x+1\})} = \sum_{x\in\mathbb{Z}/\ell\mathbb{Z}} O(\mathbb{E}(\delta_{t_n,t_{n+1}}(x) \mid \mathcal{F}_{t_n})).$$

Therefore, using that, for all $k \in \mathbb{N}_0$,

$$\kappa_k(0) = 2\epsilon_k(0) - \zeta_k(1) - \zeta_k(-1),$$

we deduce

$$\mathbb{E}(\kappa_{t_{n+1}}(0) - \kappa_{t_n}(0) \mid \mathcal{F}_{t_n}) = \sum_{y \in \mathbb{Z}/\ell\mathbb{Z}} O(\mathbb{E}(\delta_{t_n, t_{n+1}}(x) \mid \mathcal{F}_{t_n})).$$

The calculation obviously holds similarly if we replace 0 by any site $i \in \mathbb{Z}/\ell\mathbb{Z}$.

Now if

$$\liminf \delta_n / \sqrt{\alpha_n} = 0, \tag{2.11}$$

then the "drift" term in $\kappa_.(i)$, i.e. $\mathbb{E}(\kappa_k(i) - \kappa_n(i) \mid \mathcal{F}_n)$, $k \geqslant n$, after certain times n, for a site i which minimizes $Z_n(\{i, i+1\})$, will be outdone by the corresponding standard deviation of $\epsilon_.(i)$, that is $\mathbb{E}((\epsilon_\infty(i) - \epsilon_n(i))^2 \mid \mathcal{F}_n)$, which enables one to conclude, using the instability argument mentioned above, that

$$\kappa_\infty(i) = W^*(Z_\infty(\{i, i+1\}) - W^*(Z_\infty(\{i, i-1\}) \neq 0 \text{ a.s.}$$

for some $i \in \mathbb{Z}/\ell\mathbb{Z}$, and therefore that $R' \neq \mathbb{Z}/\ell\mathbb{Z}$.

This assumption (2.11) holds under weak regularity assumptions on W, which enables one to show the attracting edge property for any nondecreasing W.

2.2 Localization: positive probability results [A5]

The localization behaviour observed with the VRRW on $\mathbb{Z}$ also occurs on general graphs: the first such result was obtained by Volkov [90], which we present here, together with a generalization to arbitrary affine weight functions, in a joint work with Benaïm [A5].

For simplicity, in this section, G will denote the graph of the walk as well as its vertices, as in [A5].

Let us first introduce the notions of complete d-partite subset of G with possible loops and outer boundary.

Definition 2.2.1 *Given $d \geqslant 1$, a subset S of G will be called* complete d-partite *with possible loops, if $(S, \sim)$ is a d-partite graph on which some loops have possibly been added. That is*

$$S = V_1 \cup \ldots \cup V_d$$

with

(i) $\forall\, p \in \{1, \ldots, d\}$, $\forall\, i,\, j \in V_p$, *if* $i \neq j$ *then* $i \not\sim j$.

(ii) $\forall\, p,\, q \in \{1, \ldots, d\}$, $p \neq q$, $\forall i \in V_p$, $\forall j \in V_q$, $i \sim j$.

Given a subset A of G, we let

$$\partial A = \{j \in G \setminus A \ : \ j \sim A\}$$

be the *outer boundary* of A.

Volkov [91] showed localization with positive probability on *trapping* subsets defined as follows. Let us recall that a VRRW is a VSIRW with $W(n) := n + \Delta$, Δ'; also, R and R', defined in (2.4)–(2.5) in the introduction of Section 2, are respectively the range and asymptotic range of the process.

Definition 2.2.2 (Volkov, [90]) *A subset $G' \subseteq G$ is called a trapping subset if it consists of a complete d-partite subset $S = V_1 \cup \ldots \cup V_d$ and its outer boundary $B = \partial S$ and the following property holds: for any $y \in B$ there exist $i \in \{1, 2, \ldots, d\}$ and $x' \in S \setminus V_i$ such that $y \not\sim V_i \cup \{x'\}$.*

Theorem 6 (Volkov, [90], VRRW on general graphs) *Let $G' = S \cup B$, $S = V_1 \cup \ldots \cup V_d$ be a trapping subset of G without loops. Then, with positive probability, if $X_0 \in G'$, there exist α_i, $i \in S$, $\sum_{i \in S} \alpha_i = 1$ such that the following occurs:*

(i) $R = G'$

(ii) $Z_n(i)/n \to \alpha_i$ *for all* $i \in S$ *as* $n \to \infty$

(iii) $\displaystyle\sum_{j \in V_i} \alpha_i = 1/d$ *for all* $i \in \{1, 2, \ldots, d\}$

(iv) $\displaystyle\log Z_n(i)/\log n \to (d/(d-1)) \sum_{j \in S, j \sim i} \alpha_j.$

Our result with Benaïm [A5] shows similar behaviour for a generalized VRRW, defined as a VSIRW with

$$W_{\{i,j\}}(x) := a_{i,j} x, \ a_{i,j} = a_{j,i} > 0, \ i \sim j.$$

Its proof is based on different techniques, using stochastic approximation of ordinary differential equations. We show in passing that other trapping subsets can occur, even when $a_{i,j} = \mathbb{1}_{\{i \sim j\}}$, see Example 1.

Let us first introduce some definitions. For any $x = (x_i)_{i \in G} \in \mathbb{R}^G$, let

$$S(x) := \{i \in G / \ x_i \neq 0\}$$

be its support. Let

$$\Delta := \left\{ x \in \mathbb{R}_+^G \text{ s.t. } |S(x)| < \infty \text{ and } \sum_{i \in G} x_i = 1 \right\}$$

be the nonnegative simplex restricted to elements x of finite support.

For all $x \in \Delta$, let

$$N_i(x) := \sum_{j \in G, j \sim i} a_{i,j} x_j, \qquad H(x) = \sum_{i,j \in G, i \sim j} a_{i,j} x_i x_j = \sum_{i \in G} x_i N_i(x). \tag{2.12}$$

For all $n \in \mathbb{N}$, let

$$x_n = \left(\frac{Z_n(i) - 1}{n} \right)_{i \in G}$$

be the vector of density of occupation of the random walk at time n, which has finite support and takes values in Δ.

Let us consider the ordinary differential equation

$$\frac{dx}{dt} = F(x), \tag{2.13}$$

where

$$F(x) = (x_i[N_i(x) - H(x)])_{i \in G}. \tag{2.14}$$

Up to an adequate rescaling in time, we can show that $(x_k)_{k \in \mathbb{N}}$ approximates the ODE (2.14) under certain assumptions. The heuristics on a finite graph is that the VRRW, for n large and on small time intervals $[n, n+L]$, is close to a Markov Chain with transition probabilities $a_{ij} x_j / N_i(x)$ from i to j, which has invariant measure $(x_i N_i(x)/H(x))_{i \in G}$, so that

$$Z_{n+L}(x) - Z_n(x) \approx LF(x_n)/H(x_n).$$

The ODE (2.14) was originally introduced by Fisher, Wright and Haldane in the 1920s [41, 80], as the *replicator* dynamics in ecology: individuals are labeled by nodes of G and, at each time step, two of them are chosen, with replacement, according to the uniform distribution in the population; if i and j are chosen then, on average, $a_{i,j}$ individuals are added to population i, and $a_{j,i}$ to population j.

A point $x = (x_i)_{i \in G} \in \Delta$ is called *equilibrium* if and only if $F(x) = 0$; it is *feasible* iff $H(x) \neq 0$.

Definition 2.2.3 *For any equilibrium $x \in \Delta$, let us define the following predicates:*

$$(\mathbf{P})_x \quad \max \left(\mathsf{Sp}\, [a_{i,j} - 2H(x)]_{i,j \in S(x)} \right) \leqslant 0$$
$$(\mathbf{Q})_x \quad \max\{N_i(x) - H(x), i \in \partial S(x)\} < 0$$

Assumptions $(\mathbf{P})_x$-$(\mathbf{Q})_x$ ensure that x is a strictly stable equilibrium, in the sense that the eigenvalues of $DF(x)$, which are real, are nonpositive.

The support of such equilibria satisfies some strong "clustering" properties, described in the following Lemma 2.2.1. In the context of the replicator dynamics described above, this would imply that, generically and for symmetric payoffs (on a finite graph, if $(a_{i,j})_{i,j \in G}$ has absolutely continuous distribution w.r.t. the Lebesgue measure on symmetric matrices), a clique of species eventually prevails.

Definition 2.2.4 *For all $S \subseteq G$, let $(\mathbf{P})_S$ be the following predicate:*

- $(\mathbf{P})_S(\mathbf{a})$ *$(S, \sim)$ is a complete d-partite graph with possible loops.*
- $(\mathbf{P})_S(\mathbf{b})$ *If $i \sim i$ for some $i \in S$, then the partition containing i is a singleton.*
- $(\mathbf{P})_S(\mathbf{c})$ *If V_p, $1 \leqslant p \leqslant d$ are its d partitions, then for all p, $q \in \{1, \ldots, d\}$ and i, $i' \in V_p$, j, $j' \in V_j$, $a_{i,j} = a_{i',j'}$.*

Lemma 2.2.1 ([A5], Lemma 3) *For all $x \in \Delta$ feasible equilibrium, $(\mathbf{P})_x$ implies $(\mathbf{P})_{S(x)}$. If, for some $c > 0$, $a_{i,j} = c\mathbb{1}_{\{i \sim j\}}$ for all i, $j \in S(x)$, then the converse also holds.*

Theorem 7 ([A5], Theorem 3) *Let $x \in \Delta$, and assume that $(\mathbf{P})_x - (\mathbf{Q})_x$ holds. Then, for any neighbourhood $\mathcal{N}(x)$ of x in Δ, there is with positive probability $y \in \mathcal{N}(x)$ with $S(y) = S(x)$ and*

(i) $R' = S(x) \cup \partial S(x)$

(ii) $x_n \to y$

(iii) $\forall i \in \partial S(x),\ Z_n(i)/n^{N_i(x)/H(x)} \to C_i \in (0, \infty)$ *(random).*

Lemma 2.2.1 follows from a study of the Jacobian matrix (see [A5] Section 2.2.1). The proof of Theorem 7 makes use of two main tools. The first one is the Poisson equation, which allows us to introduce a modification z_n of x_n, taking into account the position of the particle, so that z_n will be a stochastic approximation of the ODE (2.14) after renormalization in time. Now the stable equilibria of this ODE are not isolated in general, so that convergence towards one of them with positive probability cannot be obtained by classical results (see for instance [6] Chapter 7). Our proof adapts an entropy function introduced by Akin and Losert [57] to that end; see [A5] Section 2.3 for more details.

Theorem 7 and Lemma 2.2.1 imply Theorem 6 in the case $a_{i,j} = \mathbb{1}_{j \sim i}$. Indeed, if $G' = S \cup \partial S$ is a trapping subset according to Definition 2.2.1, then, for all $x \in \Delta$ such that $S(x) = S$ and $\sum_{j \in V_i} x_j = 1/d$, it is easy to check that x is a feasible equilibrium of (2.14); now S trapping subset implies that, for all $i \in \partial S$, $N_i(x) < 1 - 1/d = H(x)$.

In the case of general a, some simple conditions can ensure the existence of trapping subsets, see for instance Theorem 4 [A5]. Even when $a_{i,j} = \mathbb{1}_{j \sim i}$, trapping subsets can occur outside the scope of Definition 2.2.1, as shown in the following Example 1.

Example 1 Assume $a_{i,j} = \mathbb{1}_{j \sim i}$, and consider a graph G on six vertices A, B, C, D, E and F, with a neighbourhood relation $\sim$ defined as follows (see Figure 1): $A \sim B \sim C \sim D \sim A$, $C \sim E \sim D$ and $E \sim F$ (recall that the graph G is symmetric). Let $x = (x_A, x_B, x_C, x_D, x_E, x_F) := (3/8, 3/8, 1/8, 1/8, 0, 0)$, then $S(x) = \{A, B, C, D\}$ and $\partial S(x) = \{E\}$. Also, x is an equilibrium of (2.13), $(\mathbf{P})_{S(x)}$ is satisfied with $V_1 = \{A, C\}$, $V_2 = \{B, D\}$, and $N_E(x) = 1/4 < H(x) = 1/2$, which implies $(\mathbf{P})_x - (\mathbf{Q})_x$, hence subsequently by Theorem 7 that $\mathcal{R} = T(x)$ with positive probability.

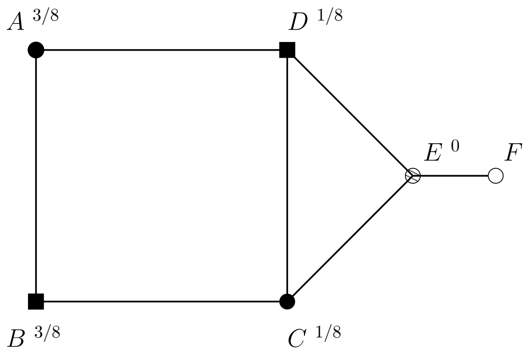


Figure 2.1: We show in Example 1, with $a_{i,j} = \mathbb{1}_{j\sim i}$, that although $T := \{A, B, C, D, E\}$ does not satisfy the assumptions of Theorem 6, the VRRW can localize on it with positive probability, using Theorem 7 and Lemma 2.2.1. The numbers indicated in superscript of vertices represent the limit proportions of visits to these vertices if $x(n)$ were to converge to the equilibrium x in the example. In this case the walk would asymptotically spend most of the time in the bipartite subset $S := V_1 \cup V_2$, where $V_1 := \{A, C\}$, $V_2 := \{B, D\}$, evenly divided between partitions V_1 and V_2, and vertex E would be seldom visited, of the order of $\sqrt{n}$ times at time n.

Now let us prove by contradiction that $T(x)$ with such x does not satisfy the assumptions of Theorem 6 above. Indeed, if $T(x) = S \cup \partial S$, then $S \subseteq \{A, B, C, D\}$ since, otherwise, F would belong to $T(x)$. Now the condition that for all $i \in \partial S$, $\exists p \in \{1, \ldots, d\}$ and $j \in S \setminus V_p$ such that $i \not\sim V_p \cup \{j\}$ implies in particular that a vertex in ∂S is not connected to at least two other vertices in S, so that $i \in \partial S$ cannot be A, B, C or D which are connected to all other but one vertex in $\{A, B, C, D\}$. Hence $S = \{A, B, C, D\}$, but then $i := E$ is connected to both partitions of S, and does not satisfy the condition mentioned last sentence, bringing a contradiction.

2.3 ERRW, VRJP and a conjecture of Diaconis [A6]

This section is devoted to the joint paper [A6] with Sabot. First we show that the edge-reinforced random walk (ERRW) is equal in law to a vertex-reinforced jump process (VRJP) with independent gamma conductances. Then we interpret the limit measure of VRJP (with fixed conductances) as a supersymmetric hyperbolic sigma model in quantum field theory [32]; one consequence is that VRJP and ERRW with large reinforcement are strongly recurrent in any dimension.

Let us first recall some earlier results. The ERRW on finite graphs is a mixture of reversible Markov chains, and the mixing measure was determined explicitly by Diaconis [29] (see also [44, 78]), which has applications in Bayesian statistics [30, 2, 3].

On acyclic graphs, it can be written as a random walk in an *independent* random environment, as was first observed by Pemantle in 1988, which enables one to deduce recurrence/transience criteria or laws of large numbers in different instances [67, 18, 45, 86].

On infinite graphs with cycles, Merkl and Rolles [60, 62, 61, 63, 79] obtained recurrence criteria and asymptotic estimates on graphs of the form $\mathbb{Z} \times G$, G finite graph, and on a modified version of the graph $\mathbb{Z}^2$, where each edge is divided into a large number of edges. However, the recurrence/transience question on $\mathbb{Z}^k$, $k \geqslant 2$, is still open.

The vertex-reinforced jump process (VRJP) on a graph $G = (V, E)$ was conceived by Werner and introduced by Davis and Volkov [25]; we define here a VRJP starting at time 0 at some vertex $i_0 \in V(G)$ with conductances $(W_e)_{e \in E(G)}$ as a continuous-time process $(Y_t)_{t \geqslant 0}$ on V, such that, if Y is at a vertex $x \in V(G)$ at time t, then, conditionally on $(Y_s, s \leqslant t)$, the process jumps to a neighbour y of x at rate $W_{\{x,y\}} L_y(t)$, where

$$L_y(t) := 1 + \int_0^t \mathbb{1}_{\{Y_s = y\}}\, ds.$$

VRJP was so far only studied on trees. In that case, a *restriction principle* observed by Davis and Volkov [26] implies the following. Let $\overline{e}$ and $\underline{e}$ be the two endpoints of an edge $e \in E$, with $\underline{e}$ closer to the initial site i_0 than $\overline{e}$: if the walk is recurrent on G, then the random variables

$$\xi_e := \lim_{t \to \infty} \frac{L_{\overline{e}}(t)}{L_{\underline{e}}(t)},\ e \in E$$

are a.s. well-defined, independent, and have inverse gaussian distribution with parameters $(W_e, 1)$, $e \in E$. This allows to show recurrence/transience criteria on trees, with CLTs in certain instances [25, 26, 19, 4].

We consider here an ERRW with (linear) weight functions $n + a_e,\ e \in E(G)$, that is, for all $n \in \mathbb{N}$, if $X_n = i$ then

$$\mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{i \sim j} \frac{a_{\{i,j\}} + Z_n(\{i,j\})}{\sum_{k \sim i} (a_{\{i,k\}} + Z_n(\{i,k\}))}, \tag{2.15}$$

and let $\mathbb{P}_{i_0}$ be the probability corresponding to the walk starting at i_0.

Our first result with Sabot is that the ERRW is equal in law to a VRJP with independent gamma conductances, seen from the times of jumps. More precisely, consider the positive continuous additive functional of (Y_s)

$$A(s) = \int_0^s \frac{1}{L_{Y_u}(u)} du = \sum_{x \in V} \log(L_x(s)),$$

and the time changed process

$$X_t = Y_{A^{-1}(t)}.$$

Let, for all $x \in V$ and $t \geqslant 0$,

$$T_x(t) = \int_0^t \mathbb{1}_{\{X_u = x\}} du$$

be the local time of $(X_t)_{t \geqslant 0}$ at site x and time t; note that $T_x(A(t)) = \log(L_x(t))$.

Note that there is a slight confusion of notation between $(X_t)_{t \geqslant 0}$ and the initial discrete-time random walk $(X_n)_{n \in \mathbb{N}_0}$. Also, for convenience we let, for all i, $j \in V(G)$, $j \sim i$, $W_{i,j} := W_{\{i,j\}}$.

Lemma 2.3.1 ([A6], Lemma 1) *The law of the process X_t is described by the following: if at time t the process X_t is at the position i, then it jumps to a neighbor j of i at rate $W_{i,j}e^{T_i(t)+T_j(t)}$.*

Theorem 8 ([A6], Theorem 1) *Let $(\tilde{X}_t)_{t\geqslant 0}$ be the continuous-time version of the ERRW with weights $(a_e)_{e\in E(G)}$, as defined in Section 2.1.3.*

Then there exists a sequence of independent random variables $W_e \sim Gamma(a_e, 1)$, $e \in E(G)$, such that, conditionally on $(W_e)_{e\in E(G)}$, $(\tilde{X}_t)_{t\geqslant 0}$ has the same law as the time modification $(X_t)_{t\geqslant 0}$ of the VRJP with weights $(W_e)_{e\in E(G)}$.

PROOF: The proof of Lemma 2.3.1 follows from elementary computations. Let us show Theorem 8. For any $e \in E(G)$, define the simple birth process $\{N_t^e, t \geqslant 0\}$ with initial population size a_e, by

$$N_t^e := a_e + \sup\left\{k \in \mathbb{N} \text{ s.t. } \sum_{i=0}^{k-1} V_i^e \leqslant t\right\}$$

population size at time t; $N_t^e - a_e$ is the number of events in the timeline e at local time t. This process is sometimes called the Yule process: by a result of D. Kendall [46], there exists $W_e := \lim N_t e^{-t}$, with distribution $Gamma(a_e, 1)$, such that, conditionally on W_e, $\{N^e_{f_{W_e}(t)}, t \geqslant 0\}$ is a Poisson process with unit parameter, where

$$f_W(t) := \log(1 + t/W);$$

hence N_e increases between times t and $t + dt$ with probability $W_e e^t\, dt = (f_{W_e}^{-1})'(t)\, dt$. A similar characterization of the timelines is also used in [A2], Lemma 4.7.

Let us now condition on $(W_e)_{e\in E(G)}$. If $\tilde{X}$ is at vertex x at time t, it jumps to a neighbour y of x at rate $W_{\{x,y\}}e^{T_x(t)+T_y(t)}$, since $T_x(t) + T_y(t)$ is the time spent adjacent to the edge $\{x, y\}$ at time t. □

The rest of the section is devoted to the study of the time-modification $(X_t)_{t\geqslant 0}$ of a VRJP with fixed conductances $(W_e)_{e\in E}$; we show that its centred occupation times converge a.s., with a limiting measure that can be computed explicitely, and that it is a mixture of time-changed Markov jump processes.

Proposition 2.3.1 ([A6], Proposition 1) *Suppose that G is finite and set $N = |V|$. The following limits exist $\mathbb{P}_{i_0}$ a.s.*

$$U_i = \lim_{t\to\infty} T_i(t) - \frac{t}{N},$$

for all $i \in V$.

Theorem 9 ([A6], Theorem 2) *Suppose that G is finite and set $N = |V|$.*
i) Under $\mathbb{P}_{i_0}$, $(U_i)_{i\in V}$ has the following density distribution on $\mathcal{H}_0 = \{(u_i), \sum u_i = 0\}$

$$\frac{1}{(2\pi)^{(N-1)/2}} e^{u_{i_0}} e^{-H(W,u)} \sqrt{D(W,u)}, \tag{2.16}$$

where

$$H(W,u) = 2\sum_{\{i,j\}\in E} W_{i,j}\sinh^2\left(\frac{1}{2}(u_i-u_j)\right)$$

and $D(W,u)$ *is any diagonal minor of the* $N\times N$ *matrix* $M(W,u)$ *with coefficients*

$$m_{i,j} = \begin{cases} W_{i,j}e^{u_i+u_j} & \text{if } i\neq j \\ -\sum_{k\in V} W_{i,k}e^{u_i+u_k} & \text{if } i=j \end{cases}$$

Let $\nu_{i_0,W}$ *be the corresponding law.*
ii) Let $(U_i)_{i\in V}$ *be a random variable in* $\mathcal{H}_0$ *distributed according to (2.16). Let* (Z_t) *be the Markov jump process starting at* i_0 *and with jump rates from* i *to* j

$$\frac{1}{2}W_{i,j}e^{U_j-U_i}.$$

Let $(l_i(t))$ *be the local times of the process* Z *at time* t. *Consider the positive continuous additive functional of* Z

$$B(t) = \int_0^t \frac{1}{2}\frac{1}{\sqrt{1+l_{Z_u}(u)}}du = \sum_{i\in V}\left(\sqrt{1+l_i(t)}-1\right),$$

and the time changed process

$$\tilde{Y}_s = Z_{B^{-1}(s)}.$$

Then the law of $\tilde{Y}$, *under* $(U_i)_{i\in V}$ *with distribution* $\nu_{i_0,W}$, *is the law of the VRJP* $(Y_s)_{s\geqslant 0}$ *with conductances* $(W_{i,j})$. *In particular, the discrete-time process associated with* (Y_s) *is a mixture of reversible Markov chains with conductances* $W_{i,j}e^{U_i+U_j}$.

N.B.: 1) the density distribution is with respect to the Lebesgue measure on $\mathcal{H}_0$ which is $\prod_{i\in V\setminus\{j_0\}} du_i$ for any choice of j_0 in V. We simple write du for any of the $\prod_{i\in V\setminus\{j_0\}} du_i$.
2) The diagonal minors of the matrix $M(W,u)$ are all equal since the sum on any line or column of the coefficients of the matrix is null.

PROOF: The heuristics of Proposition 2.3.1 is that, at “frozen” T (and jump rates $W_{i,j}e^{T_i+T_j}$), the invariant measure is uniform on all vertices, and that the jump rates increase exponentially in time. We use martingale techniques, through the Poisson equation (see [A6], Section 4.1).

Let us sketch the proof of Theorem 9 *i)*. The process X is not Markov, but $\Theta_t = (X_t,(T_i(t))_{i\in V})$ is a time-continuous Markov process on the state space $V\times\mathbb{R}^V$ with generator $\tilde{L}$ defined on C^∞ bounded functions by

$$\tilde{L}(f)(i,T) = \left(\frac{\partial}{\partial T_i}f\right)(i,T) + L(T)(f)(i), \quad \forall (x,T)\in V\times\mathbb{R}_+^V,$$

where $L(T)$ is the generator of the jump process on V at frozen T defined for $g\in\mathbb{R}^V$:

$$L(T)(g)(i) = \sum_{j\in V} W_{i,j}e^{T_i+T_j}(g(j)-g(i)), \quad \forall i\in V.$$

Let $\mathbb{P}^W_{i,T}$ be the law of X under initial condition (i,T), and let $\mathbb{P}^W_i := \mathbb{P}^W_{i,0}$. Then, letting

$$(W^T)_{i,j} = W_{i,j}e^{T_i+T_j},$$

we have

$$\mathbb{P}^W_{i,T} = \mathbb{P}^{W^T}_i.$$

Denote, for all $(\lambda_i) \in \mathcal{H}_0$

$$\begin{aligned}\Psi(i_0,T,\lambda) &= \int_{\mathcal{H}_0} e^{i<\lambda,u>}\nu^{i_0,W^T}(du)\\ &= \frac{1}{\sqrt{2\pi}^{N-1}}\int_{\mathcal{H}_0} e^{i<\lambda,u>}e^{u_{i_0}}e^{-H(W^T,u)}\sqrt{D(W^T,u)}du.\end{aligned}$$

Our aim is to show that

$$\Psi(i_0,0,\lambda) = \mathbb{E}^W_{i_0}\left(e^{i<\lambda,U>}\right).$$

□

Lemma 2.3.2 ([A6], Lemma 4) Ψ *is solution of the Feynman-Kac equation*

$$i\lambda_{i_0}\Psi(i_0,T,\lambda) + (\tilde{L}\Psi)(i_0,T,\lambda) = 0.$$

PROOF: Let $\overline{T}_i = T_i - \frac{1}{N}\sum_{j\in V} T_j$. The change of variables $\tilde{u}_i = u_i + \overline{T}_i$ yields

$$\Psi(i_0,T,\lambda) = \frac{1}{\sqrt{2\pi}^{N-1}}\int_{\mathcal{H}_0} e^{\tilde{u}_{i_0}-\overline{T}_{i_0}}e^{i<\lambda,\tilde{u}-\overline{T}>}e^{-H(W^T,\tilde{u}-\overline{T})}\sqrt{D(W^T,\tilde{u}-\overline{T})}d\tilde{u} \quad (2.17)$$

Now $H(W^T,\tilde{u}-\overline{T}) = H(W^T,\tilde{u}-T)$, since $H(W^T,u)$ only depends on the differences $u_i - u_j$. We also observe that the coefficients of the matrix $M(W^T,u)$ only contain terms of the form $W_{i,j}e^{u_i+T_i+u_j+T_j}$, hence

$$\sqrt{D(W^T,\tilde{u}-\overline{T})} = e^{\frac{N-1}{N}\sum_j T_j}\sqrt{D(W,\tilde{u})}.$$

Finally, $<\lambda,\overline{T}>=<\lambda,T>$ since $\lambda\in\mathcal{H}_0$. This implies that

$$\Psi(i_0,T,\lambda) = \frac{1}{\sqrt{2\pi}^{N-1}}\int_{\mathcal{H}_0} e^{\sum_j T_j}e^{\tilde{u}_{i_0}-T_{i_0}}e^{i<\lambda,\tilde{u}-T>}e^{-H(W^T,\tilde{u}-T)}\sqrt{D(W,\tilde{u})}d\tilde{u}. \quad (2.18)$$

An easy computation shows that

$$\frac{\partial}{\partial T_{i_0}}H(W^T,\tilde{u}-T) = e^{-(\tilde{u}_{i_0}-T_{i_0})}L(T)(e^{\tilde{u}-T})(i_0).$$

Hence,

$$
\begin{aligned}
& -\frac{\partial}{\partial T_{i_0}}\Psi(i_0,T,\lambda) \\
= & \frac{1}{\sqrt{2\pi}^{N-1}}\int_{\mathcal{H}_0}\left(i\lambda_{i_0}e^{\tilde{u}_{i_0}-T_{i_0}}+L(T)(e^{\tilde{u}-T})(i_0)\right)e^{\sum_j T_j}e^{i<\lambda,\tilde{u}-T>}e^{-H(W^T,\tilde{u}-T)}\sqrt{D(W,\tilde{u})}d\tilde{u} \\
= & i\lambda_{i_0}\Psi(i_0,T,\lambda)+(L(T)\Psi)(i_0,T,\lambda),
\end{aligned}
$$

which completes the proof. □

Therefore,

$$\Psi(i_0,0,\lambda)=\mathbb{E}_{i_0}^{W}\left(e^{i<\lambda,T(t)>}\Psi(X_t,T(t),\lambda)\right).$$

It remains to show that

$$\lim_{t\to\infty}\Psi(X_t,T(t),\lambda)=1.$$

This is a consequence of the following remarks: when t is large, so is $W^{T(t)}$, thus (u_i) is small. Hence $e^{u_{X_t}}\sim 1$, $2\sinh^2((u_j-u_i)/2)\sim\frac{1}{2}(u_i-u_j)^2$, $W_{i,j}^{T(t)}e^{u_i+u_j}\sim W_{i,j}^{T(t)}$, $e^{i<\lambda,u>}\sim 1$, and we sum a free gaussian field, whose integral is 1; see [A6], proof of Lemma 4, for more technical details.

Using Theorems 8 and 9, we can retrieve the limiting measure of ERRWs, which was initially computed by Coppersmith and Diaconis in [20] (see also [44]), as the limiting measure of VRJP with independent gamma conductances arising from Theorem 8. This explains the renormalization constant in that Coppersmith-Diaconis formula, which had remained mysterious so far.

Also, the limiting measure of VRJP described in Theorem 9 can be interpreted as a supersymmetric hyperbolic sigma model in quantum field theory [32]; note that the question of a possible link was discussed in that paper [32], suggested by Kozma, Heydenreich and Sznitman.

This interpretation enables us [A6] to deduce that VRJP and ERRW are strongly recurrent in any dimension for large reinforcement, using a localization result of Disertori and Spencer [31].

More precisely set, for all $\beta>0$,

$$I_\beta:=\sqrt{\beta}\int_{-\infty}^{\infty}\frac{dt}{\sqrt{2\pi}}e^{-\beta(\cosh t-1)},$$

which is strictly increasing in β.

Let β_c^d be defined as the unique solution to the equation

$$I_{\beta_c^d}e^{\beta_c^d(2d-2)}(2d-1)=1$$

for all $d>1$, $\beta_c^d:=\infty$ if $d=1$.

Theorem 10 (Disertori, Spencer [31], Theorem 2) *Let $G \subseteq \mathbb{Z}^d$ be a finite connected subset containing* 0*. Assume that the initial site i_0 is* 0*, and that $W_e = \beta$ for all e, with $0 < \beta < \beta_c^d$. Then there exists a universal constant $C_0 > 0$ such that, for all $x \in \mathbb{Z}^d$,*

$$\mathbb{E}\left(e^{(U_x - U_0)/2}\right) \leqslant C_0 \left[I_\beta e^{\beta(2d-2)}(2d-1)\right]^{|x|}.$$

Theorem 10 implies that the probability to leave the ball of radius n before coming back to 0 is exponentially decreasing in n, which subsequently yields the following Corollary 2.3.1. An elementary truncation argument enables us to adapt Disertori and Spencer's techniques [31] and deduce recurrence of ERRW for sufficiently large reinforcement. This question of reucurrence/transience was initially raised by Diaconis in 1986.

Corollary 2.3.1 ([A6], Corollary 1) *For $0 < \beta < \beta_c^d$, the VRJP on $\mathbb{Z}^d$ starting at* 0 *with constant conductance β is a mixture of strongly recurrent Markov chains.*

Corollary 2.3.2 ([A6], Corollary 2) *For any $d \in \mathbb{N}$, there exists $a_c^d > 0$ such that, for all $a < a_c^d$, the ERRW on $\mathbb{Z}^d$ starting at* 0 *with constant initial weight $a > 0$ is a mixture of strongly recurrent Markov chains.*

Remark 2.3.1 [[A6], Remark 5] Corollaries 2.3.1 (resp. 2.3.2) also hold on any graph of bounded degree, and for possibly non-constant conductances $(\beta_e)_{e \in E}$ with $\beta_e < \beta_c$ for some $\beta_c > 0$ (resp. weights $(a_e)_{e \in E}$ with $a_e < a_c$ for some $a_c > 0$).

Chapter 3

Brownian polymers

Let $(X_t)_{t\geqslant 0}$ be the random process defined by $X_0 := x_0 \in \mathbb{R}^d$ and

$$X_t = \sigma B_t + \int_0^t \mathrm{d}s \int_0^s f(X_s - X_u)\,\mathrm{d}u, \tag{3.1}$$

where $\sigma > 0$, $(B_t)_{t\geqslant 0}$ is a Brownian motion on $\mathbb{R}^d$ (starting in 0 at time 0), and $f : \mathbb{R}^d \to \mathbb{R}^d$ ($d \geqslant 1$) is a measurable function.

A *strong* solution (3.1) is unique if f is continuous (see Theorem 5 and Corollary 1, p. 271 in [39]), and it exists if f is Lipschitz (see Theorem 11.2 in [77]). The existence and uniqueness of a *weak* solution to (3.1) is ensured under the assumption that f is bounded, using a generalization of Girsanov theorem (see Corollary 3.5.2 in [43]).

This setting was proposed by Norris, Rogers and Williams [66] in 1987, and introduced by Durrett and Rogers [36] in 1992, as a model for the shape of a growing polymer, X_t corresponding to the location of the end of the polymer at time t. Without any assumption on the function f, the stochastic differential equation (3.1) defines a *self-interacting diffusion*, in the sense that the process X evolves in an environment changing with its prior trajectory. We will call it *self-repelling* (resp. *self-attracting*) if, for all $x \in \mathbb{R}^d$, $xf(x) \geqslant 0$ (resp. $\leqslant 0$), in other words if it is more likely to stay away from (resp. come back to) the places it has already visited before.

I solved two conjectures of Durrett and Rogers [36] on the self-repelling case, the first one (Conjecture 3) with Mountford when f has heavy tails (i.e. $|f(x)| \sim_{x\to\pm\infty} |x|^{-\beta}$, $\beta \in (0,1)$), the other one with Tóth and Valkó (Conjecture 2) when f has light tails (see assumptions of Section 3.2).

Before we consider them, let us briefly discuss previous results on the self-attracting case and alternative models. It was studied by Cranston and Le Jan [21] and Raimond [74] on $\mathbb{R}^d$, in the cases $f(x) = -bx$ and $f(x) = -bx/\|x\|$, $b > 0$: both lead to an almost-sure convergence of the process (see [40] for a generalization in the one-dimensional case)

In the non-local case $f(x) = -\operatorname{sign}(x)\mathbb{1}_{|x|\geqslant a}$, the diffusion does not converge but the paths are however bounded a.s. [21, 40].

Norris, Rogers and Williams [66] define in 1987 a Brownian motion with local time drift, with the initial motivation to propose a simple mathematical model for the self-avoiding brownian motion; the model was studied by Hu and Yor [42], Raimond and Schapira [75], who obtain central limit theorems and criteria for recurrence. The closely related Brownian Motion perturbed at extremas was introduced by Le Gall [51] in 1986, as a limit of the latter model; Le Gall and Yor [52], Carmona, Petit and Yor [17], Davis [24], Perman and Werner [71] show its existence and uniqueness, and analyse certain fine properties, for instance Hausdorff dimension of points of monotonicity in [71]. Davis [23], Werner [92] and Dolgopyat [34] show that these processes can be seen as limits in law of renormalized once-reinforced random walks or multi-excited random walks.

A model similar to (3.1) was introduced and studied in 1996 by Benaïm, Ledoux and Raimond [8], Benaïm and Raimond [9, 10], the difference being that the drift is given by an average of the past occupation (inserting a factor of $1/s$ in the first integral). Assuming that the particle lives in a compact connected smooth (C^∞) Riemannian manifold (without boundary), and that $f(X_s - X_t)$ is replaced by the gradient of a potential $\nabla V_{X_s}(X_t)$ with sufficient differentiability, they prove that the normalized occupation measure $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\,\mathrm{d}s$ asymptotically shadows the solutions of a deterministic differential equation, so that the possible limit sets of μ_t are "attractor free sets" for this equation. Depending on the structure of the interaction, various corresponding dynamics are possible; however, when the diffusion is self-repelling or weakly self-attracting, according to definitions introduced by the authors (taking into account that the particle lives in a compact set), μ_t a.s. converges toward the normalized Riemannian measure. In the symmetric case, μ_t converges almost surely to the critical set of a certain nonlinear free energy functional [10].

A self-interacting model introduced by Del Moral and Miclo [27, 28] presents some similarity with the latter model: in a discrete time setting, the evolution depends on the present position and on the occupation measure created by the path up to this instant; one can obtain sufficient conditions for a.s. convergence of the empirical measures, and provide upper bounds on the corresponding rate of convergence to the limiting measure.

Let us now describe the results and conjectures in the initial paper Durrett and Rogers [36] in 1992. First, if f is bounded and has compact support, then $|X_t|/t$ is bounded by a deterministic constant a.s.

The rest of paper deals with the one-dimensional setting $f : \mathbb{R} \to \mathbb{R}$ Lipschitz, to which we restrict ourselves in the rest of the section. Durrett and Rogers [36] prove that if f is nonnegative and $f(0) > 0$, then X_t/t is lower bounded by a deterministic constant a.s., and conjecture a strong law of large numbers, which was proved by Cranston and Mountford [22] in 1996: under the weaker condition f nonnegative and $f \not\equiv 0$, there exists $c > 0$ such that

$$\lim_{t\to\infty} \frac{X_t}{t} = c \quad \text{a.s.}$$

Next, the case of heavy-tailed f is considered. Let (A1), (A2) and (A3) be the following assumptions:

(A1) $|f(x)| \leqslant M$,

(A2) $f(x)$ is decreasing for $x \in [q, \infty)$,

(A3) $x^\beta f(x) \to l > 0$ as $x \to \infty$ with $0 < \beta < 1$.

Durrett and Rogers [36] describe the following heuristics. Given $\alpha > 0$, let $x_t = T^{-\alpha} X_{tT}$ and $W_t = T^{-1/2} B_{tT}$; then we can rewrite (3.1) as

$$x_t = T^{1/2-\alpha} W_t + T^{2-\alpha} \int_0^t \mathrm{d}s \int_0^s f\big(T^\alpha (x_s - x_u)\big)\, \mathrm{d}u.$$

If we set

$$\alpha := \frac{2}{(1+\beta)}$$

so that $2 - \alpha = \alpha\beta$ and let $T \to \infty$ we expect that a possible limit (still called x_t for simplicity) should satisfy

$$x_t = \int_0^t \mathrm{d}s \int_0^s \frac{l\, \mathrm{d}u}{(x_s - x_u)^\beta}.$$

One solution is $x_t = c_0 t^\alpha$ where c_0 satisfies

$$\alpha c_0^{\beta+1} = \int_0^1 \frac{l\, \mathrm{d}u}{(1 - u^\alpha)^\beta}. \tag{3.2}$$

The argument led to the following rigorous result.

Theorem 11 (Durrett and Rogers [36]) *Suppose* (A1)–(A3) *hold and* α *and* c_0 *are as above. Then*

$$\limsup_{t\to\infty} \frac{X_t}{t^\alpha} \leqslant c_0.$$

If, moreover, f *is nonnegative and* $f(0) > 0$*, then*

$$\frac{X_t}{t^\alpha} \to c_0 \quad a.s.$$

Conjecture 1 (Conjecture 3, Durrett and Rogers [36]) *Suppose* $f(x) = x/(1 + |x|^{\beta+1})$ *with* $0 < \beta < 1$*. Then with probability* $1/2$*,*

$$\frac{X_t}{t^\alpha} \to c_0.$$

We proved Conjecture 1 with Mountford [A7]; a sketch of the proof is given in Section 3.1.

The other conjecture [36] was initially on the case of an odd function f of compact support.

Conjecture 2 (Conjecture 2 of Durrett and Rogers [36])) *Suppose f has compact support, and $f(-x) = -f(x)$; then*

$$\frac{X_t}{t} \to 0 \quad a.s.$$

Tóth and Werner [89] later conjectured, by comparing this model with exponentially self-repelling random walks on $\mathbb{Z}$ [88], that under the same assumptions $X_t/t^{2/3}$ converges in law, which means that the particle has a super-diffusive behaviour despite the fact that it only looks at the time spent in its immediate neighbourhood.

We partially proved these conjectures with Tóth and Valkó in [A8], see Section 3.2 for more details.

3.1 Conjecture 3 of Durrett-Rogers [A7]

In this section we sketch the proof, in a joint work with Mountford [A7], of the conjecture 3 of Durrett-Rogers in [36]. The key step is the following proposition.

Proposition 3.1.1 ([A7], Proposition 1) $\mathbb{P}(\limsup_{t\to\infty} |X_t| = \infty) = 1$.

Its proof relies on the particular shape of the drift when the process remains stuck in a bounded interval. More precisely, let us define, for any $u \in \mathbb{R}_+$, the *drift function* at time u

$$g_u(x) = \int_0^u f(x - X_s)\, ds;$$

$g_u(x)$ is the drift that would be endured by the particle at time u if it were in x. Then

$$\mathrm{d}X_u = \mathrm{d}B_u + g_u(X_u)\,\mathrm{d}u.$$

For any interval I, we define the drift function restricted to contributions within that interval, i.e.

$$g_u^I(x) = \int_0^u f(x - X_s)\mathbb{1}_{X_s \in I}\, ds.$$

Fix $u \geqslant 0$ and $I = [a, b]$. Firstly, when the process remains in interval I, g_u^I gives the main contribution to the drift. As long as X_t does not leave I its behavior is, on time intervals of fixed scale starting at u, comparable to the behavior of a diffusion with drift g_u^I.

Secondly, the drift function $x \mapsto g_u^I(x)$ satisfies the following property: when $x \in I$ is close to b, then either $g_u^I(x)$ is positive or $g_u^I(y)$ is nonpositive for all $y \in [a, x]$, as implied by the following Lemma 3.1.1.

Let $x_{\max} := (1/\beta)^{1/(1+\beta)}$ be the point of change of monotonicity of f.

Lemma 3.1.1 ([A7], Proposition 1) *Let $u \in \mathbb{R}_+^*$, a, $b \in \mathbb{R}$. Suppose there exists $x_0 \in [a, b]$ such that $g_u^{[a,b]}(x_0) \leqslant 0$, and either $f(b - x_0) \leqslant f(b - a)^2$ and $b - x_0 \leqslant 1/16$, or $b - a \leqslant x_{\max}$. Then, for all $x \in [a, x_0]$, $g_u^{[a,b]}(x) \leqslant 0$.*

Lemma 3.1.1 relies on the shape of f around 0 ($f(0) = 0$, $f'(0) > 0$) and $\pm\infty$ ($|f(x)|/|x|^{-\beta}$ converges as $x \to \pm\infty$): see [A7] for details. It implies that, as long as X_t remains in I, each time it approaches the border of I the probability to leave it within a time limit depending only on the size of the interval is lower bounded. Therefore, the range of the process X_t regularly widens, which explains Proposition 3.1.1.

Then a thorough study of the local time enables us to show that, each time X_t reaches its maximum, the probability that it surpasses it by one within one unit of time is lower bounded, independently of the prior occupation measure of the process.

Finally it can be shown, using previous results of Durrett and Rogers in [36] (in particular Theorem 11 above), that conditionally on such an event, with lower bounded probability, X_t/t^α converges to c_0. Then the conclusion follows by symmetry.

3.2 Conjecture 2 of Durrett-Rogers [A8]

The aim of this section is to present the results and main arguments in the paper with Tóth and Valkó [A8], which partially shows Conjecture 2 of Durrett-Rogers in [36].

First, let us consider the following generalized polymer $(X_t)_{t\geqslant 0}$ defined by $X_0 := x_0 \in \mathbb{R}$ and

$$X_t = \sigma B_t + \int_0^t \left(\xi(X_s) + \int_0^s f(X_s - X_u)du \right) ds, \tag{3.3}$$

where $\sigma > 0$, B_t is a standard 1d Brownian motion, $f : \mathbb{R} \to \mathbb{R}$ is a function with sufficient regularity, and $\xi : \mathbb{R} \to \mathbb{R}$ is an initial drift profile with regularity (detailed below).

We assume that

$$f(x) = -b'(x),$$

where $b \in L^1(\mathbb{R}) \cap C^{(\infty)}(\mathbb{R})$ and has nonnegative Fourier transform.

Note that positive definiteness implies

$$b(-x) = b(x), \qquad \sup_{x\in\mathbb{R}} |b(x)| = b(0). \tag{3.4}$$

Now, the "drift function" at time t is

$$g_t(x) = \xi(x) + \int_0^t f(X_s - x)ds. \tag{3.5}$$

Let us study the "drift function" environment seen from the particle X_t, i.e.

$$x \mapsto \eta(t, x) := g_t(X_t + x). \tag{3.6}$$

Then $t \mapsto \eta(t) := \eta(t, \cdot)$ *is a Markov process*, on the space of smooth functions of slow increase at infinity:

$$\Omega := \{\omega \in C^\infty(\mathbb{R} \to \mathbb{R}) \, : \, (\forall k \geqslant 0, \forall l \geqslant 1) : \quad \|\, \omega \,\|_{k,l} < \infty\}. \tag{3.7}$$

where $\| \omega \|_{k,l}$ are the seminorms

$$\| \omega \|_{k,l} := \sup_{x \in \mathbb{R}} \big(1 + | x |\big)^{-1/l} \, \big| \omega^{(k)}(x) \big| \, , \qquad k \geqslant 0, \quad l \geqslant 1. \tag{3.8}$$

Given the corresponding assumptions on b, if $\xi \in \Omega$, then a solution of (3.3) exists and is unique (see Theorem 11.2 in [77]), and $\eta(t, .) \in \Omega$, for all $t \geqslant 0$.

We derive by standard Itō-calculus that

$$d\eta(t,x) = \sigma\eta'(t,x)dB(t) + \eta'(t,x)\eta(t,0)dt + \sigma^2 \frac{\eta''(t,x)}{2} dt - b'(x)dt. \tag{3.9}$$

We show in the following Theorem 12 that the *Gaussian probability measure* $\pi(d\omega)$ on Ω with mean and covariance

$$\int_\Omega \omega(x)\pi(d\omega) = 0, \qquad \int_\Omega \omega(x)\omega(y)\pi(d\omega) = b(x-y), \tag{3.10}$$

is invariant for the *Markov process* $t \mapsto \eta(t) := \eta(t, \cdot)$.

Recall that Minlos' theorem (Theorem I.10 of [83]) implies, given $x \mapsto b(x)$ with the assumed properties, that the expectations and covariances (3.10) define a unique translation invariant Gaussian probability measure $\pi(d\omega)$ on the space of tempered distributions $\mathcal{S}'(\mathbb{R})$. The regularity properties of the covariance function b imply that this measure is actually supported by the space $\Omega \subseteq \mathcal{S}'(\mathbb{R})$, see [58, 59].

Theorem 12 ([A8], Theorem 1) *The Gaussian probability measure $\pi(d\omega)$ on Ω, with mean* 0 *and covariances* (3.10) *is time-invariant and ergodic for the Ω-valued Markov process $t \mapsto \eta(t)$.*

PROOF: We only provide a formal proof of the time-invariance here; for more details, see [A8].

Let us first note that the measure π is translation-invariant, so that the action corresponding to the σdB_t part in dX_t will leave it invariant: indeed, the Laplacian ∇^2, generator of the diffusion in random scenery, is self-adjoint under π, and $\nabla^2 1\!\!1 = 0$.

Hence assume $\sigma = 0$. In order to prove that π is indeed time-stationary we have to show that for any (sufficiently smooth) test function $u(\cdot)$ the moment generating functional $\mathbb{E}\big(\exp\{\langle u, \eta(t)\rangle\}\big)$ is constant in time. Here we used the notation

$$\langle u, v\rangle := \int_{-\infty}^{\infty} v(x)u(x)dx. \tag{3.11}$$

It follows from (3.9) that

$$d\mathbb{E}\big(\exp\{\langle u, \eta(t)\rangle\}\big) = \mathbb{E}\big(d\exp\{\langle u, \eta(t)\rangle\}\big) = \mathbb{E}\big(e^{\langle u, \eta(t)\rangle}\big(-\langle u', \eta(t)\rangle\eta(t,0) + \langle u', b\rangle\big)\big)dt.$$

Let X, Y, Z be jointly Gaussian with zero mean. Then, using differentiations of the moment generating function of their joint distribution, it is easy to show that

$$\mathbb{E}\big(YZe^X\big) = \exp\{\mathbb{E}\big(X^2\big)/2\}\big(\mathbb{E}\big(YZ\big) + \mathbb{E}\big(XY\big)\mathbb{E}\big(XZ\big)\big). \tag{3.12}$$

Now note that $\mathbb{E}[\langle u,\eta(t)\rangle\langle v,\eta(t)\rangle] = \langle u, b*v\rangle$, η being a zero mean Gaussian field with covariance b, thus

$$d\mathbb{E}\big(\exp\{\langle u,\eta(t)\rangle\}\big) = e^{\frac{1}{2}\langle u,b*u\rangle}\big\{-\langle u',b\rangle - \langle u',b*u\rangle\langle u,b\rangle + \langle u',b\rangle\big\}dt = 0 \qquad (3.13)$$

since, for any test function u, $\langle u', b*u\rangle = 0$. □

The following Corollary 3.2.1 will follow from ergodicity, using irreducibility of the process (see details [A8], Section 2.3), and

$$X_t - X_0 = \sigma B(t) + \int_0^t \varphi(\eta(s))ds,$$

where

$$\varphi(\nu) := \nu(0).$$

Corollary 3.2.1 ([A8], Corollary 1) *For π-almost all initial profiles ξ,*

$$\lim_{t\to\infty}\frac{X_t}{t} = 0 \quad \text{a.s.} \qquad (3.14)$$

This partially settles Conjecture 2 of [36].

We now study the $t\to\infty$ asymptotics of the variance of displacement

$$E(t) := \mathbb{E}\big(X_t^2\big). \qquad (3.15)$$

All the following results will be meant for the process being in the stationary regime. For simplicity, we will assume $\sigma = 1$; for arbitrary $\sigma > 0$, $Y_t := X_t/\sigma$ is a Brownian polymer (3.3) with interaction function $\tilde{f}(x) := f(\sigma x)/\sigma$, so that this assumption does not restrict generality.

First let us show that, for all $t \geqslant s \geqslant 0$,

$$\mathbb{E}\big((X_t - X_s)^2\big) = t - s + \mathbb{E}\Big(\Big(\int_s^t \varphi(\eta(u))du\Big)^2\Big). \qquad (3.16)$$

This will be a consequence of the observation that the forward process $t\mapsto\eta(t)$ and flipped-backward process $t\mapsto\tilde{\eta}(t) := -\eta(-t)$ are identical in law, which is sometimes called Yaglom-reversibility, see [94, 95, 33].

More precisely let, for all s, $t\in\mathbb{R}$,

$$M(s,t) := X_t - X_s - \int_s^t \varphi(\eta(u))du = B_t - B_s. \qquad (3.17)$$

Lemma 3.2.1 ([A8], Lemma 2) *For $s\in\mathbb{R}$ fixed the process $[s,\infty)\ni t\mapsto M(s,t)$ is a forward martingale with respect to the* forward filtration $\{\mathcal{F}_{(-\infty,t]} : t\geqslant s\}$ *of the process $t\mapsto\eta(t)$. For $t\in\mathbb{R}$ fixed the process $(-\infty,t]\ni s\mapsto M(s,t)$ is a backward martingale with respect to the* backward filtration $\{\mathcal{F}_{[s,\infty)} : s\leqslant t\}$ *of the process $t\mapsto \eta(t)$.*

PROOF: The first statement follows from definitions. Let us prove the second one: (1) For any $s \leqslant t$, there is a Borel function $F_{s,t}$ mapping a.s. $(\eta(u))_{s\leqslant u\leqslant t}$ to $X(t)-X(s)$. By symmetry, $F_{-t,-s}$ maps the flipped-backward process $(\eta(-u))_{-t\leqslant u\leqslant -s}$ to

$$\tilde{X}_{-s} - \tilde{X}_{-t} = X_s - X_t. \tag{3.18}$$

(2) $t \mapsto \eta(t)$ and $t \mapsto \tilde{\eta}(t) = -\eta(-t)$ are identical in law.
(3) The function $\omega \mapsto \varphi(\omega)$ is odd with respect to the flip map $\omega \mapsto -\omega$.

Putting these facts together (in this order) we obtain

$$\begin{aligned}\lim_{h\to 0} \mathbf{E}\big(\frac{X_{s-h}-X_s}{-h}\big|\mathcal{F}_{[s,\infty)}\big) &= -\lim_{h\to 0} \mathbf{E}\big(\frac{\tilde{X}_{-s+h}-\tilde{X}_{-s}}{h}\big|\tilde{\mathcal{F}}_{(-\infty,-s]}\big) \\ &= -\varphi(\tilde{\eta}(-s)) = \varphi(\eta(s)).\end{aligned} \tag{3.19}$$

□

From Lemma 3.2.1 it follows that

$$\mathbb{E}\big((X_t - X_s)^2\big) = \mathbb{E}\big((M(s,t))^2\big) + \mathbb{E}\big(\big(\int_s^t \varphi(\eta(u))du\big)^2\big), \tag{3.20}$$

which implies (3.16).

Now, let

$$\rho^2 := \int_{-\infty}^{\infty} p^{-2}\hat{b}(p)dp \leqslant \infty. \tag{3.21}$$

Conversely, as stated in the following Theorem 13, we can show that X_t behaves at least diffusively if $\rho^2 < \infty$; note that this requires $\hat{b}(0) = 0$ in particular.

Theorem 13 ([A8], Theorem 2) *Let ρ^2 be the constant defined in* (3.21). *Then*

$$1 \leqslant \varliminf_{t\to\infty} t^{-1}E(t) \leqslant \varlimsup_{t\to\infty} t^{-1}E(t) \leqslant 1+\rho^2. \tag{3.22}$$

Let, for all $\lambda > 0$,

$$\hat{E}(\lambda) := \int_0^{\infty} e^{-\lambda t}E(t)dt. \tag{3.23}$$

Let us consider the following *infrared bounds* for the correlation function $\hat{b}(p)$: for some $-1 < \alpha < 1$

$$C_1 := \varlimsup_{p\to 0} |p|^{-\alpha}\hat{b}(p) < \infty, \qquad C_2 := \varliminf_{p\to 0} |p|^{-\alpha}\hat{b}(p) > 0. \tag{3.24}$$

Obviously, $C_2 \leqslant C_1$.

Theorem 14 ([A8], Theorem 3) *If for some $-1 < \alpha < 1$ the infrared bounds* (3.24) *hold, then*

$$\varlimsup_{\lambda\to 0} \lambda^{(5-\alpha)/2}\hat{E}(\lambda) \leqslant C_3 < \infty, \tag{3.25}$$

and

$$\underline{\lim}_{\lambda \to 0} \lambda^{(9-2\alpha+\alpha^2)/4} \hat{E}(\lambda) \geqslant C_4 > 0, \tag{3.26}$$

where the constants C_3 and C_4 depend only on α, C_1 and C_2.

The upper bound on $\hat{E}(\lambda)$ can be converted into an upper bound on $E(t)$, using the following result Tauberian result.

Lemma 3.2.2 (Quastel and Valkó [73]) *There exists an explicit finite constant C such that*

$$E(t) \leqslant C t^{-1} \hat{E}(t^{-1}). \tag{3.27}$$

In summary the upper bound on $\hat{E}(\lambda)$ (3.25) can be converted into

$$\overline{\lim}_{t \to \infty} t^{-(3-\alpha)/2} E(t) \leqslant C_3' < \infty. \tag{3.28}$$

and the lower bound on $\hat{E}(\lambda)$ essentially means

$$\underline{\lim}_{t \to \infty} t^{-(5-2\alpha+\alpha^2)/4} E(t) \geqslant C_4' > 0. \tag{3.29}$$

Chapter 4

Stochastic algorithms

Stochastic approximation algorithms are meant to describe the state of a system which gradually adapts to its environment over time. A discrete-time stochastic algorithm $(x_n)_{n\in\mathbb{N}}$ taking values in $\mathbb{R}^d$ adapted to a filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ will often be assumed to satisfy a recursion of the form

$$x_{n+1} - x_n = \gamma_n F(x_n) + c_n(\epsilon_{n+1} + r_{n+1}), \tag{4.1}$$

where (ϵ_n) is a ($\mathbb{F}$-adapted) martingale increment, and where (γ_n), (c_n) and (r_n) are $\mathbb{F}$-adapted and small, in a sense that will be made more specific later.

Another setting considered here consists in recursions of the type

$$x_{n+1} - x_n = \gamma_n y_n + c_n(\epsilon_{n+1} + r_{n+1}),$$

where y_n is not necessarily a function of x_n. If it is possible to control the effect of certain small perturbations of the system on x_n and y_n, then a.s. nonconvergence towards some limit situations can be deduced: see Sections 2.1.2 and 2.1.4 for instance (and the corresponding papers [A1,A2]), where x_n and y_n are one-dimensional, with a technique involving a coupling.

The study of *stochastic approximation* started in the early 50s with the seminal papers of Robbins and Monro [76], Kiefer and Wolfowitz [47]), and was the subject of numerous works in signal processing, adaptive control (Kushner and Yin, Ljung and Söderström [49, 55, 56]) and recursive estimation (Nevelson and Khaminskii [64]). This theory more recently found interesting applications in neural networks (Fort and Pagès [37], White [93]), simulated annealing (Duflo [35]), game theory (Fudenberg and Levine [38]), and new developments in signal processing, through the use of methods based on Monte Carlo simulations, with Markov Chain Monte Carlo and sequential Monte Carlo approaches (Cappé, Moulines and Ryden [16]).

The behaviour of a stochastic algorithm is, under certain assumptions, comparable to the solutions of the ordinary differential equation (ODE)

$$\dot{x} = F(x).$$

The method was introduced by Ljung in 1977 ([54]) and developed, amongst others, by Kushner, Clark and Yin [48, 49], Benveniste, Métivier and Priouret [11], Duflo [35]. Benaïm and Hirsch [6, 7] could show, under weak assumptions on the dynamics, that the limit sets of the algorithm belong to a class of sets fully determined by the corresponding dynamical system, called internally chain recurrent sets.

I started my PhD on questions of nonconvergence towards unstable sets for the corresponding ODE. This problem had already been extensively studied, amongst others by Pemantle [68, 69], Priouret, Brandière and Duflo [12, 13, 14], and Benaïm [6]. My contribution was to emphasize in [A9,A10], described Section 4.1, that the convergence or not of the algorithm to the set mainly depends on the nature of the perturbation, and not on the dynamics itself. In particular non-convergence towards the unstable set is obtained in Theorem 15 under minimal assumptions in dimension one, if the perturbation is "exciting" enough.

The corresponding heuristics was useful in the study of VRRW on the integers, see Sections 2.1.2 and 2.1.4. Also, for strongly edge-reinforced random walks with Limic, localization on more than one edge can be interpreted as convergence towards 0 of a process that is "almost" a martingale, as explained Section 2.1.5, so that the techniques in the proof of Theorem 15 Section 4.1 can be adapted: see for instance Section 2.3 [A3], and Section 3 [A4].

Conversely, the study of some unstable limiting behaviour of VRRW where the "excitation" assumption was not satisfied (see Section 4 of [A1]) stimulated some work on the two-armed bandit problem, at the end of my PhD ([87] Chapter 4) and with Lamberton and Pagès in the case of i.i.d. payoffs [A11], and more recently with Vandekerkhove on ergodic payoffs [A12]: these results are presented in Section 4.2. On a theoretical level, they imply that, if the martingale increment $(\epsilon_n)_{n\in\mathbb{N}}$ is weak and irregular, which corresponds to slowly decreasing step sequences in the two-armed bandit setting, then the algorithm can converge to the unstable set with positive probability.

I also studied a few models of learning or adaptive behaviour with stochastic approximations techniques.

First, in a joint work with Benaïm and Schreiber [A15], we proposed models of evolution based on generalized urns processes, in order to understand the relative importance of natural selection and random genetic drift in finite but growing populations. This paper is not presented here, since I did not work on it or develop related techniques after my PhD.

Second, the study of VRRW as a stochastic approximation of the *replicator dynamics* with Benaïm [A5] enabled us to obtain results of localization with positive probability, which are presented Section 2.2.

The two other applications of stochastic algorithms considered here are online learning algorithms in [A13] and reinforcement learning in signaling game [A14]: they are presented in Sections 4.3 and 4.4.

4.1 Unstable traps [A9,A10]

I recall here a one-dimensional result obtained at the end of my PhD [A10], which could be adapted in several of my other works [A1,A2,A3,A4]. The techniques for its proof also enabled generalization of some previous results of nonconvergence towards unstable equilibria or periodic orbits and normally hyperbolic sets, see Theorem 2 [A9].

We consider real random variables (X_n), (Y_n), adapted to the filtration $\mathbb{F}$ and satisfying

$$X_{n+1} = Y_n + c_{n+1}(\epsilon_{n+1} + r_{n+1}) \quad \text{if } X_n \in V \tag{4.2}$$

Let V be a neighbourhood of 0 in $\mathbb{R}$, let (ϵ_n) and (r_n) $\mathbb{F}$-adapted real sequences, let (c_n) be a nonnegative deterministic sequence having infinitely many positive terms.

Let, for all $n \in \mathbb{N}$, $\alpha_n = \sum_{i=n}^{+\infty} c_i^2$. Let **A-1** be the following assumption on the perturbation:

- $E(\epsilon_{n+1}|\mathcal{F}_n) = 0$, $\liminf_{n\to\infty} E(\epsilon_{n+1}^2|\mathcal{F}_n) > 0$ et
  $\exists a > 2 \ \limsup_{n\to\infty} E(|\epsilon_{n+1}|^a|\mathcal{F}_n) < +\infty$
- $\sum_n r_n^2 < +\infty$.

Theorem 15 ([A10], Theorem 3.2.1) *Let (X_n) and (Y_n) be random variables satisfying (4.2). Assume* **A-1** *and, for all $n \in \mathbb{N}$, $|Y_n| \geqslant |X_n|$; then*

$$P(\lim_{n\to+\infty} X_n = 0) = 0.$$

Let us shortly sketch the proof of Theorem 15 when $\sum c_i^2 < \infty$ and $r_i = 0$: the general case is more difficult, but follows the same simple heuristics. First, it is easy to show that, for all k, $n \in \mathbb{N}$, $k \geqslant n$,

$$\mathbb{E}(X_k^2|\mathcal{F}_n) \geqslant X_n^2 + \sum_{i=n+1}^{k} c_i^2 \mathbb{E}(\epsilon_i^2|\mathcal{F}_n).$$

Let $T := \inf\{k \geqslant n \text{ s.t. } |X_k| \geqslant L(\sum_{i=k}^{\infty} c_i^2)^{1/2}\}$. Using an upper bound on $|X_{i+1} - Y_i|$, we deduce that, for any $L > 0$, there exists a constant $C > 0$ such that

$$\mathbb{P}\left(T < \infty \,|\, \mathcal{F}_n\right) \geqslant C.$$

Then Doob's inequality enables one to conclude that there exists a constant $D > 0$ such that, using Markov's inequality,

$$\mathbb{P}\left(\liminf |X_k| = 0 \,|\, \mathcal{F}_T\right) \leqslant \frac{\mathbb{E}(\sum_{i=T+1}^{\infty} \epsilon_i^2|\mathcal{F}_T)}{X_T^2} \leqslant \frac{D}{L^2}.$$

Indeed the numerator in the fraction would be $\mathbb{E}((X_\infty - X_T)^2|\mathcal{F}_T)$ if we had $X_k = Y_k$, and here the algorithm pushes X_k further away from 0 so that the probability that its $\liminf$ is 0 has to be smaller: see [A10] for details.

In summary we conclude that $\mathbb{P}(\liminf |X_k| \neq 0|\mathcal{F}_n)$ is lower bounded as n goes to infinity, which enables us to conclude, since it converges to $\mathbb{1}_{\{\liminf |X_k| \neq 0\}}$ a.s.

4.2 Narendra two-armed bandit algorithm [A11,A12]

The so-called two-armed bandit is a device with two arms, each one yielding an outcome in $\{0,1\}$ at each time step, irrespective of the strategy of the player, who faces the challenge of choosing the best one without loosing too much time on the other.

The Narendra algorithm is a stochastic procedure devised to that end, which was initially introduced by Norman, Shapiro and Narendra [65] in the fields of mathematical psychology and learning automata.

Formally, the Narendra two-armed bandit algorithm is defined as follows. At each time step $n \in \mathbb{N}$, we play source A (resp. source B) with probability X_n (resp. $1-X_n$), where $X_0 = x \in (0,1)$ is fixed and X_n is updated according to the following rule, for all $n \geqslant 0$:

$$X_{n+1} = \begin{cases} X_n + \gamma_{n+1}(1-X_n) & \text{if } U_{n+1} = A \text{ and } \eta_{A,n+1} = 1 \\ (1-\gamma_{n+1})X_n & \text{if } U_{n+1} = B \text{ and } \eta_{B,n+1} = 1 \\ X_n & \text{otherwise,} \end{cases} \tag{4.3}$$

where $(\gamma_n)_{n\geqslant 1}$ is a deterministic sequence taking values in $(0,1)$, U_{n+1} is the random variable corresponding to the label of the arm played at time $n+1$, and $\eta_{\ell,n+1}$ denotes the (deterministic) payoff, taking values in $\{0,1\}$, of source $\ell \in \{A,B\}$ at time $n+1$.

Note that, if the payoffs $\eta_{\ell,n+1}$, instead of being deterministic, were i.i.d. with probabilities θ_ℓ, then X_n would satisfy a recursion of the form (4.1), with $F(x) := (\theta_A - \theta_B)x(1-x)$. Assuming $\theta_A > \theta_B$, 0 would then be an unstable equilibrium of the corresponding dynamics, and a.s. convergence of X_n to 1 would be expected if $\theta_A > \theta_B$.

We first obtained a fallibility result with Lamberton and Pagès [A11] in the i.i.d. payoffs, which extends the setting of deterministic payoffs, as shown in [A12] with Vandekerkhove.

Theorem 16 ([A12], Theorem 4) *Assume* $\sum_{n\geqslant 0}\prod_{k=1}^{n}(1-\gamma_k\eta_{B,k}) < \infty$. *Then*

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = 0\right) > 0.$$

Theorem 16 does not require any information on the limit proportion of successes from A, but it is a fallibility result only when this limit proportion is greater than the one for B. Its proof is simple, and based on a Borel-Cantelli argument.

In the case where $(\eta_{B,k})_{k\geqslant 0}$ is an i.i.d. sequence of random variables, then

$$\mathbb{E}_x\left(\sum_{n\geqslant 0}\prod_{k=1}^{n}(1-\gamma_k\eta_{B,k})\right) = \sum_{n\geqslant 0}\prod_{k=1}^{n}(1-\gamma_k\theta_B) < \infty$$

ensures that the third condition of Theorem 16 is fulfilled a.s. This corresponds to the fallibility result Theorem 1 (b) in [A11].

Let us now discuss assumptions that ensure infallibility.

Step sequence Conditions. Let, for all $n \in \mathbb{N} \cup \{\infty\}$, $\Gamma_n = \sum_{k=1}^{n} \gamma_k$.

Let **(S1)** and **(S2)** be the following assumptions on the step sequence $(\gamma_n)_{n\in\mathbb{N}}$:

(S1) $(\gamma_n)_{n\geqslant 1}$ is nonincreasing and $\Gamma_\infty = \infty$;

(S2) $\gamma_n = O(\Gamma_n e^{-\theta_B \Gamma_n})$.

Let **(S)** be the set of conditions **(S1)-(S2)**.

Theorem 17 ([A11], Theorem 1 (c)) *Assume $(\eta_{i,k})_{k\geqslant 0}$, $i \in \{A, B\}$ are i.i.d. sequences of random variables, with mean θ_A and θ_B, $\theta_A > \theta_B$, that* **(S2)** *holds and $\Gamma_\infty = \infty$. Then X_n converges to 1 a.s.*

The result was later improved by Lamberton and Pagès, who showed in [50], still in this i.i.d. case with $\theta_A > \theta_B$, that if the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is nonincreasing, then

$$\mathbb{P}(\lim X_n = 1) = 1 \iff \mathbb{P}\left(\sum_{n\geqslant 0} \prod_{k=1}^{n} (1 - \gamma_k \eta_{B,k}) < \infty\right) = 0;$$

note that this provides, together with Theorem 16, a necessary and sufficient condition for infallibility in that case.

If we assume for instance that

$$\gamma_n := (\frac{c}{c+n})^\alpha, \ c > 0, \alpha \in (0, 1]$$

then it follows from Theorems 16 and 17 that X_n converges to 1 a.s. if and only if $\alpha = 1$ and $c \leqslant \theta_B^{-1}$.

The proof of Theorem 17 draws on a similar heuristics as Theorem 15, with the difference that the variance vanishes when X_n approaches 0: so the proof of the first step, namely that X_n will eventually become larger than the standard deviation of the remaining perturbation, relies on different techniques.

We generalized Theorem 17 with Vandekerkhove to payoffs which are no longer i.i.d., under the assumption that the empirical means of $(\eta_{i,k})_{k\geqslant 0}$ converge at rate at least $\log(n+1)^{-1-\epsilon}$, $\epsilon > 0$.

Ergodic Conditions. Let **(E)** be the assumption that the outputs of arms A and B satisfy

$$\textbf{(E)} \quad \frac{1}{n}\sum_{k=1}^{n} \eta_{A,k} \underset{n\to\infty}{\longrightarrow} \theta_A, \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n} \eta_{B,k} \underset{n\to\infty}{\longrightarrow} \theta_B,$$

where θ_A, $\theta_B \in (0, 1)$. Let, for all $n \in \mathbb{N}$,

$$R_n := \max_{\ell\in\{A,B\}} \left| \sum_{i=1}^{n} (\eta_{\ell,i} - \theta_\ell) \right|.$$

Given a map $\phi : \mathbb{N} \longrightarrow \mathbb{R}_+$ and $\theta_A, \theta_B \in (0,1)$, let us denote by **(E$\phi$)** the assumption that $R_n/\phi(n) \underset{n\to\infty}{\longrightarrow} 0$.

Let **(E1)** and **(E2)** be condition **(Eϕ)**, respectively with the following assumption on ϕ:

(E1) ϕ is nondecreasing concave on $[k_0, \infty)$ for some $k_0 \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} \gamma_n \phi(n) < \infty$.

(E2) $\phi(n) = \dfrac{n}{(\log(n+2))^{1+\varepsilon}}$ for some $\varepsilon > 0$.

Note that **(E)** corresponds to **(Eϕ)** with $\phi(n) = n$, $n \in \mathbb{N}$, for which **(E1)** holds for instance in the case of a step sequence $\gamma_n = c/(c+n)$, $c > 0$. It is also possible to show that **(S)-(E2)** implies **(E1)** (see Lemma 1 [A12]).

Theorem 18 ([A12], Theorem 2) *Under assumptions* **(S1)-(E1)***, the Narendra sequence* $(X_n)_{n\in\mathbb{N}}$ *converges* $\mathbb{P}_x - a.s$ *towards* 0 *or* 1 *as* n *tends to infinity.*

Theorem 19 ([A12], Theorem 3) *Under assumptions* **(S)-(E2)** *and* $\theta_A > \theta_B$*, the Narendra sequence* $(X_n)_{n\in\mathbb{N}}$ *converges* $\mathbb{P}_x - a.s$ *towards* 1 *as* n *tends to infinity.*

Recall that the above conditions **(E1)** and **(E2)** are purely deterministic. If we let the sequences $(\eta_{A,i})_{i\in\mathbb{N}}$ and $(\eta_{B,i})_{i\in\mathbb{N}}$ be distributed as i.i.d. sequences with expectations θ_A and θ_B, then **(E2)** almost surely occurs as a consequence of the law of iterated logarithm. If we further assume **(S)** and $\theta_A > \theta_B$, then Theorem 19 implies that the algorithm $(X_n)_{n\in\mathbb{N}}$ almost surely converges to 1, which generalizes Theorem 17 in the case nonincreasing step sequences $(\gamma_n)_{n\in\mathbb{N}}$.

In practice, the Narendra algorithm is used in the context of performance assessment, in applications either in automatic control or in financial mathematics, and the i.i.d. assumption looks rather unrealistic, since the performance depends in general on parameters that evolve slowly and randomly in time.

The proof of Theorem 19 is quite technical, and relies on carefully chosen integrations by parts, in order to translate information on the sequences $(\eta_{\ell,i})_{i\in\mathbb{N}}$, $\ell \in \{A, B\}$ into some on the asymptotic behaviour of the algorithm.

4.3 Online Learning as Stochastic Approximation of Regularization Paths [A13]

Consider the following classical problem of learning from examples: given a sequence of i.i.d. random samples $(z_t = (x_t, y_t))_{t\in\mathbb{N}}$ drawn from a probability measure ρ on $X \times Y$, one seeks to approximate the *regression function*

$$f_\rho(x) := \int_Y y d\rho_{Y|x},$$

i.e. the conditional expectation of y given x.

The quality of the estimate one can obtain depends on the regularity of f_ρ, measured through a Mercer Kernel $K : X \times X \longrightarrow \mathbb{R}$ (continuous, symmetric and positive semidefinite). The Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}_K$ is defined as the closure of the linear span of the set of functions $\{K_x := K(x,.),\ x \in X\}$, with the inner product, denoted as $< .,. >_K$, satisfying $< K_x, K_y >_K = K(x,y)$.

Recall the reproducing property $< K_x, f >= f(x)$, for all $x \in X,\ f \in \mathcal{H}_K$, which implies in particular that $\|f\|_\infty \leqslant \kappa \|f\|_K$, where $\kappa := \sup_{x\in X} \sqrt{K(x,x)}$.

In "batch" learning algorithms, one way to approximate f_ρ, knowing $(z_t)_{1\leqslant t\leqslant m}$, is to perform a Tikhonov regularization, in order to trade off bias against variance, as usually in statistical learning, i.e. choose, for some $\lambda > 0$,

$$f_{z,\lambda} := \mathsf{argmin}_{f\in\mathcal{H}_K} \left\{ \frac{1}{m} \sum_{i=1}^{m} (f(x_i) - y_i)^2 + \lambda \|f\|_K^2 \right\}.$$

Note that $f_{z,\lambda}$ exists and is unique by convexity, and that λ can be chosen as a function of the regularity assumed on f_ρ, and of m.

More precisely, let ρ_X be the induced marginal probability measure from ρ on X, and let $L_K : \mathcal{L}^2(\rho_X) \longrightarrow \mathcal{L}^2(\rho_X)$ be the self-adjoint operator defined by

$$L_K(f)(x) = \int_X K(x,y) f(y) d\rho_X(y) =< K_x, f >_{\mathcal{L}^2(\rho_X)}, x \in X,$$

which is positive and compact, so that we can define (through any orthornomal system), the operators $L_K^r : \mathcal{L}^2(\rho_X) \longrightarrow \mathcal{L}^2(\rho_X)$ for all $r \in \mathbb{R}_+$. We assume that there exists $M_\rho > 0$ such that $\rho((x,y) : |y| \leqslant M_\rho) = 1$.

Let $\mathsf{Cst}(a_1, a_2, \ldots, a_p)$ denote a positive constant depending only on $a_1,\ a_2,\ \ldots\ a_p$, and let Cst denote a universal positive constant.

Smale and Zhou performed in 2007 the following analysis on "batch" learning Tikhonov regularization algorithms.

Theorem 20 (Smale and Zhou [85], Theorem 2) *Assume $L_K^{-r} f_\rho \in L^2(\rho_X)$ for some $1/2 < r \leqslant 1$. Let $\lambda := (3\kappa M_\rho / \|L_K^{-r} f_\rho\|_\rho)^{2/(1+2r)} m^{-1/(1+2r)}$. Then, for all $\delta \in (0,1)$, with confidence $1-\delta$,*

$$\|f_{z,\lambda} - f_\rho\|_K \leqslant 4 \log(2/\delta) (3\kappa M_\rho)^{(2r-1)/(2r+1)} \|L_K^{-r} f_\rho\|_\rho^{2/(1+2r)} m^{-(2r-1)/(4r+2)}.$$

The choice of λ in Theorem 20 is unrealistic, because f_ρ is not known, but one could however have some initial information on the regularity of f_ρ.

Our goal with Yao was to obtain similar bounds for *online learning algorithms,* which are recursive, contrary to *batch learning algorithms* which process the data once and for all at some fixed time m. We could obtain the same rate [A13], with a careful choice of the step sequences.

More precisely, we analyze algorithms of the type

$$f_t = f_{t-1} - \gamma_t [(f_{t-1}(x_t) - y_t) K_{x_t} + \lambda_t f_{t-1}], \qquad \text{for some } f_0 \in \mathcal{H}_K, \text{ e.g. } f_0 = 0,$$

with gain sequences $(\lambda_t)_{t\in\mathbb{N}}$ and $(\gamma_t)_{t\in\mathbb{N}}$ taking values in $\mathbb{R}_+\setminus\{0\}$, originally introduced by Smale and Yao in [84], and further studied by Yao in [96]. The recursion can be interpreted as a stochastic gradient descent

$$f_t = f_{t-1} - \mathsf{grad}\ V_{z_t}^{\lambda_t}(f_{t-1}),$$

where

$$V_z^{\lambda}(f) := \frac{1}{2}[(f(x)-y)^2 + \lambda\|f\|_k^2]$$

for all $f \in \mathcal{H}_K$, $z \in X \times Y$ and $\lambda \in \mathbb{R}_+$. One of the advantages of such algorithms, besides being adaptive, is their computational complexity, which is quadratic in time in the worst case, and can be linear at the cost of a large memory allocation. In comparison, the batch learning Tikhonov regularization scheme typically involves the inverse of a matrix, which is $O(t^3)$ in the worst case.

We optimize the choice of $(\lambda_t)_{t\in\mathbb{N}}$ and $(\gamma_t)_{t\in\mathbb{N}}$, as a function of the regularity of f_ρ. We choose $f_0 := 0$, and

$$\gamma_t := (t+t_0)^{-\frac{2r}{2r+1}}, \quad \lambda_t := (t+t_0)^{-\frac{1}{2r+1}}, \tag{4.4}$$

for some $t_0 := \mathsf{Cst}(\kappa)$.

Theorem 21 ([A13], Theorem B) *Assume $L_K^{-r} f_\rho \in \mathcal{L}^2(\rho_X)$ for some $1/2 < r \leqslant 3/2$. Then, with confidence $1-\delta$,*

$$\|f_t - f_\rho\|_K \leqslant \mathsf{Cst}(\kappa, M_\rho, \|L_K^{-r} f_\rho\|_{\mathcal{L}^2(\rho_X)}) \left(\log\frac{2}{\delta}\right) t^{-\frac{2r-1}{4r+2}},$$

Theorem 22 ([A13], Theorem C) *Assume $L_K^{-r} f_\rho \in \mathcal{L}^2(\rho_X)$ for some $1/2 < r \leqslant 1$. Then, with confidence $1-\delta$,*

$$\|f_t - f_\rho\|_{\mathcal{L}^2(\rho_X)} \leqslant \mathsf{Cst}(\kappa, M_\rho, \|L_K^{-r} f_\rho\|_{\mathcal{L}^2(\rho_X)}) \left(\log\frac{2}{\delta}\right)^2 t^{-\frac{r}{2r+1}}.$$

The exponent in t in the $\mathcal{H}_K$-norm rate is the same as the best known one in batch learning, obtained by Smale and Zhou [85], and the mean square distance convergence rate is optimal in the sense that it reaches the minimax and individual lower rates (see for instance Caponnetto and de Vito [15]).

The choice of the gain sequences in (4.4) is derived from the analysis of the algorithm as a stochastic approximation of a Tikhonov regularization path converging to the regression function.

The proof is based on the one hand on some martingale and reverse-martingale expansions, and on the other hand on probabilistic exponential inequalities on Banach spaces provided of Pinelis [72], which allow to extend finite-dimensional techniques in the study of rate of convergence of stochastic approximation to the infinite-dimensional online algorithm considered here.

The paper is still under progress, in two respects: first, the $(\log(2/\delta))^2$ factor in Theorem 22 improves the $\sqrt{1/\delta}$ factor in Theorem C [A13], and we are finishing the writing of this improvement. Second, the constants involved in Theorems 21 and 22 can be improved, if we multiply γ_t and λ_t in our current choice by a factor depending on the regularity of f_ρ: we are currently investigating this.

4.4 Reinforcement learning in signaling game [A14]

We study learning and creation of a common language through reinforcement learning. Let $\mathcal{S}_1$ be a set of states, and let $\mathcal{S}_2$ be a set of signals, with $|\mathcal{S}_1| = M$, $|\mathcal{S}_2| = N$; let $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$. A Sender and Receiver have a set of urns which they use to make decisions:

For any state i, the Sender has a State Urn i, with balls of N different colors, one per signal ($V(n,i,j)$ balls of colour j at time n).

For any signal j, Receiver has a Signal Urn j with balls of M different colors, one per state ($V(n,j,i)$ balls of colour i at time n).

The players privilege strategies that yield better payoff, through reinforcement. More precisely, for any state i, signal j and time n, let

$$\begin{aligned} V(n,i,j) :=&\ \text{number of balls of colour/signal } j \text{ in state Urn } i \text{ at time } n \\ =&\ \text{number of balls of colour/state } i \text{ in signal Urn } i \text{ at time } n \end{aligned}$$

The model is described as follows.

1 **Initial setting.** For any $i \in \mathcal{S}_1$, $j \in \mathcal{S}_2$, we assume that $V(0,i,j) = V(0,j,i) > 0$ is fixed.

2 **Reinforcement learning.** At each time step, Sender observes a certain state i from set of states $\mathcal{S}_1$; we assume here that all states arise with equal probability $1/M$. Then Sender randomly chooses a signal, his probability of drawing j being

$$\frac{V(n,i,j)}{\sum_{l\in\mathcal{S}_2} V(n,i,l)}.$$

Receiver observes the signal he receives (let us call it j) and then randomly chooses a state k with probability

$$\frac{V(n,k,j)}{\sum_{l\in\mathcal{S}_1} V(n,l,j)}.$$

3 **Updating rule.** Both Sender and Receiver receive payoffs when the state chosen by Receiver matches the state observed by Sender. For any $i \in \mathcal{S}_1$, $j \in \mathcal{S}_2$,

$$V(n+1,i,j) := \begin{cases} V(n,i,j)+1 & \text{if Sender observes state } i \text{ and chooses} \\ & \text{signal } j \text{, and Receiver chooses state } i; \\ V(n,i,j) & \text{if else.} \end{cases}$$

For simplicity, for all $n \in \mathbb{N}$, i, $j \in \mathcal{S}_1$ (or $\mathcal{S}_2$), let $V(n,i,j) = 0$.

Note that it follows directly from the model that, for all $n \in \mathbb{N}$, i, $j \in \mathcal{S}$, $V(n,i,j) = V(n,j,i)$.

For all $n \in \mathbb{N}$ and i, $j \in \mathcal{S}$, let

$$T_n := \sum_{k \in \mathcal{S}_1, l \in \mathcal{S}_2} V(n,k,l)$$

be the number of successes up to time n, and let

$$\begin{aligned} x_{ij}^n &:= \frac{V(n,i,j)}{T_n}, \;\; x_i^n := \sum_{j \in \mathcal{S}_2} x_{ij}^n, \;\; x_j^n := \sum_{i \in \mathcal{S}_1} x_{ij}^n \\ X_n &:= (x_{ij}^n)_{i \in \mathcal{S}_1, j \in \mathcal{S}_2}; \end{aligned} \tag{4.5}$$

X_n takes values in

$$\Delta := \left\{ (x_{ij})_{i \in \mathcal{S}_1, j \in \mathcal{S}_2} : x_{ij} \geqslant 0, \sum_{i \in \mathcal{S}_1, j \in \mathcal{S}_2} x_{ij} = 1, \quad x_{ij} = x_{ji} \right\};$$

given $x \in \Delta$ and i, $j \in \mathcal{S}_2$, define x_{ij}, x_i and x_j as in (4.5).

In the particular case where $\mathcal{S}_1 = \{1,2\}$, $\mathcal{S}_2 = \{A,B\}$, Argiento, Pemantle, Skyrms and Volkov obtained the following

Theorem 23 (Argiento, Pemantle, Skyrms and Volkov [1]) *Almost surely, either $X_n = (x_{1A}, x_{1B}, x_{2A}, x_{2B})$ converges towards $(1/2, 0, 0, 1/2)$ or $(0, 1/2, 1/2, 0)$ as n tends to infinity.*

Let $F(X) : \Delta \longrightarrow T\Delta$ be defined by

$$F(X) := \left(x_{ij} \Big(\frac{x_{ij}}{x_i x_j} - H(X) \Big) \right)_{i \in \mathcal{S}_1, j \in \mathcal{S}_2}$$

with the convention that $F(X)_{ij} = 0$ if $x_{ij} = 0$. Let, for all $X \in \Delta$,

$$H(X) = \sum_{i \in \mathcal{S}_1,\, j \in \mathcal{S}_2 \, s.t. \, x_{ij} > 0} \frac{x_{ij}^2}{x_i x_j} = \frac{1}{2} \sum_{i,j \in \mathcal{S} \, s.t. \, x_{ij} > 0} \frac{x_{ij}^2}{x_i x_j}.$$

Note that H is not continuous on the boundary of the simplex.

We obtained the following results with Hu and Skyrms [A14].

Theorem 24 ([A14], Theorem 2.1) *The communication potential process $(H(x_n))_{n \in \mathbb{N}}$ is a bounded submartingale, and hence converges a.s.*

The proof of Theorem 24 is quite technical; let us however show that H is a Lyapounov function for the ODE (4.9). Let

$$\partial\Delta = \{x \in \Delta : x_i = 0 \text{ for some i}\}.$$

Note that $\partial\Delta$ is not the topological boundary of Δ.

Assume $x \in \Delta \setminus \partial\Delta$:

$$\begin{aligned}
\nabla H \cdot F\,(x) \;&=\; \sum_{i,j\in\mathcal{S}} \frac{x_{ij}}{x_i x_j}\Big(\frac{(x_{ij})^2}{x_i x_j} - x_{ij}H(x)\Big) \\
&\quad - \frac{(x_{ij})^2}{2(x_i)^2 x_j}\Big(\sum_{k\in\mathcal{S}} \frac{(x_{ik})^2}{x_i x_k} - x_{ik}H(x)\Big) \\
&\quad - \frac{(x_{ij})^2}{2x_i (x_j)^2}\Big(\sum_{k\in\mathcal{S}} \frac{(x_{jk})^2}{x_j x_k} - x_{jk}H(x)\Big) \\
&= \sum_{i,j\in\mathcal{S}} \frac{(x_{ij})^3}{(x_i)^2(x_j)^2} - \sum_{i,j,k\in\mathcal{S}} \frac{(x_{ij})^2(x_{ik})^2}{(x_i)^3 x_j x_k} \qquad (4.6)\\
&\quad - \Big(\sum_{i,j\in\mathcal{S}} \frac{(x_{ij})^2}{x_i x_j} - \sum_{i,j,k\in\mathcal{S}} \frac{(x_{ij})^2 x_{ik}}{(x_i)^2 x_j}\Big)H(x). \qquad (4.7)
\end{aligned}$$

Using that $\sum_{k\in\mathcal{S}} x_{ik}/x_i = 1$,

$$(4.7) \;=\; \sum_{i,j\in\mathcal{S}} \frac{(x_{ij})^2}{x_i x_j}\Big(1 - \sum_{k\in\mathcal{S}} \frac{x_{ik}}{x_i}\Big)H(x) \;=\; 0.$$

Using the symmetry between j and k, we obtain

$$\begin{aligned}
(4.6) \;&=\; \sum_{i,j,k\in\mathcal{S}} \frac{(x_{ij})^3 x_{ik}}{(x_i)^3(x_j)^2} - \sum_{i,j,k\in\mathcal{S}} \frac{(x_{ij})^2(x_{ik})^2}{(x_i)^3 x_j x_k} \\
&= \frac{1}{2}\sum_{i,j,k\in\mathcal{S}} \frac{x_{ij}x_{ik}}{x_i}\left(\frac{x_{ij}}{x_i x_j} - \frac{x_{ik}}{x_i x_k}\right)^2. \qquad (4.8)
\end{aligned}$$

Lemma 4.4.1 ([A14], Lemma 3.1) *For all $n \in \mathbb{N}$,*

$$X_{n+1} - X_n = \frac{1}{(1+T_n)M}F(X_n) + \eta_{n+1}$$

where $(\eta_n)_{n\in\mathbb{N}}$ is a martingale increment, with $|\eta_n| \leqslant 2/(1+T_n)$.

Let

$$\Gamma := \{X \in \Delta \text{ s.t. } F(X) = 0\}$$

be the set of equilibria of the ODE

$$\frac{dX}{dt} = F(X). \qquad (4.9)$$

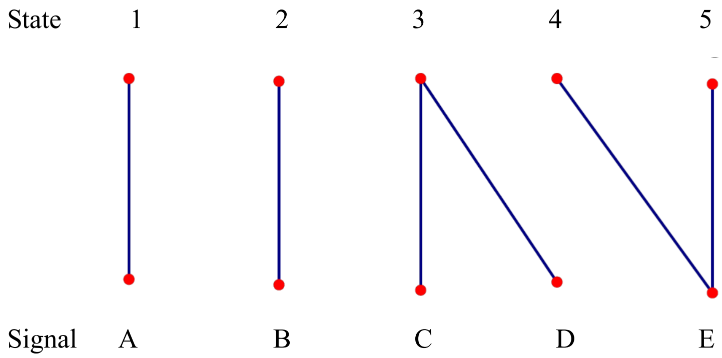


Figure 4.1: Here $M = N$, $(P)_{\mathcal{G}}$ holds, with both a synonym and an informational bottleneck.

Theorem 25 ([A14], Theorem 2.2) *(X_n) converges to Γ a.s.*

The proof of Theorem 25 is very technical, since we have to deal with the possibility that X_n approaches $\partial\Delta$, where F is not continuous anymore.

Given a graph $\mathcal{G}$ on $\mathcal{S}_1 \cup \mathcal{S}_2$, let $(P)_{\mathcal{G}}$ be the following property:

- if we let $\mathcal{C}_1$, ... $\mathcal{C}_d$ be its connected components then, for every $i \in \{1, \dots, d\}$, $\mathcal{C}_i \cap \mathcal{S}_1$ or $\mathcal{C}_i \cap \mathcal{S}_2$ is a singleton.
- each vertex has a corresponding edge.

We call *synonym* (resp. *informational bottleneck* or *polysemy*) a state (resp. signal) associated to several signals (resp. states), or the corresponding set of adjacent signals (resp. states). Obviously $M \neq N$ ensures the existence of at least one synonym or polysemy.

Given $x \in \Delta$, let $\mathcal{G}_x$ be the weighted bipartite graph with vertices $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$, and adjacency $\sim$ as follows:

$$\forall\, i \in \mathcal{S}_1,\ j \in \mathcal{S}_2,\ i \sim j \iff x_{ij} > 0.$$

Note that, if x is not in the topological boundary of Δ, then $\mathcal{G}_x$ is the complete 2-partite graph with partitions $\mathcal{S}_1$ and $\mathcal{S}_2$.

Given $x \in \Delta$, and even if $M = N$, property $(P)_{\mathcal{G}_x}$ allows for synonyms or informational bottlenecks, and does not ensure that the system is optimal as a communication system, i.e. that $H(x)$ reaches the maximum of H.

Theorem 26 ([A14], Theorem 2.3) *For all $\mathcal{G}$ on $\mathcal{S}_1 \cup \mathcal{S}_2$ s.t. $(P)_{\mathcal{G}}$ holds, we have, with positive probability,*

(a) $x_n \to x$ *s.t.* $\mathcal{G}_x = \mathcal{G}$.

(b) $\forall i, j \in \mathcal{S}$, $V(\infty, i, j) = \infty \iff \{i, j\}$ *is an edge of* $\mathcal{G}$.

We also show in Proposition 5.2 [A14] that $x \in \Delta \setminus \partial\Delta$ is linearly stable if and only if $(P)_{\mathcal{G}_x}$ holds.

Interestingly, the result (and proof) of Theorem 26 resembles localization results for vertex-reinforced random walks [70, 90], [A5,A1,A2].

We are currently studying, with my PhD student Daniel Kious, a generalization of this model to social networks: on a graph G, i decides to speak to j, within its neighbours, with a probability proportional to the number of times the communication between i and j succeeded, the same for j, and the communication succeeds if the two events occur and i or j has been chosen at that time by Nature to communicate. These probabilities to be chosen to communicate are arbitrary (p_i for i).

Then the signaling model can be seen as the case of a bipartite graph; we are currently extending the results of this section to that setting.

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