# Diffusivity bounds for 1d Brownian polymers

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**Abstract**

We study the asymptotic behavior of a self interacting one dimensional Brownian polymer first introduced by Durrett and Rogers [5]. The polymer describes a stochastic process with a drift which is a certain average of its local time.

We show that a smeared out version of the local time function as viewed from the actual position of the process is a Markov process in a suitably chosen function space, and that this process has a Gaussian stationary measure. As a first consequence this enables us to partially prove a conjecture about the law of large numbers for the end-to-end displacement of the polymer formulated in [5].

Next we give upper and lower bounds for the variance of the process under the stationary measure, in terms of the qualitative infrared behavior of the interaction function. In particular we show that in the locally self-repelling case (when the process is essentially pushed by the negative gradient of its own local time) the process is super-diffusive.

## 1 Introduction

### 1.1 Historical background

Let $(X(t))_{t\geq 0}$ be the random process defined by $X(0) := x_0 \in \mathbb{R}$ and

$$X(t) = B(t) + \int_0^t \int_0^s \left(\xi(X(s)) + f(X(s) - X(u))du\right) ds, \tag{1}$$

where $B(t)$ is a standard 1d Brownian motion, $f : \mathbb{R} \to \mathbb{R}$ is a function with sufficient regularity, and $\xi : \mathbb{R} \to \mathbb{R}$ is an initial drift profile with regularity (detailed below).

This process $X(t)$ was introduced by Durrett and Rogers [5] as a model for the location of the end of a growing polymer at time $t$, in the case of zero initial profile ($\xi \equiv 0$).

It is phenomenologically instructive to write the driving mechanism on the right hand side of (1) in terms of the occupation time density (local time) of the process $X(t)$:

$$X(t) = B(t) + \int_0^t \Big\{\xi(X(s)) + \int_{-\infty}^{\infty} f(z)L(s, X(s) - z)dz\Big\}ds, \tag{2}$$

where

$$L(s,y) := \partial_y \int_0^s 1\!\!1_{\{X(u)<y\}} du. \tag{3}$$

Various choices of the function $f$ have been analyzed in detail and mathematically deep, sometimes phenomenologically surprising results have been obtained in the papers [2], [3], [16]. For a detailed survey of the problem see [16]. However, satisfactory understanding of the asymptotic behavior of the process (1) has not been reached in many interesting cases.

In particular, the following conjecture has remained open so far:

**Conjecture 1.** *(Durrett and Rogers [5]) Suppose $f$ has sufficient fast decay at infinity, and*

$$f(-x) = -f(x), \qquad \textit{and} \qquad \operatorname{sgn}(f(x)) = \operatorname{sgn}(x). \tag{4}$$

*Then $X(t)/t \to 0$ a.s.*

Tóth and Werner later conjectured that, under the same assumptions, $X(t)/t^{2/3}$ converges in law, by analogy with the discrete space-time self-repelling random walk on $\mathbb{Z}$ which displays this $t^{2/3}$ asymptotic behaviour (with identification of the limiting distribution, see [23]), and with a continuous space-time process arising as a scaling limit constructed in [25]. These studies were stimulated by so-called *true self-avoiding random walk* (TSAW) introduced in the physics literature by Amit, Parisi and Peliti [1].

We partially prove Conjecture 1, and obtain asymptotic lower and upper bounds in the stationary regime which translate, in the case (4), into

$$\varliminf_{t\to\infty} t^{-5/4}\mathbf{E}\big(X(t)^2\big) > 0, \qquad \varlimsup_{t\to\infty} t^{-3/2}\mathbf{E}\big(X(t)^2\big) < \infty, \tag{5}$$

where the lower bound is meant in the sense of Laplace transform (see details later). We also show that the process $X(t)$ behaves diffusively, for functions $f$ satisfying a certain summability condition (see (24)). Our argument is based on the study of an underlying Markov process living in the path space, which has invariant Gaussian measure.

In the follow-up paper [7] the analogous polymer model in dimensions $d \geq 3$ is investigated. There full CLT is proved for the locally self-repelling case in those dimensions, using Kipnis-Varadhan theory and a relaxation of Varadhan's sector condition. As explained in that paper, technical parts of that method do not apply (so far) in lower dimensions.

## 1.2 Assumptions on $f$

We assume throughout the paper that the Brownian polymer processes (1) are under the assumption that the function $f$ is the *negative gradient of an absolutely integrable smooth function of positive type*: that is

$$f(x) = -b'(x), \tag{6}$$

where $b \in L^1(\mathbb{R}) \cap C^{(\infty)}(\mathbb{R})$ and has nonnegative Fourier transform. Note that positive definiteness implies

$$b(-x) = b(x), \qquad \sup_{x \in \mathbb{R}} |b(x)| = b(0). \tag{7}$$

Actually it is sufficient to assume that $b$ is infinitely differentiable at $x = 0$. Since $b \in L^1(\mathbb{R})$ and $\hat{b}(p) \geq 0$ it follows that $\hat{b}$ has finite moments of all order: for all $k \in \mathbb{N}$

$$\int_{-\infty}^{\infty} |\, p\,|^k \, \hat{b}(p) dp < \infty, \tag{8}$$

and hence it follows that actually $b \in C^{\infty}(\mathbb{R})$).

Note that the regularity assumption is much more than really needed, we assume it in order to make the technical arguments shorter.

## 1.3 Underlying Markov Process and invariant Gaussian measure

First, we let $t \mapsto \zeta(t, x)$ be the "drift function" environment at time $t$ (i.e. $\zeta(t, x)$ is the drift that would be endured by the particle at time $t$ if it were in $x$):

$$\zeta(t, x) = \zeta(0, x) + \int_0^t b'(X(s) - x) ds. \tag{9}$$

Then (1) reads

$$X(t) = X(0) + B(t) + \int_0^t \zeta(s, X(s)) ds. \tag{10}$$

In other words,

$$dX(t) = dB(t) + \zeta(t, X(t))dt, \qquad d\zeta(t, x) = b'(X(t) - x)dt. \tag{11}$$

Now let $\eta$ be the environment profile as seen from the moving point $X(t)$, i.e.

$$x \mapsto \eta(t, x) := \zeta(t, X(t) + x). \tag{12}$$

Then $t \mapsto \eta(t) := \eta(t, \cdot)$ *is a Markov process*, on the space of smooth functions of slow increase at infinity:

$$\Omega := \{\omega \in C^{\infty}(\mathbb{R} \to \mathbb{R}) \, : \, (\forall k \geq 0, \forall l \geq 1) : \ \ \|\, \omega \,\|_{k,l} < \infty\}. \tag{13}$$

where $\|\, \omega \,\|_{k,l}$ are the seminorms

$$\|\, \omega \,\|_{k,l} := \sup_{x \in \mathbb{R}} \big(1 + |\, x \,|\big)^{-1/l} \, \big|\, \omega^{(k)}(x) \,\big| \,, \qquad k \geq 0, \ \ l \geq 1. \tag{14}$$

$\Omega$ endowed with these seminorms $\|\, \omega \,\|_{k,l}$, $k \geq 0$, $l \geq 1$, is a Fréchet space.

Note that the existence and uniqueness of a pathwise *strong* solution of (1) is standard, see for instance Theorem 11.2 in [20]. Furthermore, given the corresponding assumptions on $b$, if $\zeta(0, .) \in \Omega$ then $\zeta(t, .) \in \Omega$, for all $t \geq 0$.

Using (11) with the definition (12), we derive by standard Ito-calculus that

$$d\eta(t,x) = \eta'(t,x)dB(t) + \eta'(t,x)\eta(t,0)dt + \frac{\eta''(t,x)}{2}dt - b'(x)dt. \tag{15}$$

We show in the following Theorem 1 that the unique *Gaussian probability measure* $\pi(d\omega)$ on $\Omega$ with mean and covariance

$$\int_\Omega \omega(x)\pi(d\omega) = 0, \qquad \int_\Omega \omega(x)\omega(y)\pi(d\omega) = b(x-y), \tag{16}$$

is invariant for the *Markov process* $t \mapsto \eta(t) := \eta(t,\cdot)$.

Recall that Minlos theorem [22] implies, given $x \mapsto b(x)$ with the assumed properties, that the expectations and covariances (16) define uniquely a Gaussian probability measure $\pi(d\omega)$, supported by the space $\Omega$ [14, 15].

A natural realization of the measure $\pi(d\omega)$ is the following: Let $c : \mathbb{R} \to \mathbb{R}$ be the *unique* function of positive type for which $b = c * c$ and let $w'(y)$ be standard white noise on the line $\mathbb{R}$. Let

$$\omega(x) := \int_R c(x-y)w'(y)dy. \tag{17}$$

Then the random element $\omega(\cdot) \in \Omega$ will have exactly the distribution $\pi(d\omega)$.

Note that the group of spatial translations

$$\mathbb{R} \ni z \mapsto \tau_z : \Omega \to \Omega, \qquad (\tau_z\omega)(x) := \omega(x+z) \tag{18}$$

acts naturally on $\Omega$ and preserves the probability measure $\pi(d\omega)$. As can be seen from the representation (17), the dynamical system $(\Omega, \pi(d\omega), \tau_z : z \in \mathbb{R})$ is ergodic.

**Theorem 1.** *The Gaussian probability measure $\pi(d\omega)$ on $\Omega$, with mean $0$ and covariances (16) is time-invariant and ergodic for the $\Omega$-valued Markov process $t \mapsto \eta(t)$.*

Theorem 1 is proved in Subsection 2.3; we also provide in Section 1.5 a short formal proof of it.

Now define the function $\varphi : \Omega \to \mathbb{R}$,

$$\varphi(\omega) := \omega(0). \tag{19}$$

Note that (9), (10) and (12) imply

$$X(t) - X(0) = B(t) + \int_0^t \varphi(\eta(s))ds. \tag{20}$$

The law of large numbers is therefore a direct consequence of ergodicity.

**Corollary 1.** *For $\pi$-almost all initial profiles $\zeta(0,\cdot)$,*

$$\lim_{t\to\infty} \frac{X(t)}{t} = 0 \quad \text{a.s.} \tag{21}$$

This partially settles Conjecture 2 of [5].

## 1.4 Diffusivity bounds on $X(t)$

All results in the sequel will be meant for the process being in the stationary regime described last subsection (i.e. $\zeta(0,\cdot)\in\Omega$ distributed according to $\pi$).

We now study the $t\to\infty$ asymptotics of the variance of displacement

$$E(t) := \mathbf{E}\big(X(t)^2\big). \tag{22}$$

First, we use a special kind of time-reversal symmetry, sometimes called Yaglom-reversibility, see [26, 27, 4], to show in Section 3.1 that, under the general assumptions of Section 1.2, for any $s<t$, the random variables $B(t)-B(s)$ and $\int_s^t \varphi(\eta(u))du$ are *uncorrelated*, and hence

$$\mathbf{E}\big((X(t)-X(s))^2\big) = t-s+\mathbf{E}\Big(\Big(\int_s^t \varphi(\eta(u))du\Big)^2\Big). \tag{23}$$

Furthermore, if the following summability condition holds,

$$\rho^2 := \int_{-\infty}^{\infty} p^{-2}\hat{b}(p)dp < \infty, \tag{24}$$

then the process $X(t)$ behaves diffusively, as stated in the following Theorem 2, shown in Section 3.2. Note that (24) is a condition on the infrared ($|p|\ll 1$) asymptotics of the spectrum $\hat{b}(p)$.

**Theorem 2.** *Let $\rho^2$ be the constant defined in* (24). *Then*

$$1 \le \varliminf_{t\to\infty} t^{-1}E(t) \le \varlimsup_{t\to\infty} t^{-1}E(t) \le 1+\rho^2. \tag{25}$$

**Remarks:** (1) The upper bound in (25) is informative only when the integral on the right hand side of (24) is finite, which does not hold for instance in the self-repelling case $f=-b'$ of the form (4), where $\hat{b}(0)>0$.
(2) This result is short of proving the full CLT. Namely that $\sigma^2 := \lim_{t\to\infty} t^{-1}E(t)$ exists, it is between the bounds given in (25), and $t^{-1/2}X(t) \Rightarrow N(0,\sigma^2)$. Recall that in the follow-up paper [7] full CLT is proved for the locally self-repelling Brownian polymer in $d\ge 3$. The proof relies on Kipnis-Varadhan theory and a relaxation of Varadhan's sector condition. As explained in that paper, technical parts of that method can't be applied (so far) in lower dimensions.

Let, for all $\lambda>0$,

$$\hat{E}(\lambda) := \int_0^\infty e^{-\lambda t}E(t)dt, \tag{26}$$

and let $D$ be the *diffusivity*, as usually defined: $D(t) := t^{-1}E(t)$.

One can easily show (by a simple change of variables) that, for $\nu>0$,

$$\Big\{E(t)\sim Ct^{2\nu},\ \ t\gg 1\Big\} \Rightarrow \Big\{\hat{E}(\lambda)\sim C'\lambda^{-2\nu-1},\ \ \lambda\ll 1\Big\}. \tag{27}$$

The following Theorem 3 shows infrared bounds for the Laplace transform $\hat{E}(\lambda)$ as $\lambda \to 0$, based on the *resolvent method*, first used by Landim, Quastel, Salmhofer and Yau in [12] to provide superdiffusive estimates on the diffusivity of asymmetric simple exclusion process in one and two dimensions.

Then Lemma 1, shown in a different context in [19] but readily translated for our purposes (see also [13, 10, 19]), enables us to convert the *upper* bound on $\hat{E}(\lambda)$ into an upper bound on $E(t)$, without the need of extra regularity assumption, as is usually required in Tauberian theorems. Its proof relies on the estimate of the variance of additive functionals of Markov processes using the $H_{-1}$ norm.

More precisely, let us consider the following *infrared bounds* for the correlation function $\hat{b}(p)$: for some $-1 < \alpha < 1$

$$C_1 := \varlimsup_{p \to 0} |p|^{-\alpha} \hat{b}(p) < \infty, \qquad C_2 := \varliminf_{p \to 0} |p|^{-\alpha} \hat{b}(p) > 0. \tag{28}$$

Of course, $C_2 \le C_1$.

**Theorem 3.** *If for some $-1 < \alpha < 1$ the infrared bounds* (28) *hold, then*

$$\varlimsup_{\lambda \to 0} \lambda^{(5-\alpha)/2} \hat{E}(\lambda) \le C_3 < \infty, \tag{29}$$

*and*

$$\varliminf_{\lambda \to 0} \lambda^{(9-2\alpha+\alpha^2)/4} \hat{E}(\lambda) \ge C_4 > 0, \tag{30}$$

*where the constants $C_3$ and $C_4$ depend only on $\alpha$, $C_1$ and $C_2$.*

**Lemma 1.** *There exists an explicit finite constant $C$ such that*

$$E(t) \le C t^{-1} \hat{E}(t^{-1}). \tag{31}$$

**Remarks:** (1) By Lemma 1 the bound (29) can be converted into

$$\varlimsup_{t \to \infty} t^{-(3-\alpha)/2} E(t) \le C_3' < \infty. \tag{32}$$

(2) Although we cannot translate the lower bound on $\hat{E}(\lambda)$ into a asymptotic lower bound on $E(t)$, by (27) the bound (30) essentially means

$$\varliminf_{t \to \infty} t^{-(5-2\alpha+\alpha^2)/4} E(t) \ge C_4' > 0. \tag{33}$$

(3) The locally self avoiding case corresponds to $\alpha = 0$. In this case our results give

$$C_4'' t^{5/4} \le E(t) \le C_3'' t^{3/2}, \tag{34}$$

with some constants $C_4'' > 0$, $C_3'' < \infty$. Here the first inequality is meant in the sense of Laplace transforms. Recall that in this particular case, the conjectured order in [25] is $E(t) \asymp t^{4/3}$.
(4) We make the following conjecture:

**Conjecture 2.** *Under the conditions of Theorem 3 the true asymptotic order is*

$$E(t) = \mathbf{E}\big(X(t)^2\big) \asymp t^{\frac{4}{3+\alpha}}. \tag{35}$$

**Remark:** This conjecture is formally in agreement with the order of the limit proved in [16] under superballistic scaling, for slowly decaying (with distance) self-interaction functions $f$, and the corresponding conjectures formulated in [5, 25].

## 1.5 Formal proof of Theorem 1

In order to prove that $\pi$ is indeed time-stationary we have to show that for any (sufficiently smooth) test function $u(\cdot)$ the moment generating functional $\mathbf{E}\big(\exp\{\langle u, \eta(t)\rangle\}\big)$ is constant in time. Here we used the notation

$$\langle u, v\rangle := \int_{-\infty}^{\infty} v(x)u(x)dx. \tag{36}$$

(Note that starting from Section 2 the brackets $\langle\cdot,\cdot\rangle$ will have a different meaning, see (43)). It follows from (15) that

$$\begin{aligned} d\mathbf{E}\big(\exp\{\langle u, \eta(t)\rangle\}\big) &= \mathbf{E}\big(d\exp\{\langle u, \eta(t)\rangle\}\big) \\ &= \mathbf{E}\Big(e^{\langle u,\eta(t)\rangle}\Big(\frac{1}{2}\langle u'', \eta(t)\rangle + \frac{1}{2}\langle u', \eta(t)\rangle^2 - \langle u', \eta(t)\rangle\eta(t,0) + \langle u', b\rangle\Big)\Big)dt. \end{aligned} \tag{37}$$

Let $X, Y, Z$ be jointly Gaussian with zero mean. Then it is easy to show (by differentiations of the moment generating function of their joint distribution) that

$$\mathbf{E}\big(Ye^X\big) = \exp\{\mathbf{E}\big(X^2\big)/2\}\mathbf{E}\big(XY\big), \tag{38}$$

$$\mathbf{E}\big(YZe^X\big) = \exp\{\mathbf{E}\big(X^2\big)/2\}\big(\mathbf{E}\big(YZ\big) + \mathbf{E}\big(XY\big)\mathbf{E}\big(XZ\big)\big). \tag{39}$$

Using these identities, if $\eta$ is a zero mean Gaussian field with covariance $b$ (as it is assumed), the right hand side of (37) can be computed explicitly to deduce

$$e^{\frac{1}{2}\langle u, b*u\rangle}\Big\{\frac{1}{2}\langle u'', b*u\rangle + \frac{1}{2}\langle u', b*u'\rangle + \frac{1}{2}\langle u', b*u\rangle^2 - \langle u', b*u\rangle\langle u, b\rangle\Big\}dt. \tag{40}$$

Note that for any test function $u$ we have $\langle u', b*u\rangle = 0$. Thus, after one integration by parts we note that the previous expression is always 0 which shows that $\mathbf{E}\big(\exp\{\langle u, \eta(t)\rangle\}\big)$ is indeed constant in time.

**Remark:** It is not hard to check that translation invariant Gaussian fields with nonzero centering and the same covariances:

$$\int_\Omega \omega(x)\pi(d\omega) = v \in \mathbb{R}, \qquad \int_\Omega \omega(x)\omega(y)\pi(d\omega) - v^2 = b(x-y) \tag{41}$$

are also time-stationary (and ergodic) for the process $t \mapsto \eta(t)$. If we start our process with these initial distributions then the corresponding laws of large numbers

$$\lim_{t\to\infty} \frac{X(t)}{t} = v \quad \text{a.s.} \tag{42}$$

hold, which means *ballistic behavior* of the process $t \mapsto X(t)$. We will not pursue these regimes in the present note.

# 2 Spaces and operators

## 2.1 Spaces

The natural formalism for the proofs of our theorems is that of Fock space and Gaussian Hilbert spaces. We follow the usual notation of Euclidean quantum field theory, see e.g. [22].

Endow the space of real valued smooth functions of rapid decrease (Schwartz space) $\mathcal{S} = \mathcal{S}(\mathbb{R})$ with the inner product

$$\langle u, v\rangle := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x)b(x-y)v(y)dxdy = \int_{-\infty}^{\infty} \hat{u}(-p)\hat{v}(p)\hat{b}(p)dp < \infty, \tag{43}$$

and let $\mathcal{V}$ be the closure of $\mathcal{S}(\mathbb{R})$ with respect to this Euclidean norm.

We denote $\mathcal{H} := \mathcal{L}^2(\Omega, \pi)$. Then

$$\phi : \mathcal{S} \to \mathcal{H} : \qquad \phi(v)(\omega) := \int_{-\infty}^{\infty} \omega(x)v(x)dx. \tag{44}$$

is an isometric embedding of $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ in $\mathcal{H}$:

$$\|\phi(v)\|_{\mathcal{H}}^2 = \|v\|_{\mathcal{V}}^2, \tag{45}$$

so $\phi$ extends as an isometric embedding of $\mathcal{V}$ into the Gaussian subspace of $\mathcal{H}$.

The Hilbert space $\mathcal{H}$ is naturally graded

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n \oplus \cdots, \tag{46}$$

where

$$\mathcal{H}_0 := \{c1\!\!1,\ c \in \mathbb{R}\}, \tag{47}$$
$$\mathcal{H}_1 := \{\phi(v),\ v \in \mathcal{V}\}, \tag{48}$$
$$\mathcal{H}_n := \operatorname{span}\{ :\!\phi(v_1)\dots\phi(v_n)\!:,\ v_1,\dots,v_n \in \mathcal{V}\}. \tag{49}$$

Here and throughout the rest of the paper $:X_1 \dots X_n:$ denotes the *Wick product* of the jointly Gaussian random variables $(X_1,\dots,X_n)$. For basics of Fock space and Wick products see e.g. [22], [8].

## 2.2 Operators

We use the standard notations of Fock spaces. Given a (bounded or unbounded) closed linear operator $A$ over the basic Hilbert space $\mathcal{V}$ its second quantized version over the Hilbert space $\mathcal{H}$ will be denoted $d\Gamma(A)$. This latter one acts on Wick monomials as follows

$$d\Gamma(A)\ {:}\phi(v_1)\cdots\phi(v_j)\cdots\phi(v_n){:} = \sum_{j=1}^{n}\ {:}\phi(v_1)\cdots\phi(Av_j)\cdots\phi(v_n){:}\,, \tag{50}$$

and it is extended by linearity and graph closure.

A particularly important linear operator over $\mathcal{V}$ is the differentiation with respect to the $x$-variable:

$$\partial v(x) := v'(x). \tag{51}$$

This is an unbounded skew self-adjoint (and hence closed) operator defined on the dense domain

$$\mathrm{Dom}(\partial) = \{v \in \mathcal{V} : \int_{-\infty}^{\infty} p^2 |\hat{v}(p)|^2 \hat{b}(p) dp < \infty\}. \tag{52}$$

We denote the second quantization of $\partial$ by

$$\nabla := d\Gamma(\partial). \tag{53}$$

Then $\nabla$ is also unbounded and skew self-adjoint (and hence closed) operator over $\mathcal{H}$. We shall also need the operator $\nabla^2$ acting on $\mathcal{H}$. (Note that this is *not* the second quantization of $\partial^2$.)

Given an element $u \in \mathcal{V}$ the creation and annihilation (or: raising and lowering) operators associated to it are

$$a^*(u) : \mathcal{H}_n \to \mathcal{H}_{n+1}, \qquad a(u) : \mathcal{H}_n \to \mathcal{H}_{n-1}, \tag{54}$$

acting on Wick monomials as

$$a^*(u)\ {:}\phi(v_1)\dots\phi(v_n){:} = {:}\phi(u)\phi(v_1)\dots\phi(v_n){:}\,, \tag{55}$$

$$a(u)\ {:}\phi(v_1)\dots\phi(v_n){:} = \sum_{j=1}^{n}\langle u, v_j\rangle\ {:}\phi(v_1)\dots\phi(v_{j-1})\phi(v_{j-1})\dots\phi(v_n){:}\ . \tag{56}$$

For basics about creation, annihilation and second quantized operators see e.g. [22] or [8]. In particular note that, for $F \in \mathcal{C}$ and $u \in \mathcal{V}$ such that $b * u \in \Omega$ the following identities hold

$$\big((a^*(u) + a(u))F\big)(\omega) = \big(\phi(u)F\big)(\omega), \tag{57}$$

$$\big(a(u)F\big)(\omega) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\big(F(\omega + \varepsilon\, b * u) - F(\omega)\big). \tag{58}$$

These identities are easily checked on Wick monomials and extended by linearity. Identity (57) means that the sum of the creation and annihilation operators corresponding to an element of the basic space $\mathcal{V}$ is the multiplication operator with the isometric Gaussian embedding of that vector. The meaning of (58) is that the annihilation operator $a(u)$ is actually a "directional derivative" in the direction $b * u \in \Omega$. This latter one is a particular case of a well-known identity from Malliavin calculus, see e.g. [8]. We will also use the following straightforward commutation relation:

$$[\nabla, a(u)] = a(u'). \tag{59}$$

We define the unitary involution $J$ on $\mathcal{H}$:

$$Jf(\omega) := f(-\omega), \qquad J\restriction_{\mathcal{H}_n} = (-1)^n I \restriction_{\mathcal{H}_n} . \tag{60}$$

The subspace of smooth functions

$$\mathcal{C} := \{F(\phi(v_1), \dots, \phi(v_k)) : F \in C_0^\infty(\mathbb{R}^k \to \mathbb{R}),\ v_1, \dots, v_k \in \mathcal{S}\} \subset \mathcal{H} \tag{61}$$

is a *common core* for all (unbounded) operators defined above and used in the sequel. They act on functions of this form as follows:

$$\nabla F(\phi(v_1), \dots, \phi(v_k)) = \sum_{l=1}^{k} \partial_l F(\phi(v_1), \dots, \phi(v_k))\phi(v_l'), \tag{62}$$

$$\begin{aligned}\nabla^2 F(\phi(v_1), \dots, \phi(v_k)) = \sum_{l,m=1}^{k} \partial^2_{l,m} F(\phi(v_1), \dots, \phi(v_k))\phi(v_l')\phi(v_m') \\ + \sum_{l=1}^{k} \partial_l F(\phi(v_1), \dots, \phi(v_k))\phi(v_l''),\end{aligned} \tag{63}$$

$$a(u)F(\phi(v_1), \dots, \phi(v_k)) = \sum_{l=1}^{k} \partial_l F(\phi(v_1), \dots, \phi(v_k))\langle u, v_l\rangle, \tag{64}$$

$$a^*(u)F(\phi(v_1), \dots, \phi(v_k)) = \phi(u)F(\phi(v_1), \dots, \phi(v_k)) - a(u)F(\phi(v_1), \dots, \phi(v_k)). \tag{65}$$

Notice that $\nabla$ is the infinitesimal generator of the *unitary group of spatial translations* while $\nabla^2/2$ is the infinitesimal generator of the Markovian semigroup of *diffusion in random scenery*

$$\exp\{z\nabla\} = T_z, \qquad T_z f(\omega) := f(\tau_z\omega), \tag{66}$$

$$\exp\{t\nabla^2/2\} = Q_t, \qquad Q_t f(\omega) := \int \frac{\exp\{-z^2/(2t)\}}{\sqrt{2\pi t}} f(\tau_z\omega)dz. \tag{67}$$

### 2.3 The infinitesimal generator, stationarity, Yaglom-reversibility, ergodicity

We denote

$$P_t : \mathcal{H} \to \mathcal{H}, \qquad P_t f(\omega) := \mathbf{E}\big(f(\eta(t))\big|\eta(0) = \omega\big). \tag{68}$$

Then $[0, \infty) \ni t \mapsto P_t \in \mathcal{B}(\mathcal{H})$ is a positivity preserving contraction semigroup on $\mathcal{H}$.

Given $f = F(\phi(v_1), \ldots, \phi(v_k)) \in \mathcal{C}$, from (11), (12) and using (62)–(65), one can compute

$$\lim_{t \to 0} \frac{\mathbf{E}\big(f(\eta(t) - f(\eta(0)))\big|\eta(0) = \omega\big)}{t} = \left(\frac{1}{2}\nabla^2 + \phi(\delta)\nabla + a(\delta')\right) f(\omega). \tag{69}$$

This operator is extended from $\mathcal{C}$ by graph closure. Now, using the commutation relation (59) we obtain the *infinitesimal generator* of the semigroup $P_t$:

$$G := \frac{1}{2}\nabla^2 + a^*(\delta)\nabla + \nabla a(\delta). \tag{70}$$

The adjoint of the generator is

$$G^* := \frac{1}{2}\nabla^2 - a^*(\delta)\nabla - \nabla a(\delta). \tag{71}$$

For later use we introduce notation for the symmetric (self-adjoint) and antisymmetric (skew-self-adjoint) parts of the generator

$$S := -\frac{1}{2}(G + G^*) = -\frac{1}{2}\nabla^2 \tag{72}$$

$$A := \frac{1}{2}(G - G^*) = a^*(\delta)\nabla + \nabla a(\delta) =: A_+ + A_-. \tag{73}$$

Note that

$$S : \mathcal{H}_n \to \mathcal{H}_n, \quad A_+ : \mathcal{H}_n \to \mathcal{H}_{n+1}, \quad A_- : \mathcal{H}_n \to \mathcal{H}_{n-1}, \quad A_- = -A_+^*, \tag{74}$$

and

$$S \restriction_{\mathcal{H}_0} = 0, \qquad A_+ \restriction_{\mathcal{H}_0} = 0, \qquad A_- \restriction_{\mathcal{H}_0 \oplus \mathcal{H}_1} = 0. \tag{75}$$

*Proof of Theorem 1 and Corollary 1.* It is clear that

$$G^* 1\!\!1 = 0, \tag{76}$$

and hence it follows that $\pi$ is indeed stationary distribution of the process $t \mapsto \eta(t)$ and $G^*$ is itself the infinitesimal generator of the stochastic semigroup $P_t^*$ of the time reversed process.

Proving ergodicity is easy. For any $f \in \mathcal{H}$ the Dirichlet form of the process $t \mapsto \eta(t)$ is given by

$$\mathcal{D}(f) := -(f, Gf) = -(f, \frac{1}{2}\nabla^2 f) = \frac{1}{2} \parallel \nabla f \parallel^2. \tag{77}$$

where $(\cdot, \cdot)$ and $\| \cdot \|$ denote the scalar product and $L_2$ norm in $\mathcal{H}$. So,

$$\big\{\mathcal{D}(f) = 0\big\} \;\Leftrightarrow\; \big\{\nabla f = 0\big\} \;\Leftrightarrow\; \big\{f = \text{const.} \;\; \pi\text{-a.s.}\big\}, \tag{78}$$

since $z \mapsto \tau_z$ acts ergodically on $(\Omega, \pi)$.

Corollary 1 follows directly (20), by the ergodic theorem. $\square$

The generator $G$ is, of course, not reversible, but the so-called *Yaglom-reversibility* [26, 27, 4] holds:

$$G^* = JGJ. \tag{79}$$

This identity means that the stationary forward process $(-\infty,\infty) \ni t \mapsto \eta(t)$ and the flipped backward process

$$(-\infty,\infty) \ni t \mapsto \tilde{\eta}(t) := -\eta(-t) \tag{80}$$

obey the same law. We will call $t \mapsto \tilde{\eta}(t)$ the *flipped-backward process.*

# 3 Diffusive bounds

Throughout this section we assume (24) and we prove Theorem 2.

## 3.1 Diffusive lower bound

For $-\infty < s \le t < \infty$ denote

$$M(s,t) := X(t) - X(s) - \int_s^t \varphi(\eta(u))du = B(t) - B(s). \tag{81}$$

**Lemma 2.** *For $s \in \mathbb{R}$ fixed the process $[s,\infty) \ni t \mapsto M(s,t)$ is a forward martingale with respect to the* forward filtration $\{\mathcal{F}_{(-\infty,t]} : t \ge s\}$ *of the process $t \mapsto \eta(t)$. For $t \in \mathbb{R}$ fixed the process $(-\infty,t] \ni s \mapsto M(s,t)$ is a backward martingale with respect to the* backward filtration $\{\mathcal{F}_{[s,\infty)} : s \le t\}$ *of the process $t \mapsto \eta(t)$.*

*Proof.* There is nothing to prove about the first statement: the integral on the right hand side of (81) was chosen exactly so that it compensates the conditional expectation of the infinitesimal increments of $X(t)$.

We turn to the second statement of the lemma. We use the following facts:
(1) The displacements are reverted on the flipped-backward trajectories $t \mapsto \tilde{\eta}(t)$ defined in (80):

$$\tilde{X}(t) - \tilde{X}(s) = -X(t) + X(s). \tag{82}$$

(2) The forward process $t \mapsto \eta(t)$ and flipped-backward process $t \mapsto \tilde{\eta}(t)$ are identical in law.
(3) The function $\omega \mapsto \varphi(\omega)$ is odd with respect to the flip map $\omega \mapsto -\omega$.

Putting these facts together (in this order) we obtain

$$\begin{aligned} \lim_{h\to 0} \mathbf{E}\big(\frac{X(s-h) - X(s)}{h} \big| \mathcal{F}_{[s,\infty)}\big) &= \lim_{h\to 0} \mathbf{E}\big(\frac{-\tilde{X}(-s+h) + \tilde{X}(-s)}{h} \big| \tilde{\mathcal{F}}_{(-\infty,-s]}\big) \\ &= -\varphi(\tilde{\eta}(-s)) = \varphi(\eta(s)). \end{aligned} \tag{83}$$

$\square$

From Lemma 2 it follows that

$$\begin{aligned}\mathbf{E}\big((X(t)-X(s))^2\big) &= \mathbf{E}\big((M_{[s,t]})^2\big)+\mathbf{E}\Big(\big(\int_s^t \varphi(\eta(u))du\big)^2\Big) \\ &= t-s+\mathbf{E}\Big(\big(\int_s^t \varphi(\eta(u))du\big)^2\Big).\end{aligned} \tag{84}$$

Hence the lower bound in (25).

## 3.2 Diffusive upper bound

First we recall a general result about the limiting variance of additive functionals integrated along the trajectory of a stationary and ergodic Markov process.

Let $t \mapsto \eta(t)$ be a stationary and ergodic Markov process on the abstract probability space $(\Omega, \pi)$. Denote the infinitesimal generator acting on $\mathcal{L}^2(\Omega, \pi)$ and its adjoint by $G$, respectively, by $G^*$. These might be unbounded operators, but it is assumed that they have a common core of definition. Denote the symmetric (self-adjoint), respectively, the antisymmetric (skew-self-adjoint) part of the infinitesimal generator by

$$S := -\frac{1}{2}(G+G^*), \qquad A := \frac{1}{2}(G-G^*). \tag{85}$$

Let $t \mapsto \xi(t)$ be the *reversible* Markov process on the same state space $(\Omega, \pi)$ which has the infinitesimal generator $-S$.

The following lemma is proved in [21]. See also the survey papers [18], [11] and further references cited therein.

**Lemma 3.** *Let $\varphi \in \mathcal{L}^2(\Omega, \pi)$ with $\int \varphi d\pi = 0$. Then*

$$\overline{\lim_{t\to\infty}}\, t^{-1}\mathbf{E}\Big(\big(\int_0^t \varphi(\eta(s))ds\big)^2\Big) \le \lim_{t\to\infty} t^{-1}\mathbf{E}\Big(\big(\int_0^t \varphi(\xi(s))ds\big)^2\Big). \tag{86}$$

In our particular case

$$S = -\frac{1}{2}\nabla^2, \tag{87}$$

and the reversible process $t \mapsto \xi(t)$ will be the so-called *diffusion in random scenery* process, see e.g. [9] or the more recent survey [6]. That means:

$$\xi(t) := \tau_{Z_t}\omega, \tag{88}$$

where $t \mapsto Z_t$ is a standard Brownian motion, independent of the field $\omega$. The function $\varphi : \Omega \to \mathbb{R}$ is $\varphi(\omega) = \omega(0)$. Thus the upper bound in (86) will be

$$\lim_{t\to\infty} t^{-1}\mathbf{E}\Big(\big(\int_0^t \varphi(\xi(s))ds\big)^2\Big) = \lim_{t\to\infty} t^{-1}\mathbf{E}\Big(\big(\int_0^t \omega(Z_s)ds\big)^2\Big) = \int_{-\infty}^{\infty} p^{-2}\,\hat{b}(p)dp. \tag{89}$$

Here the last step is just explicit computation, with expectation taken over the Brownian motion $Z(t)$ *and* over the random scenery $\omega$. The straightforward details are left for the reader. $\square$

## 4 Superdiffusive bounds

From (84) it follows that

$$E(t)=t+\mathbf{E}\big(\big(\int_0^t \varphi(\eta(s))ds\big)^2\big)=t+2\int_0^t (t-s)\mathbf{E}\big(\varphi(\eta(s))\varphi(\eta(0))\big)ds. \tag{90}$$

Taking the Laplace transform of the previous equation we get

$$\hat{E}(\lambda)=\lambda^{-2}\left(1+2(\varphi,(\lambda-G)^{-1}\varphi)\right). \tag{91}$$

We will estimate $(\varphi,(\lambda-G)^{-1}\varphi)$ using the following variational formula, see e.g. [12].

$$(\varphi,(\lambda-G)^{-1}\varphi)=\sup_{\psi\in\mathcal{H}}\big\{2(\varphi,\psi)-(\psi,(\lambda+S)\psi)-(A\psi,(\lambda+S)^{-1}A\psi)\big\}. \tag{92}$$

### 4.1 Superdiffusive upper bounds

*Proof of Theorem 3 – upper bound.* The upper bound will follow from simply dropping the last term on the right hand side of (92):

$$(\varphi,(\lambda-G)^{-1}\varphi)\le\sup_{\psi\in\mathcal{H}}\big\{2(\varphi,\psi)-(\psi,(\lambda+S)\psi)\big\}=(\varphi,(\lambda+S)^{-1}\varphi). \tag{93}$$

Note that – modulo a Tauberian inversion – this is equivalent to the argument in subsection 3.2.

Using (67) and (72) we write the resolvent of $-S$ as

$$(\lambda+S)^{-1}=\int_{-\infty}^{\infty}\int_0^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\lambda t-z^2/(2t)}\,T_z dt dz=\int_{-\infty}^{\infty}g_\lambda(z)T_z dz, \tag{94}$$

where the function $g_\lambda(z)$ and its Fourier transform $\hat{g}_\lambda(p)$ are

$$g_\lambda(z)=\frac{1}{\sqrt{2\lambda}}e^{-\sqrt{2\lambda}|z|},\qquad \hat{g}_\lambda(p)=\frac{1}{\sqrt{2\pi}}\frac{1}{\lambda+p^2/2}. \tag{95}$$

Hence, by Parseval formula,

$$(\varphi,(\lambda+S)^{-1}\varphi)=\int_{-\infty}^{\infty}g_\lambda(z)\mathbf{E}\big(\omega(0)\omega(z)\big)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{b}(p)}{\lambda+p^2/2}dp.$$

By (28) we can choose $\delta>0$ so that for $|p|<\delta$

$$\frac{C_2}{2}|p|^\alpha\le\hat{b}(p)\le 2C_1|p|^\alpha. \tag{96}$$

Then

$$\begin{aligned}(\varphi,(\lambda+S)^{-1}\varphi)&\le C_1\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{|p|^\alpha}{\lambda+p^2/2}dp+\sqrt{\frac{2}{\pi}}\int_{|p|>\delta}p^{-2}\hat{b}(p)dp\\&=\lambda^{\frac{\alpha-1}{2}}C_1\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{|q|^\alpha}{1+q^2/2}dq+\sqrt{\frac{2}{\pi}}\int_{|p|>\delta}p^{-2}\hat{b}(p)dp.\end{aligned} \tag{97}$$

Since both integrals in (97) are finite (as $|\alpha|<1$), the upper bound (29) follows from (91), (93) and (97). □

## 4.2 Superdiffusive lower bounds

*Proof of Theorem 3 – lower bound.* Lower bounds are obtained by taking on the right hand side of (92) the supremum over the subspace $\mathcal{H}_1$ only:

$$(\varphi,(\lambda-G)^{-1}\varphi)\geq \sup_{\psi\in\mathcal{H}_1}\big\{2(\varphi,\psi)-(\psi,(\lambda+S)\psi)-(A\psi,(\lambda+S)^{-1}A\psi)\big\} \tag{98}$$

$$= \sup_{\psi\in\mathcal{H}_1}\big\{2(\varphi,\psi)-(\psi,(\lambda+S)\psi)-(A_+\psi,(\lambda+S)^{-1}A_+\psi)\big\}. \tag{99}$$

The last identity is due to (75).

We write $\psi\in\mathcal{H}_1$ as

$$\psi=\int_{-\infty}^{\infty}u(x)\omega(x)dx, \tag{100}$$

with $u$ even function and compute the three terms on the right hand side of (99). The first two are straightforward:

$$(\varphi,\psi)=\int_{-\infty}^{\infty}u(x)\mathbf{E}\big(\omega(0)\omega(x)\big)dx=\int_{-\infty}^{\infty}\hat{b}(p)\hat{u}(p)dp \tag{101}$$

$$\begin{aligned}(\psi,(\lambda+S)\psi)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(\lambda u(x)u(y)+\frac{1}{2}u'(x)u'(y)\big)\mathbf{E}\big(\omega(x)\omega(y)\big)dxdy\\&=\int_{-\infty}^{\infty}\big(\lambda+p^2/2\big)\hat{b}(p)\hat{u}(p)^2dp\end{aligned} \tag{102}$$

In order to compute the third term we first note that

$$A_+\psi=\int_{-\infty}^{\infty}u'(x)\ {:}\omega(0)\omega(x){:}\ dx, \tag{103}$$

and hence

$$\begin{aligned}&(A_+\psi,(\lambda+S)^{-1}A_+\psi)=\\&\qquad=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_\lambda(z)u'(x)u'(y)\mathbf{E}\big(\ {:}\omega(0)\omega(x){:}\ {:}\omega(z)\omega(z+y){:}\ \big)dxdydz\\&\qquad=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_\lambda(z)u'(x)u'(y)\big(b(z)b(z+y-x)+b(z+y)b(z-x)\big)dxdydz\\&\qquad=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\hat{b}(p)\hat{b}(q)}{\lambda+(p-q)^2/2}\big(p\hat{u}(p)-q\hat{u}(q)\big)^2dqdp\\&\qquad\leq\int_{-\infty}^{\infty}\hat{b}(p)p^2\hat{u}(p)^2K(\lambda,p)dp,\end{aligned} \tag{104}$$

where

$$K(\lambda,p):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{b}(q)}{\lambda+(p+q)^2/2}dq=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{b}(q-p)}{\lambda+q^2/2}dq. \tag{105}$$

In the last step we used the Cauchy-Schwartz inequality and the fact that $\hat{b}(\cdot)$ is nonnegative even function.

From (99), (101), (102) and (104) it follows that

$$(\varphi, (\lambda - G)^{-1}\varphi) \geq \int_{-\infty}^{\infty} \frac{\hat{b}(p)}{\lambda + p^2/2 + K(\lambda, p)p^2} dp. \tag{106}$$

Next we give an *upper* bound for $K(\lambda, p)$. Let $\delta$ be chosen so that the bounds (96) hold and assume that $\lambda < \delta^2/4$. Then

$$\begin{aligned}
1\!\!1_{\{|p|<\lambda^{1/2}\}} K(\lambda, p) &\leq C_1 \sqrt{\frac{2}{\pi}} 1\!\!1_{\{|p|<\lambda^{1/2}\}} \int_{-\infty}^{\infty} \frac{|q-p|^\alpha}{\lambda + q^2/2} dq + \sqrt{\frac{2}{\pi}} \int_{|q|>\delta/2} q^{-2} \hat{b}(q-p) dq \\
&\leq \lambda^{(\alpha-1)/2} C_1 \sqrt{\frac{2}{\pi}} \sup_{|r|<1} \int_{-\infty}^{\infty} \frac{|q-r|^\alpha}{1 + q^2/2} dq + \sqrt{\frac{2}{\pi}} \int_{|q|>\delta/2} q^{-2} \hat{b}(q-p) dq \qquad (107) \\
&\leq C \lambda^{(\alpha-1)/2}, \qquad (108)
\end{aligned}$$

with some $C < \infty$, for $\lambda$ sufficiently small. The last inequality holds since the integrals in (107) are bounded.

From (106) and (108) it follows that for sufficiently small $\lambda$

$$\begin{aligned}
(\varphi, (\lambda - G)^{-1}\varphi) &\geq \int_{|p|\leq\lambda^{1/2}} \frac{\hat{b}(p)}{\lambda + C\lambda^{(\alpha-1)/2} p^2} dp \\
&\geq \frac{C_2}{2} \int_{|p|\leq\lambda^{1/2}} \frac{|p|^\alpha}{\lambda + C\lambda^{(\alpha-1)/2} p^2} dp \\
&= \lambda^{-(1-\alpha)^2/4} \frac{C_2}{2} \int_{|r|\leq\lambda^{(\alpha-1)/4}} \frac{|r|^\alpha}{1 + Cr^2} dr \\
&\geq C\lambda^{-(1-\alpha)^2/4}, \qquad (109)
\end{aligned}$$

with some $C > 0$, for $\lambda$ sufficiently small.

The lower bound (30) follows from (91) and (109).

$\square$

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