

Gauge Theory of “Topological Phases” of Matter¹

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Credits

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Plan of Lectures

1. Introduction: Goal and Purpose of Lectures
2. The Chiral Anomaly in 2 Dimensions
3. Conductance Quantization in Ideal Quantum Wires
4. Anomalous Chiral Edge Currents in Incompressible Hall Fluids
5. Induced Chern-Simons Terms in Three-Dimensional Theories
6. Chiral Spin Currents in Planar Topological Insulators
7. The Chiral Anomaly in 4 Dimensions
8. Chiral Magnetic Effect, Axion Electrodynamics
9. 3D Topological Insulators and Weyl Semi-Metals
10. Summary, Open Problems

Abstract

I start with a description of the goals of the analysis – developing a “gauge theory of states of matter” – and a survey of the chiral anomaly, including a sketch of an application to quantum wires.

I then review some basic elements of the theory of the quantum Hall effect in 2D electron gases. In particular, I discuss the role of anomalous chiral edge currents and of anomaly inflow in 2D insulators with explicitly or spontaneously broken time reversal, i.e., in Hall- and Chern insulators. The topological Chern-Simons action yielding the correct response eqs. for the 2D bulk of such materials and the anomalous edge action are exhibited. A classification of “abelian” Hall insulators is outlined.

After some remarks on induced Chern-Simons terms, I analyze chiral photonic wave guides and chiral edge spin-currents as well as the bulk response equations in time-reversal invariant 2D topological insulators.

The “chiral magnetic effect” in 3D systems and axion-electrodynamics are reviewed next. A short digression into the theory of 3D topological insulators, including “axionic insulators”, follows. I conclude with some remarks on Weyl semi-metals, which exhibit the chiral magn. effect.

Some open problems are presented at the end.

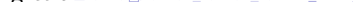
1. Introduction: Goal and Purpose of Lectures

- ▶ Our main goal is to use concepts and results from Gauge Theory, Current Algebra, and General Relativity, in order to develop a *“Gauge Theory of Phases/States of Matter”*, which complements the *Landau Theory of Phases and Phase Transitions* when there are no local order parameters available to characterize some states of interest, and which yields information on current Green functions, whence on transport coefficients (conductivities).
- ▶ Show on interesting examples how that theory can be used to classify (“topologically protected”) correlated *bulk- and surface states* of *interacting* systems of condensed matter when $\nexists$ local order parameters.
- ▶ Key tools to develop a *“Gauge Theory of Phases of Matter”* are:
 - *“Effective Actions”* = generating functionals of connected current Green functions $\leftrightarrow$ transport coeffs., in particular *conductivities*;
 - implications of gauge invariance $\leftrightarrow$ current conservation (Ward ids.), locality & power counting on form of *Effective Actions*;
 - Gauge Anomalies and their cancellations $\leftrightarrow$ edge (surface) degrees of freedom $\leftrightarrow$ “holography”; etc.

Applications to Condensed-Matter Physics

- ▶ Introduce & study these field-theoretic notions and concepts, and discuss the following applications of the “*Gauge Theory of States of Matter*”; (a list of references to some of my work will be given at the end):
 - Conductance quantization in ideal quantum wires
 - Theory of the Fractional Quantum Hall Effect
 - Theory of chiral states of light in wave guides
 - Time-reversal invariant 2D “topological” insulators and superconductors; chiral edge spin currents
 - Chiral magnetic effect²; higher-dimensional cousins of the QHE³, 3D topological insulators, Weyl semi-metals, etc.
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- ▶ Applications in other areas of physics, in particular in cosmology

²Found in a preliminary form by A. Vilenkin; see Alekseev, Cheianov, JF.

³They have also been studied by O. Zilberberg et al. 

Digression on Effective Actions

Consider a quantum-mechanical system with degrees of freedom described by fields $\bar{\psi}, \psi, \dots$ over a space-time, Λ , which is equipped with a metric $g_{\mu\nu}$ of signature $(-1, 1, 1, 1)$. Its dynamics is assumed to be derivable from an action functional $S(\bar{\psi}, \psi, \dots; g_{\mu\nu})$. We assume that there is a conserved vector current (density) J^μ , with $\nabla_\mu J^\mu = 0$. If the current J^μ is charged, i.e., is carried by electrically charged degrees of freedom, it couples to the electromagnetic field, which we describe by its vector potential A_μ . Then the action of the system is given by

$$S(\bar{\psi}, \psi, \dots; g_{\mu\nu}, A) := S(\bar{\psi}, \psi, \dots; g_{\mu\nu}) + \int_\Lambda d^4x \sqrt{-g} J^\mu(x) A_\mu(x), \quad (1.1)$$

where $g = \det(g_{\mu\nu})$. The *Effective Action* of the system on a space-time Λ with metric $g_{\mu\nu}$ and in an external electromagnetic field with vector potential A_μ is then defined by the functional integral

$$\begin{aligned} S_{\text{eff}}(g_{\mu\nu}, A_\mu) &:= -i\hbar \ln \left(\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left[\frac{i}{\hbar} S(\bar{\psi}, \psi, \dots; g_{\mu\nu}, A_\mu)\right] \right) \\ &+ \quad (\text{divergent}) \text{ const.} \end{aligned} \quad (1.2)$$

Properties of S_{eff}

A precise definition of the right side in (1.2) requires specifying initial and final field configurations, e.g., corresp. to ground-states of the system.

Next, we review some properties of S_{eff} :

1. The variational derivatives of S_{eff} with respect to A_μ are given by connected current Green functions:

$$\frac{\delta S_{\text{eff}}(g_{\mu\nu}, A_\mu)}{\delta A_\mu(x)} = \langle J^\mu(x) \rangle_{g,A}, \quad (1.3)$$

and

$$\frac{\delta^2 S_{\text{eff}}(g_{\mu\nu}, A_\mu)}{\delta A_\mu(x) \delta A_\nu(y)} = \langle J^\mu(x) J^\nu(y) \rangle_{g,A}^c, \quad (1.4)$$

where $\langle (\cdot) \rangle_{g,A} = \dots$, etc.

2. Let us consider the effect of a gauge transformation, $A_\mu \mapsto A_\mu + \partial_\mu \chi$, where χ is an arbitrary smooth function on Λ , on the effective action, S_{eff} . After an integration by parts we find that

$$\frac{\delta S_{\text{eff}}(g_{\mu\nu}, A_\mu + \partial_\mu \chi)}{\delta \chi(x)} = \nabla_\mu \langle J^\mu(x) \rangle_{g,A} = 0 \quad (1.5)$$

vanishes, because J^μ is conserved. Thus, S_{eff} is **invariant under gauge transformations**!

Properties of S_{eff} – ctd.

3. We may also vary S_{eff} with respect to the metric $g_{\mu\nu}$:

$$\frac{\delta S_{\text{eff}}(g_{\mu\nu}, A_\mu)}{\delta g_{\mu\nu}(x)} = \langle T^{\mu\nu}(x) \rangle_{g,A},$$

where $T^{\mu\nu}$ is the energy-momentum tensor of the system. Using local energy-momentum conservation, i.e., $\nabla_\mu T^{\mu\nu} = 0$, we find that $S_{\text{eff}}(g_{\mu\nu}, A_\mu)$ is **invariant under coordinate transformations** on Λ .

A general (possibly curved) metric $g_{\mu\nu}$ can be used to describe defects – dislocations and disclinations – in a condensed-matter system. – Invariance of S_{eff} under Weyl rescalings of the metric (i.e., under local variations of the density) would imply that $\langle T^\mu_\mu(x) \rangle_{g,A} \equiv 0 \leftrightarrow$ **scale-invariance** (criticality) of the system.

4. If a system exhibits an energy gap above its ground-state, i.e., if it is an “insulator”, then the zero-temperature connected current Green functions have good decay properties in space and time. In the **scaling limit**, i.e., in the limit of very large distances and very low frequencies, its effective action then approaches a functional that is a space-time integral of **local, gauge-invariant polynomials** in A_μ and derivatives of A_μ .

Form of effective actions in the scaling limit

These terms can be organized according to their scaling dimensions, (power counting).

Properties 1. through 4. enable us to determine the general form of **effective actions**, S_{eff} , (of insulators) in the scaling limit.

Example: We consider an insulator with broken parity and time-reversal confined to a flat 2D region. Then $S_{\text{eff}}(A_\mu)$ tends to

$$\frac{\sigma_H}{2} \int_\Lambda A \wedge dA + \frac{1}{2} \int_\Lambda d^3x \sqrt{-g} [\underline{E}(x) \cdot \varepsilon \underline{E}(x) - \mu^{-1} B(x)^2] + \dots,$$

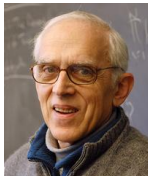
as the scaling limit is approached, where σ_H is the Hall conductivity, ε is the tensor of dielectric constants, and μ is the magnetic susceptibility. –

Note: Chern-Simons term **not** gauge-invariant if $\partial\Lambda \neq \emptyset \rightarrow$ *holography*!

We also use generalizations of these concepts for **non-abelian** gauge fields and currents that are only covariantly conserved. Such gauge fields may represent “real” external fields; but also “virtual” ones merely serving to develop the response theory needed to determine transport coefficients.

These matters are discussed in detail in my 1994 Les Houches lectures.

2. The Chiral Anomaly



Consider a system of relativistic, massless, charged fermions in a space-time of dimension $2n, n = 1, 2, \dots$. We consider the vector current, J^μ , and the axial current, J_5^μ , of this system. The vector current turns out to be conserved:

$$\partial_\mu J^\mu = 0 \leftrightarrow \text{gauge invariance of theory}$$

But the axial current is anomalous: In 2D,

$$\partial_\mu J_5^\mu = \frac{\alpha}{2\pi} E, \quad \alpha := \frac{e^2}{\hbar}, \quad [J_5^0(\vec{y}, t), J^0(\vec{x}, t)] = i \frac{\alpha}{2\pi} \delta'(\vec{x} - \vec{y}), \quad (2.1)$$

where α is the finestructure constant and E is the electric field.

Chiral Anomaly – ctd.

In 4D:

$$\partial_\mu J_5^\mu = \frac{\alpha}{\pi} \vec{E} \cdot \vec{B} \quad (\propto \text{instanton density})$$

and

$$[J_5^0(\vec{y}, t), J^0(\vec{x}, t)] = i \frac{\alpha}{4\pi} \vec{B}(\vec{y}, t) \cdot \nabla_{\vec{y}} \delta(\vec{x} - \vec{y}),$$

where $\vec{E}$ is the electric field and $\vec{B}$ the magnetic induction. For massive fermions, there are terms $\propto$ fermion masses contributing to $\partial_\mu J_5^\mu$.

—

We now derive the formulae in Eq. (2.1), (setting $\hbar = 1$). We consider a system on 2D Minkowski space, Λ . Let i be the 1-form dual to the vector current density J^μ . Then

$$\partial_\mu J^\mu = 0 \Leftrightarrow di = 0.$$

By Poincaré's lemma,

$$i = \frac{Q}{2\pi} d\varphi, \quad \varphi \text{ a scalar field, } Q = \text{“charge”}.$$

Chiral anomaly in 2D – potential of conserved current

Thus

$$J^\mu = \frac{Q}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \varphi \quad (2.2)$$

In 2D, (given an arbitrary metric to raise and lower indices),

$$J_5^\mu = \varepsilon^{\mu\nu} J_\nu \stackrel{(2)}{=} \frac{Q}{2\pi} \partial^\mu \varphi, \quad (2.3)$$

(see Schwinger, Seiler, and others). Suppose that $E = 0$, mass $m = 0$.

Then

$$\partial_\mu J_5^\mu = 0 \stackrel{(2.3)}{\Leftrightarrow} \square \varphi = 0, \quad (2.4)$$

i.e., φ is a massless free field. $\rightarrow$ Lagrangian QFT with action given by

$$S(\varphi) = \frac{1}{4\pi} \int_\Lambda d^2x \sqrt{-g} \partial_\mu \varphi(x) \partial^\mu \varphi(x). \quad (2.5)$$

Momentum, ϖ , canonically conjugate to φ , ($g_{\mu\nu}$ flat, for simplicity):

$$\varpi(x) = \frac{\delta S(\varphi)}{\delta(\partial_0 \varphi(x))} = \frac{1}{2\pi} \frac{\partial \varphi(x)}{\partial t} = -Q^{-1} J^1(x).$$

Bosonization of Fermi fields

By (2.3),

$$J_5^0 = Q\varpi, \quad J_5^1 = \frac{Q}{2\pi} \frac{\partial \varphi}{\partial \underline{x}}.$$

Equal-time canonical commutation relations on Fock space,

$$[\varpi(t, \underline{x}), \varphi(t, \underline{y})] = -i\delta(\underline{x} - \underline{y}),$$

imply an “anomalous current commutator”:

$$[J^0(t, \underline{x}), J_5^0(t, \underline{y})] = i \frac{Q^2}{2\pi} \delta'(\underline{x} - \underline{y}). \quad (2.6)$$

Chiral currents: $J_{\ell/r}^\mu := J^\mu \pm J_5^\mu.$

Chiral Fermi fields: Define

$$\begin{aligned} \psi_{\ell/r}^{(q)}(x) &= : \exp\left\{\pm 2\pi i \frac{q}{Q} \int_{\underline{x}}^{\infty} i_{\ell/r}(x^0, \underline{y})\right\} : \\ &= : \exp 2\pi i q \left[\pm \frac{\varphi(x)}{2\pi} + \int_{\underline{x}}^{\infty} \varpi(x^0, \underline{y}) d\underline{y}\right] : \end{aligned} \quad (2.7)$$

El. charge: $Q \cdot q$; statistics: $e^{\pm i\pi q^2}$ (Weyl rel.) $\rightarrow$ Fermi field if $q = 1$!

Coupling to an external abelian gauge field

Electric field $E(x)$ can be derived from vector potential $A_\mu(x)$:

$$E(x) = \varepsilon^{\mu\nu}(\partial_\mu A_\nu)(x).$$

Now, replace $S(\varphi)$ in (2.5) by

$$\begin{aligned} S(\varphi; A) &:= \frac{1}{4\pi} \int_{\Lambda} \partial_\mu \varphi \partial^\mu \varphi d^2x + \int_{\Lambda} J^\mu A_\mu d^2x \\ &= \frac{1}{4\pi} \int_{\Lambda} \{ \partial_\mu \varphi \partial^\mu \varphi + 2Q \varepsilon^{\mu\nu} \partial_\nu \varphi A_\mu \} d^2x \\ &= \frac{1}{4\pi} \int_{\Lambda} \{ \partial_\mu \varphi \partial^\mu \varphi + 2Q \varphi E \} d^2x. \end{aligned}$$

Can be derived from theory of Dirac fermions coupled to vector potential, (by convergent perturbation theory in $\int_{\Lambda} J^\mu A_\mu d^2x$). $\Rightarrow$ Field equation for φ becomes $\square\varphi(x) = QE(x)$. Hence

$$\boxed{\partial_\mu J_5^\mu = \frac{Q^2}{2\pi} E(x)} \quad (2.8)$$

which is the chiral anomaly in 2D!

3. Conductance quantization in quantum wires

– with A. Alekseev and V. Cheianov, 1998 –

Conserved chiral charges: The currents

$$\hat{J}_{\ell/r}^{\mu} := J_{\ell/r}^{\mu} \mp \frac{Q}{2\pi} \varepsilon^{\mu\nu} A_{\nu}$$

are conserved, $(\partial_{\mu} \hat{J}_{\ell/r}^{\mu} = 0)$, but *not* gauge-invariant. However, the chiral charges

$$N_{\ell/r} := \int \hat{J}_{\ell/r}^0(t, \underline{x}) d\underline{x} \quad (3.1)$$

are *not only conserved*, but also *gauge-invariant*!

Consider a very long wire containing a 1D **interacting electron gas** ($Q = -e$) connected to electron reservoirs on the left end and the right end; assume that there are **no back-scattering processes** converting left-moving electrons into right-moving ones (or conversely), and that $E = 0$. This system has a conserved vector current, $J^{\mu} = \frac{e}{2\pi} \varepsilon^{\mu\nu} \partial_{\nu} \varphi$, a conserved axial current, J_5^{μ} , and two conserved charges, N_{ℓ} and N_r . Let H denote the Hamiltonian of the electron gas.

Equilibrium state and equilibrium current

The equilibrium state of the electron gas at inverse temperature β is given by the density matrix

$$P_{\mu_\ell, \mu_r} := \Xi_{\beta, \mu_\ell, \mu_r}^{-1} \exp(-\beta H_{\mu_\ell, \mu_r}), \quad (3.2)$$

where $\Xi_{\beta, \mu_\ell, \mu_r}$ = partition function, μ_ℓ and μ_r denote the chemical potentials of reservoirs on the right end of the wire (injecting left-moving electrons into the wire) and on the left end of the wire, respectively, and

$$H_{\mu_\ell, \mu_r} := H - \mu_\ell N_\ell - \mu_r N_r.$$

Expectations with respect to P_{μ_ℓ, μ_r} are denoted by $\langle (\cdot) \rangle_{\mu_\ell, \mu_r}$. We then find the following formula for the current, I , through the wire:

$$\begin{aligned} I := \langle J^1(x) \rangle_{\mu_\ell, \mu_r} &= -\frac{e}{2\pi} \left\langle \frac{\partial \varphi(x)}{\partial t} \right\rangle_{\mu_\ell, \mu_r} \\ &= i \frac{e}{2\pi} \langle [H, \varphi(x)] \rangle_{\mu_\ell, \mu_r} \quad (\text{Heisenberg Eq. of motion}) \\ &= \frac{ie}{2\pi} \langle [H_{\mu_\ell, \mu_r}, \varphi(x)] + [\mu_\ell N_\ell + \mu_r N_r, \varphi(x)] \rangle_{\mu_\ell, \mu_r} \end{aligned}$$

Quantized conductance

The expectation $\langle [H_{\mu_\ell, \mu_r}, \varphi(x)] \rangle_{\mu_\ell, \mu_r}$ vanishes, as follows from (3.2)! Using Eq. (3.1) and the anomalous commutator (2.6), we find that the remaining terms in the expression for the current I add up to

$$\begin{aligned} I &= -\frac{ie^2}{2\pi}(\mu_\ell - \mu_r) \int \langle [\varpi(t, \underline{y}), \varphi(t, \underline{x})] \rangle_{\mu_\ell, \mu_r} d\underline{y} \\ &= -\frac{e^2}{2\pi}(\mu_\ell - \mu_r), \text{ by CCR.} \end{aligned} \quad (3.3)$$

Notice that $-(\mu_\ell - \mu_r) =: \Delta V$ is the voltage drop through the wire. Re-installing Planck's constant $\hbar$, we find that

$$I = \frac{e^2}{2\pi\hbar} \Delta V.$$

Since electrons have spin $\frac{1}{2}$, there are actually two species of charged particles ("spin-up" and "spin-down") per filled band in the wire. Thus,

$$I = 2n \frac{e^2}{h} \Delta V, \quad \text{for a wire with } n \text{ filled bands.}$$

(Generalizations for wires with impurities (Bachas-F): ↗ K. Gawedzki) 

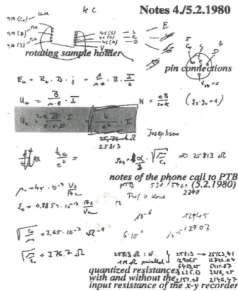
4. Anomalous chiral edge currents in incompressible Hall fluids

In this section we outline the *general theory of the QHE*.

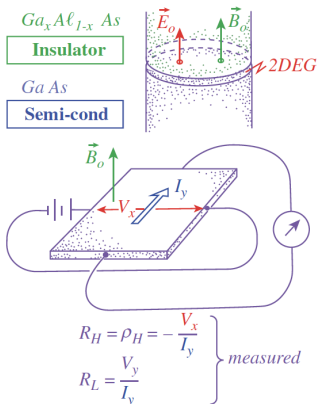


From von Klitzing's lab journal (→ 1985 Nobel Prize in Physics):

K. von Klitzing



Setup and experimental data



Experimental behavior of the Hall conductivity

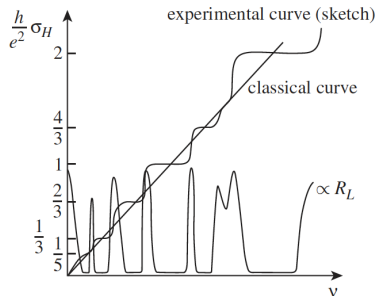


Fig. 1. Schematic representation of the experimental setup and of experimental results on the QHE.

Observations: $R_L = 0 \Leftrightarrow (v, \sigma_H) \in \text{plateau}$; plateau heights $\in \frac{e^2}{h} \mathbb{Q}$; the cleaner the sample, the more numerous are the observed plateaux and the narrower they are; if $\frac{h}{e^2} \sigma_H \notin \mathbb{Z}$ there appear to exist fractional electric charges.

Electrodynamics of 2D incompressible electron gases

Basic quantities: 2D electron gas confined to sample $\Omega \subset xy$ -plane, in magnetic field $\vec{B}_0 \perp \Omega$. Filling factor ν chosen such that $R_L = 0$.⁴ Study the response of 2D EG to small perturbing e.m. field, $\vec{E} \parallel \Omega$, $\vec{B} \perp \Omega$, with

$$\vec{B}^{tot} := \vec{B}_0 + \vec{B}, \quad B := |\vec{B}|, \quad \underline{E} := (E_1, E_2).$$

We now review the electrodynamics of 2D “incompressible” ($R_L = 0$) electron gases. Field tensor:

$$F = \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & -B \\ -E_2 & B & 0 \end{pmatrix} = dA, \quad (A: \text{e.m. vector potential})$$

Current:

$$j^\mu(x) := \langle J^\mu(x) \rangle_A, \quad \mu = 0, 1, 2,$$

(reference to metric $g_{\mu\nu}$, chosen to be flat, omitted.)

Here are the basic equations:

⁴To show that, for *interacting* 2D EG, $\exists \nu$'s such that $R_L = 0$ is a very hard problem of quantum many-body theory! ...

Electrodynamics ... - ctd.

(1) *Hall's Law* – phenomenological

$$\underline{j}^k(x) = \sigma_H \varepsilon^{k\ell} \underline{E}_\ell(x), \text{ assuming } R_L = 0 \rightarrow \text{broken } P, T! \quad (4.1)$$

(2) *Charge conservation* – fundamental

$$\frac{\partial}{\partial t} \rho(x) + \underline{\nabla} \cdot \underline{j}(x) = 0. \quad (4.2)$$

(3) *Faraday's induction law* – fundamental

$$\frac{\partial}{\partial t} B_3^{\text{tot}}(x) + \underline{\nabla} \wedge \underline{E}(x) = 0. \quad (4.3)$$

Combining (1) through (3), we get

$$\frac{\partial}{\partial t} \rho \stackrel{(2)}{=} -\underline{\nabla} \cdot \underline{j} \stackrel{(1)}{=} -\sigma_H \underline{\nabla} \wedge \underline{E} \stackrel{(3)}{=} \sigma_H \frac{\partial}{\partial t} B. \quad (4.4)$$

Integrate (4.4) in t , with integration constants chosen as follows:

$$j^0(x) := \rho(x) + e \cdot n, \quad B(x) = B_3^{\text{tot}}(x) - B_0 \quad \Rightarrow$$

Electrodynamics ... -ctd.

(4) Chern-Simons Gauss law

$$j^0(x) = \sigma_H B(x). \quad (4.5)$$

Eqs. (4.1) and (4.5) yield

$$\boxed{j^\mu(x) = \sigma_H \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x)} \quad (4.6)$$

which is a generally covariant relation between current density and field tensor. $\rightarrow$ Puzzle:

$$0 \stackrel{(2)}{=} \partial_\mu j^\mu \stackrel{(3),(6)}{=} \varepsilon^{\mu\nu\lambda} (\partial_\mu \sigma_H) F_{\nu\lambda} \neq 0, \quad (4.7)$$

wherever $\sigma_H \neq \text{const.}$, e.g., at $\partial\Omega$.

Solution of Puzzle:

j^μ is **bulk current density** $\neq$ conserved **total** electric current density!

$$j_{\text{tot}}^\mu = j_{\text{bulk}}^\mu + j_{\text{edge}}^\mu, \quad \partial_\mu j_{\text{tot}}^\mu = 0, \quad \text{but} \quad \partial_\mu j_{\text{bulk}}^\mu \stackrel{(4.7)}{\neq} 0. \quad (4.8)$$

Note:

$$\text{supp } j_{\text{edge}}^\mu = \text{supp}\{\nabla\sigma_H\} \supseteq \partial\Omega, \quad j_{\text{edge}} \perp \nabla\sigma_H.$$

Anomalous edge current

Combining (4.7) (with $j^\mu = j_{bulk}^\mu$) with (4.8), we find that

$$\partial_\mu j_{edge}^\mu \stackrel{(4.8)}{=} -\partial_\mu j_{bulk}^\mu|_{\text{supp}\{\nabla\sigma_H\}} \stackrel{(4.6)}{=} -\sigma_H E_\parallel|_{\text{supp}\{\nabla\sigma_H\}} \quad (4.9)$$

Chiral anomaly in 1 + 1 dimensions !

Eq. (4.9) is an example of “holography”. Apparently, j_{edge}^μ is an anomalous chiral current in 1 + 1 dimensions.

Here is a classical-physics argument determining the chirality of j_{edge}^μ : At the edge of the sample the Lorentz force acting on electrons must be cancelled by the force confining them to the interior of the sample. Thus

$$\frac{e}{c} B^{tot} v_\parallel^k = \varepsilon^{k\ell} \frac{\partial V_{edge}}{\partial x^\ell},$$

where V_{edge} is the potential of the force confining electrons to the interior of the sample $\rightarrow$ equation for chiral motion, (“skipping orbits”).

Analogous phenomenon in classical physics: **Hurricanes**:

$\vec{B} \rightarrow \vec{\omega}_{earth}$, Lorentz force $\rightarrow$ Coriolis force, $V_{edge} \rightarrow$ air pressure.

An expression for the Hall conductivity σ_H

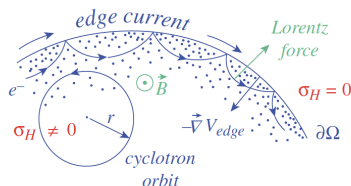


Fig. 2. Skipping orbits of electrons moving along the boundary/edge of a 2D electron gas confined to a disk — electrons near $\partial\Omega$ perform a chiral motion.

From the theory of the chiral anomaly in $1 + 1$ dimensions we infer that

$$\partial_\mu j_{edge}^\mu = -\frac{e^2}{h} \left(\sum_{\text{edge modes } \alpha} Q_\alpha^2 \right) E_{\parallel} \stackrel{\text{with (4.9)}}{\Rightarrow} \sigma_H = \frac{e^2}{h} \sum_{\text{species } \alpha} Q_\alpha^2 \quad (4.10)$$

where eQ_α is the “charge” (see (2.2), (2.5)) of the edge current, J_α^μ , corresponding to species α of clockwise-chiral edge modes; (similar contributions from counter-clockwise chiral modes, but with **reversed** sign!) $\rightarrow$ **Halperin's chiral edge currents**. – Apparently, if $\sigma_H \notin \frac{e^2}{h} \mathbb{Z}$ then $\exists$ **fractionally charged currents** propagating along the edge!

Bulk effective action of a 2D Hall insulator

Consider a 2D electron gas in a neutralising ionic background subject to a constant transversal magnetic field $\vec{B}_0$. Electrons are confined to a region Ω in the xy -plane. The space-time of the system is given by $\Lambda = \mathbb{R} \times \Omega$. We suppose that electrons are coupled to an external em vector potential $A = A_0 dt + A_1 dx_1 + A_2 dx_2$ describing a small perturbing em field ($\underline{E}||\Omega, B$).

We assume that the 2D EG is an insulator, i.e., that the **longitudinal conductance vanishes**. It is then easy to determine the form of the effective action, $S_{\text{eff}}(A)$, of this system as a functional of the external vector potential A in the limiting regime of very large distances and very low frequencies (**scaling limit**), as explained in the Introduction:

$$\begin{aligned} S_{\text{eff}}(A) &= \frac{\sigma_H}{2} \int_{\Lambda} A \wedge [dA + K] + \text{boundary term} \\ &+ \frac{1}{2} \int_{\Lambda} d^3x \{ \underline{E}(x) \cdot \varepsilon \underline{E}(x) - \mu^{-1} B^2(x) \} + \dots \end{aligned} \quad (4.11)$$

where the coefficient, σ_H , of the **topological Chern-Simons action** turns out to be the **Hall conductivity**, K is the Gauss curvature 2-form of the sample, (and ε = tensor of dielectric consts., μ = magn. permeability).

Bulk effective action – ctd.

The presence of the Chern-Simons term on the right side of (4.11) can also be inferred from Eq. (4.6): Omitting curvature terms ($\leftrightarrow$ “shift”),

$$\begin{aligned} j_{bulk}^\mu &= \langle J^\mu(x) \rangle_A \equiv \frac{\delta S_\Lambda(A)}{\delta A_\mu(x)} \\ &\stackrel{(4.6)}{=} \sigma_H \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x), \quad x \notin \partial\Lambda. \end{aligned}$$

$$\Rightarrow S_\Lambda(A) = \frac{\sigma_H}{2} \int_\Lambda A \wedge dA + \text{boundary term}$$

That there *must* be a boundary term is a consequence of the fact that the Chern-Simons bulk term is **not** gauge-invariant on a space-time Λ with non-empty boundary $\partial\Lambda$: Under a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \chi$, the Chern-Simons action changes by a boundary term

$$\frac{\sigma_H}{2} \int_{\partial\Lambda} [\chi dA] \quad (4.12)$$

This anomaly must be cancelled by the anomaly of a boundary term!

Edge effective action

Returning to Eq. (4.10), we guess that the boundary term must be the generating functional of the connected Green functions of the anomalous chiral edge currents J_α^μ , $\alpha = 1, 2, \dots$, introduced there, where α labels the different species of charged chiral edge modes. The charge of J_α^μ has been denoted by eQ_α .

Let v_α denote the propagation speed of the chiral modes that give rise to the edge current J_α^μ . This propagation speed plays the role of the “speed of light” in 2D current algebra. We introduce “light-cone coordinates”, u^+, u^- , on $\partial\Lambda$. Let $a := A_\parallel$ denote the em vector potential restricted to the 1 + 1-dimensional boundary $\partial\Lambda$ of space-time. Then $a = a_+ du^+ + a_- du^-$. The eff. action of the chiral edge current J_α^μ is then given by

$$\frac{(eQ_\alpha)^2}{h} \Gamma_{\partial\Lambda}^{(\alpha)}(a), \text{ with } \Gamma_{\partial\Lambda}^{(\pm)}(a) := \frac{1}{2} \int_{\partial\Lambda} [a_+ a_- - 2a_\pm \frac{\partial_\pm^2}{\square} a_\pm] du^+ du^-, \quad (4.13)$$

where, in the last term on the right side of (4.13), the subscript “+” is chosen if the modes that give rise to the current J_α^μ propagate clockwise, and “−” is chosen if they propagate counter-clockwise; (dependence on α though chirality of mode α and propagation speed v_α !)

Anomaly inflow and anomaly cancellation

I propose as an exercise to the audience to verify that the anomaly (4.12), namely the term $\frac{\sigma_H}{2} \int_{\partial\Lambda} [\chi dA]|_{\partial\Lambda}$, is cancelled by the anomaly of the edge effective action,

$$\sigma_H \Gamma_{\text{edge}}(a) := \sum_{\text{species } \alpha} \frac{(eQ_\alpha)^2}{h} \Gamma_{\partial\Lambda}^{(\alpha)}(a),$$

under a gauge transformation $a \rightarrow a + d\chi|_{\partial\Lambda}$ if and only if

$$\sigma_H = \frac{e^2}{h} \sum_{\alpha} Q_\alpha^2.$$

Note that, for simplicity, it is assumed here and in the following that all edge modes have the *same* chirality; otherwise, we would have to insert appropriate signs into these formulae. –

Whatever has been said here about Hall insulators also applies to so-called **Chern insulators**, which break reflection- and time-reversal invariance *even* in the absence of a magnetic field; e.g., because of magnetic impurities in the bulk of the material; ($\nearrow$ Haldane model).

Classification of “abelian” Hall fluids & Chern insulators

Here I sketch a general classification of 2D insulators with broken P and T exhibiting quasi-particles with *abelian braid statistics*.⁵ Let $\mathcal{J}$ denote the total electric current density (bulk + edge), which is conserved:

$\partial_\mu \mathcal{J}^\mu = 0$. – In the following we use units such that $\frac{e^2}{\hbar} = 1$.

Ansatz:

$$\mathcal{J} = \sum_{\alpha=1}^N Q_\alpha \mathcal{J}_\alpha, \quad (4.14)$$

where the currents $\mathcal{J}_\alpha$ are assumed to be canonically normalized and conserved, w. charges $Q_\alpha \in \mathbb{R}$. On a 3D space-time $\Lambda = \Omega \times \mathbb{R}$, a conserved current $\mathcal{J}$ can be derived from a vector potential, B : If i denotes the 2-form dual to $\mathcal{J}$ then $\partial_\mu \mathcal{J}^\mu = 0 \Rightarrow di = 0$, hence

$$i = \frac{1}{\sqrt{2\pi}} dB,$$

where the vector potential B is a 1-form. It is determined by i up to the gradient of a scalar function, β : B and $B + d\beta$ yield the *same* i .

⁵States exhibiting quasi-particles with non-abelian braid statistics are discussed in my work with Pedrini, Schweigert and Walcher.

Chern-Simons action of conserved currents in an insulator

For a 2D insulator with broken time reversal (T), the effective field theory of the currents $(\mathcal{J}_\alpha)_{\alpha=1}^N$ must be *topological* in the scaling limit (large distances, low frequencies). If reflection in lines and T are broken the “most relevant” term in the action of the potentials, $\underline{B} := (B_\alpha)_{\alpha=1}^N$, of the currents $\mathcal{J}_\alpha$ is the Chern-Simons term

$$S_\Lambda(\underline{B}, A) := \sum_{\alpha=1}^N \int_\Lambda \left\{ \frac{1}{2} B_\alpha \wedge dB_\alpha + A \wedge \frac{Q_\alpha}{\sqrt{2\pi}} dB_\alpha \right\} + \text{bd. terms} + \dots, \quad (4.15)$$

where A is the em vector potential, and the boundary terms must be added to cancel the anomalies of the Chern-Simons term under the “gauge trsfs.” $B_\alpha \rightarrow B_\alpha + d\beta_\alpha$, $A \rightarrow A + d\chi$. – Carrying out the oscillatory Gaussian integrals over the potentials B_α , we find

$$\int \exp(iS_\Lambda(\underline{B}, A)) \prod_{\alpha=1}^N \mathcal{D}B_\alpha = \exp \left(i\sigma_H \left[\frac{1}{2} \int_\Lambda A \wedge dA + \Gamma_{\text{edge}}(A_\parallel) \right] \right), \quad (4.16)$$

where $\sigma_H = \frac{1}{2\pi} \sum_{\alpha=1}^N Q_\alpha^2$, (see (4.13)!)

Classification of 2D “abelian” Hall insulators – bulk degrees of freedom

Physical states of the Chern-Simons theory with action as in (4.15) can be constructed from Wilson networks – lines can be *flux tubes* – contained in the half space $\Lambda_- := \Omega \times \mathbb{R}_-$ whose lines/tubes end in Ω . Given a network, W , let $|W\rangle$ denote the physical state corresponding to W ; (the map $W \rightarrow |W\rangle$ is “many to one”!). Let $\Theta(W)$ denote the network contained in $\Lambda_+ := \Omega \times \mathbb{R}_+$ arising from W by reflection in Ω , followed by complex conjugation. If W_1 and W_2 are two such networks with the property that their intersections with Ω , more precisely their *fluxes* through Ω , coincide (see blackboard) we may consider the *gauge-invariant* network, $W_1 \circ \Theta(W_2)$, arising by multiplying W_1 with $\Theta(W_2)$; (graphically: concatenation at coinciding points/regions in Ω). Then the scalar product of the state $|W_1\rangle$ with the state $|W_2\rangle$ is given by

$$\langle W_2 || W_1 \rangle := \int (W_1 \circ \Theta(W_2))(\underline{B}) \exp(iS_\Lambda(\underline{B}, A)) \prod_{\alpha=1}^N \mathcal{D}B_\alpha. \quad (4.17)$$

Fact (easy to verify): In the scaling limit, the Hamiltonian of a Hall insulator corresp. to (4.15) vanishes. Thus, excitations are “*static*”!

Classification of 2D “abelian” Hall insulators – charges of physical states

The operator, $Q_{\mathcal{O}}$, measuring the electric charge stored in states inside a region $\mathcal{O}$ of the sample space Ω is given in terms of Wilson loop “ops.”:

$$\exp(i\varepsilon Q_{\mathcal{O}}) := \exp\left(i\varepsilon \int_{\mathcal{O}} \mathcal{J}^0 d^2x\right) = \exp\left(i \sum_{\alpha=1}^N \varepsilon \frac{Q_{\alpha}}{\sqrt{2\pi}} \int_{\partial\mathcal{O}} B_{\alpha}\right), \varepsilon \in \mathbb{R}.$$

Because the ground-state energy of a Hall insulator is separated from the rest of the energy spectrum by a **positive (mobility) gap**, **electric charge** is a good quantum number to label its physical states (at zero temperature). In other words, the charge operators

$$Q_{\mathcal{O}}, \text{ and } Q := \lim_{\mathcal{O} \nearrow \Omega} Q_{\mathcal{O}}$$

are well defined on physical states (at zero temperature).⁶

The electric charges contained in a region $\mathcal{O} \subset \Omega$, denoted $q_{\mathcal{O},1}, q_{\mathcal{O},2}$, of two states $|W_1\rangle, |W_2\rangle$ with the property that $W_1 \circ \Theta(W_2)$ is gauge-invariant are *identical*: $q_{\mathcal{O},1} = q_{\mathcal{O},2} \equiv q_{\mathcal{O}}$.

⁶The same conclusion is reached by noticing that all Wilson loop expectations have perimeter decay and then invoking “t’Hooft duality”. 

Classification of 2D “abelian” Hall insulators – connection between charge and statistics

The charge $q_{\mathcal{O}}$ contained in $\mathcal{O}$ is given by

$$\begin{aligned} \exp(i\varepsilon q_{\mathcal{O}}) \langle W_2 \| W_1 \rangle &= \\ &= \int (W_1 \circ \Theta(W_2))(\underline{B}) \exp(i\varepsilon Q_{\mathcal{O}}) \exp(iS_{\Lambda}(\underline{B}, A)) \prod_{\alpha=1}^N \mathcal{D}B_{\alpha}, \quad (4.18) \end{aligned}$$

If a Wilson network W creates a physical state $|W\rangle$ describing n electrons or holes located inside a region $\mathcal{O} \subset \Omega$ from the ground-state of a Hall insulator then the charge $q_{\mathcal{O}} \equiv q_{\mathcal{O}}(W)$ contained in $\mathcal{O}$ is given by $q_{\mathcal{O}}(W) = -n + 2k$, where k is the number of holes in $\mathcal{O}$. If the charge, $-n + 2k$, deposited in $\mathcal{O}$ by an excitation W creating $n - k$ electrons and k holes in $\mathcal{O}$ is odd, i.e., if n is **odd**, then the excitation created by W inside $\mathcal{O}$ must have **Fermi-Dirac statistics**, if n is **even** it must have **Bose-Einstein statistics**.

Classification of 2D “abelian” Hall insulators – statistics and braiding

More precisely: Let W and W' be two Wilson networks creating excitations with the *same* number of electrons and holes, but located at *disjoint* points inside a region $\mathcal{O} \subset \Omega$, with $q_{\mathcal{O}}(W) = q_{\mathcal{O}}(W') = n \bmod 2$. Let $\widetilde{W}$ be an *arbitrary* Wilson network with the property that $(W \cdot W') \circ \Theta(\widetilde{W})$ and $\mathcal{B}_{\mathcal{O}}(W \cdot W') \circ \Theta(\widetilde{W})$ are *gauge-invariant*, where $\mathcal{B}_{\mathcal{O}}(W \cdot W')$ arises from $W \cdot W'$ by braiding all lines of the two networks with endpoints inside $\mathcal{O}$, and *only* those, in such a way that the endpoints of all lines of W ending inside $\mathcal{O}$ are exchanged with the endpoints of all lines of W' ending in $\mathcal{O}$, but *without any lines crossing each other*; (see blackboard). Then

$$\langle \widetilde{W} | \mathcal{B}_{\mathcal{O}}(W \cdot W') \rangle = \exp(i\pi n^2) \langle \widetilde{W} | (W \cdot W') \rangle.$$

This is the standard *connection between electric charge and statistics* in systems of electrons.

Classification of 2D “abelian” Hall insulators in terms of odd-integral lattices

Consider a Wilson network W with just a single line, γ_p , starting at some point in Λ_- and ending at a point, p , in a region $\mathcal{O}$, and let $q := (q^\alpha)_{\alpha=1}^N$ denote the quantum numbers (fluxes) dual to the potentials B_α attached to this line. This line corresponds to the “operator”

$$\exp\left(i \sum_{\alpha=1}^N \sqrt{2\pi} q^\alpha \int_{\gamma_p} B_\alpha\right).$$

It follows from Eq. (4.18) that

$$q_{\mathcal{O}}(W) = \sum_{\alpha=1}^N Q_\alpha q^\alpha = Q \cdot q. \quad (4.19)$$

It is almost obvious that the *quantum numbers*, $q = (q^\alpha)_{\alpha=1}^N$, *corresp. to multi-electron/hole excitations, form a module, Γ , over $\mathbb{Z}$ of rank N* , i.e., a *lattice of rank N* . The “vector” $Q = (Q_1, \dots, Q_N)$ is an integer-valued $\mathbb{Z}$ -linear functional on Γ , i.e., an element of the dual lattice, Γ^* .

Classification of 2D “abelian” Hall insulators – Hall lattices

The lattice Γ is equipped with an odd-integral quadratic form,

$$\langle q^{(1)}, q^{(2)} \rangle := \sum_{\alpha} q^{(1)\alpha} \cdot q^{(2)\alpha}, \quad q^{(1)}, q^{(2)} \in \Gamma.$$

This is seen as follows: Braiding two lines with quantum numbers $q^{(1)} = q^{(2)} = q \in \Gamma$ yields a phase factor $\exp(i\pi \langle q, q \rangle)$, which must be $= 1$ if $Q \cdot q$ is even, and $= -1$ if $Q \cdot q$ is odd. $\square$

If q is the vector of quantum numbers corresponding to a single electron/hole then

$$Q \cdot q = \mp 1, \text{ and } \exp(i\pi \langle q, q \rangle) = -1. \quad (4.20)$$

Thus Q is a “visible” vector of Γ^* . Since $Q \in \Gamma^*$ and Γ is an (odd-) integral lattice, it follows that

$$\boxed{\frac{h}{e^2} \sigma_H = Q \cdot Q \equiv \sum_{\alpha=1}^N Q_{\alpha}^2 \in \mathbb{Q}} \quad (4.21)$$

→ Must classify $(\Gamma, Q \in \Gamma^*)$, using *invariants* of these data! (See F-Studer-Thiran, 1992-1994; Les Houches 1994 – separate lecture).

Classification – edge degrees of freedom

Chiral anomaly (4.13) $\Rightarrow$ several (N) species of gapless quasi-particles propagating along edge $\leftrightarrow$ described by N chiral scalar Bose fields $\{\varphi^\alpha\}_{\alpha=1}^N$ with propagation speeds $\{v_\alpha\}_{\alpha=1}^N$, such that

1. Chiral electric edge current operator & Hall conductivity

$$J_{edge}^\mu = e \sum_{\alpha=1}^N \frac{Q_\alpha}{\sqrt{2\pi}} \partial^\mu \varphi^\alpha, \quad Q = (Q_1, \dots, Q_N), \quad \sigma_H = \frac{e^2}{h} Q \cdot Q^T$$

2. Multi-electron/hole states loc. along edge created by vertex ops.

$$: \exp i \left(\sum_{\alpha=1}^N \sqrt{2\pi} q_\alpha \varphi^\alpha \right) :, \quad q = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} \in \Gamma, \quad j = 1, \dots, N. \quad (4.22)$$

Charge $\leftrightarrow$ Statistics $\Rightarrow \Gamma$ an **odd-integral lattice** of rank N . Hence:

3. Classifying data are

$$\{ \Gamma; \text{“visible” } Q \in \Gamma^*; v = (v_\alpha)_{\alpha=1}^N; \text{“CKM matrix” } \}$$

$\Gamma^* \ni q^* \leftrightarrow$ quasi-particles w. **abelian braid statistics!**

Success of classification

A large class of Hall insulators is classified by the data derived above:

Γ = odd-integral lattice, Q = visible vector in Γ^* , ... $\Rightarrow \frac{h}{e^2} \sigma_H \in \mathbb{Q}$; etc.

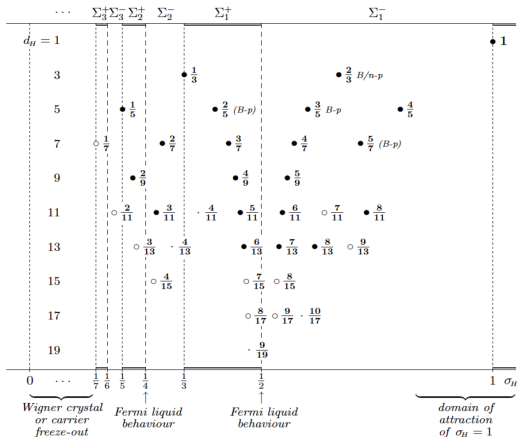


Fig. 3. Observed Hall fractions $\sigma_H = n_H/d_H$ in the interval $0 < \sigma_H \leq 1$ and their experimental status in single-layer 2D electron gases exhibiting the quantum Hall effect.

Classification of “non-abelian” Hall insulators: See F-P-S-W!

5. Induced Chern-Simons Terms in Three-Dimensional Theories

We consider a relativistic quantum field theory of an odd number of 2-component Dirac fermions, (ψ_α) , with masses M_α , $\alpha = 1, 2, \dots, 2n+1$, propagating on a three-dimensional space-time, $\Lambda (= \Omega \times \mathbb{R})$, and minimally coupled to an electromagnetic vector potential A . This theory breaks time reversal, T , and reflection in lines, P . Integrating over the degrees of freedom of these Dirac fermions, we find that the effective action of the vector potential A is given by

$$\begin{aligned} S_\Lambda(A) &= \sum_{\alpha=1}^{2n+1} \ell n[\det_{ren}((\partial_\mu + A_\mu)\gamma^\mu + M_\alpha)] \\ &= \sum_{\alpha=1}^{2n+1} Tr \ell n(\mathbf{1} + G_{M_\alpha} A_\mu \gamma^\mu), \end{aligned} \quad (5.1)$$

where G_M is the propagator of a free 2-component Dirac fermion with mass $M \neq 0$ propagating in Λ . One may then expand the logarithm on the right side of (5.1) in powers of A .

The effective action of the electromagnetic field

For large M , the leading term in $\text{Tr} \ln(\mathbf{1} + G_M A_\mu \gamma^\mu)$ is the one quadratic in A , which can be calculated without difficulty.⁷ It is given by

$$\text{sgn}(M) \frac{e^2}{8\pi\hbar} \int_\Lambda A \wedge dA + \text{boundary term}, \quad (5.2)$$

i.e., by a Chern-Simons term corresponding to a Hall conductivity

$\sigma_H = \frac{1}{2} \cdot \frac{e^2}{h}$. Terms of higher order in A tend to 0, as $M \rightarrow \infty$.

I will not reproduce the calculations leading to (5.2); but see Redlich's papers.

If the electromagnetic field is treated as dynamical one must add the Maxwell term to the induced Chern-Simons term (5.2), in order to get the full effective action, which is given by

$$\begin{aligned} S_\Lambda(A) &= \int_\Lambda [\varepsilon \underline{E}^2 - \mu^{-1} B^2] d^3x + \\ &+ \text{sgn}(M) \left\{ \frac{e^2}{8\pi\hbar} \int_\Lambda A \wedge dA + \Gamma_{\partial\Lambda}(A_\parallel) \right\}. \end{aligned} \quad (5.3)$$

⁷(Unpublished work on QED_3 , by J. Magnen, the late R. Sénéor and myself in 1976). Explicit expressions were published by Deser, Jackiw and Templeton, and by Redlich.

Massive photons and Dirac quasi-particles

$S_\Lambda(A)$ in (5.3) is quadratic in A . It therefore suffices to calculate 2-point functions. If we choose $\Lambda = \mathbb{R}^3$ then the imaginary-time (euclidian) 2-pt. functions of the components, $F_{\mu\nu}$, of the electromagnetic field tensor are analytic in momentum space ($\propto (k^2 + \text{cst.}e^4)^{-1}$). This is an easy exercise left to the reader. Thus, photons turn out to have a *strictly positive mass* $\propto e^2$.

If space-time Λ has a boundary then the effective action of the electromagnetic field has a boundary term given by the anomalous chiral action $\Gamma_{\partial\Lambda}(A_\parallel)$ cancelling the anomaly of the Chern-Simons term in (5.3) (+ irrelevant terms), as discussed in (4.13) and (4.16).

It is argued that, in certain planar systems of condensed matter, there exist quasi-particles with low-energy properties mimicking those of 2-component Dirac fermions. An example is “doped” graphene; (see, e.g., lectures by G. Semenoff). Other examples will be discussed in later sections. The low-energy properties of such systems can be described by QED_3 , as introduced above.

Dualities in planar systems

Dualities

In planar systems (three space-time dimensions), the em vector potential A and the vector potential, B , of the conserved el. current, $\mathcal{J} = \nabla \wedge B$, are *dual* to one another. Under the replacements

$$A \mapsto B, \quad B \mapsto A,$$

conventional time-reversal inv. 2D insulators are mapped to *2D superconductors*, and *electronic Hall- or Chern insulators* to *gapped photonic wave guides* exhibiting extended chiral electromagnetic surface waves; and conversely. This is seen using functional Fourier transformation; (see F-S-T, Les Houches 1994).

Here we consider the duality between Hall- or Chern insulators and *gapped photonic wave guides*. We define

$$\tilde{S}_\Lambda(B) := \frac{1}{2\sigma_H} \int_\Lambda B \wedge dB + \text{bd. term} + \text{less relevant terms}, \quad (5.4)$$

where $\sigma_H := \frac{e^2}{4\pi\hbar}$. Then we have the duality

Chern-Simons $QED_3 \leftrightarrow$ Quantum Theory of Currents in Hall insulators

Gapped photonic wave guides

This is elucidated by Functional Fourier Transformation:

$$e^{i S_{\Lambda}(A)} = \mathcal{N}^{-1} \int e^{i \tilde{S}_{\Lambda}(B)} e^{i \int_{\Lambda} A \wedge dB} \mathcal{D}B, \quad (5.5)$$

where $\mathcal{N}$ is a normalization factor, and conversely. We may view the current driven through a wave guide with broken time-reversal invariance as a “*classical control variable*”, while the electromagnetic field is treated as *dynamical* and is quantized. Then we have the response equations:

$$\begin{aligned} \langle F_{\mu\nu}(x) \rangle_B &= \varepsilon_{\mu\nu\lambda} \frac{\delta \tilde{S}_{\Lambda}(B)}{\delta B_{\lambda}(x)} \\ &= \sigma_H^{-1} \varepsilon_{\mu\nu\lambda} j^{\lambda}(x). \end{aligned} \quad (5.6)$$

The boundary term on the right side of Eq. (5.4) is – as we already know from (4.13), ... – given by $\frac{1}{\sigma_H} \Gamma_{\partial\Lambda}^{(\pm)}(B|_{\partial\Lambda})$, with

$$\Gamma_{\partial\Lambda}^{(\pm)}(b) := \frac{1}{2\sigma_H} \int_{\partial\Lambda} [b_+ b_- - 2b_{\pm} \frac{\partial_{\pm}^2}{\square} b_{\pm}] du^+ du^-,$$

Concluding remarks

in light-cone coordinates (u^+, u^-) on $\partial\Lambda$, with $B|_{\partial\Lambda} = b_+ du^+ + b_- du^-$. The sign of σ_H and the choice of $\pm$ depends on the chirality of the edge waves. This is the generating functional of Green functions of the em field of gapless quantized edge waves propagating chirally around the boundary of the wave guide.

There would be various further topics to be discussed, such as the *theory of rotating Bose gases* (which started with my work with Studer and Thiran – see, e.g., Les Houches 1994. Further work was carried out by N. Cooper et al., N. Rougerie, J. Yngvason et al., ...), or the role of *gravitational anomalies* (see, e.g., the work of S. Klevtsov and P. Wiegmann) related to heat transport; etc.

Five-dimensional QED – a close cousin of (5.1) through (5.3) – will have a brief appearance in Sect. 8.

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6. Chiral Spin Currents in Planar Topological Insulators

So far, we have ignored electron spin, in spite of the fact that there are 2D EG exhibiting the fractional quantum Hall effect where electron spin plays an important role. (We won't study these systems; but see refs..)

Here we consider *time-reversal invariant 2D topological insulators* (2D TI) exhibiting chiral spin currents. – We start from the

Pauli equation for a spinning electron:

$$i\hbar D_0 \Psi_t = -\frac{\hbar^2}{2m} g^{-1/2} D_k g^{1/2} g^{kl} D_l \Psi_t, \quad (6.1)$$

where m is the (effective) mass of an electron, (g_{kl}) = metric of sample space(-time), an orthonormal frame bundle is introduced on space-time enabling one to define spinors, ($\uparrow$ and $\downarrow$):

$$\psi_t(x) = \begin{pmatrix} \psi_t^\uparrow(x) \\ \psi_t^\downarrow(x) \end{pmatrix} \in L^2(\Omega, d \text{ vol.}) \otimes \mathbb{C}^2 : \text{ 2-component Pauli spinor}$$

Furthermore,

$$i\hbar D_0 = i\hbar \partial_t + e\varphi - \underbrace{\vec{W}_0 \cdot \vec{\sigma}}_{\text{Zeeman coupling}}, \quad \vec{W}_0 = \mu c^2 \vec{B} + \dots \quad (6.2)$$

$U(1)_{em} \times SU(2)_{spin}$ - gauge invariance

$$\frac{\hbar}{i} D_k = \frac{\hbar}{i} \nabla_k + eA_k - \vec{W}_k \cdot \vec{\sigma} + \dots, \quad (6.3)$$

where φ is the electrostatic potential, $\vec{\nabla}$ is the covariant gradient, $\vec{A}$ is the vector potential, and the dots stand for terms arising in a moving frame (ignored in the following), and

$$\vec{W}_k \cdot \vec{\sigma} := \underbrace{[(-\tilde{\mu}\vec{E} + \dots) \wedge \vec{\sigma}]_k}_{\text{spin-orbit interactions}}, \quad (6.4)$$

and $\tilde{\mu} = \mu + \frac{e\hbar}{4mc^2}$, (the last term due to Thomas precession).

We observe that the Pauli equation (6.1) displays perfect

$U(1)_{em} \times SU(2)_{spin}$ - gauge invariance.

We now consider an interacting gas of electrons confined to a region Ω of a 2D plane, with $\vec{B} \perp \Omega$ and $\vec{E} \parallel \Omega$. Then the $SU(2)$ - connection, $\vec{W}_\mu$, is given by

$$W_\mu^3 \cdot \sigma_3, \quad \text{with } W_\mu^K \equiv 0, \text{ for } K = 1, 2. \quad (6.5)$$

Effective action of a 2D T-invariant topological insulator

From (6.5) we conclude that parallel transport of Pauli spinors splits into parallel transport for spin $\uparrow$ and for spin $\downarrow$. The component $\psi^\uparrow$ of a Pauli spinor Ψ couples to the *abelian* connection $a + w$, while $\psi^\downarrow$ couples to $a - w$, where

$$a_\mu = -eA_\mu, \text{ and } w_\mu = W_\mu^3, \quad (\text{see (6.2) -- (6.4)}).$$

Under *time reversal*, T ,

$$a_0 \rightarrow a_0, \ a_k \rightarrow -a_k, \text{ but } w_0 \rightarrow -w_0, \ w_k \rightarrow w_k. \quad (6.6)$$

The dominant term in the effective action of a *2D T-inv. topological insulator*, with $\vec{W}$ as in (6.5), is a *Chern-Simons term*. If either $w \equiv 0$ or $a \equiv 0$ a Chern-Simons term in a or in w *alone* would *not* be T -invariant. If $w \equiv 0$ the dominant term would thus be given by

$$S_\Lambda(A) = \int_\Lambda dt \, d^2x \{ \varepsilon \underline{E}^2 - \mu^{-1} B^2 \}, \quad (6.7)$$

which is the effective action of a conventional insulator.

The Chern-Simons effective action

In the presence of both a and w a combination of **two** Chern-Simons terms **is** T -invariant:

$$\begin{aligned} S_{\Lambda}(a, w) &= \frac{\sigma}{2} \int_{\Lambda} \{ (a + w) \wedge d(a + w) - (a - w) \wedge d(a - w) \} \\ &= \sigma \int_{\Lambda} \{ a \wedge dw + w \wedge da \}, \end{aligned} \quad (6.8)$$

up to **boundary terms**. (Note that, for $\vec{W}$ as in (6.2), (6.4), (6.5), one recovers (6.7)!)⁸ The gauge fields a and w transform **independently** under gauge transformations (preserving (6.5)), and the action (6.8) is **anomalous** under these gauge transformations on a 2D sample Ω with non-empty boundary. We have learned that the anomalous boundary action,

$$\sigma [\Gamma_{\partial\Omega \times \mathbb{R}}^+((a + w)_{||}) - \Gamma_{\partial\Omega \times \mathbb{R}}^-((a - w)_{||})], \quad (6.9)$$

cancels the anomalies of the bulk action. This boundary action is the generating functional of connected Green functions of **two counter-propagating chiral edge currents**.

⁸The effective action (6.8) first appeared in a paper w. Studer in 1993.

Chiral edge spin currents

One of the two counter-propagating edge currents has spin $\uparrow$ (in $+3$ -direction $\perp \Omega$), the other one has spin $\downarrow$. Thus, a net *chiral spin current*, s_{edge}^3 , can be excited to propagate along the edge.

The bulk response equations (analogous to the Hall-Chern-Simons law (4.6)) are given by

$$j^k(x) = 2\sigma\varepsilon^{k\ell}\partial_\ell B(x), \quad s_3^\mu(x) = \frac{\delta S_\Lambda(a, w)}{\delta w_\mu(x)} = 2\sigma\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}(x) \quad (6.10)$$

The second equation could again be used to deduce that there *must* exist edge spin-currents.

We should ask what kinds of quasi-particles in the bulk of such materials could produce the bulk Chern-Simons terms in (6.8). Given our findings in Sect. 5, it is tempting to argue that a *2D time-reversal invariant topological insulator* with a bulk effective action as given in (6.8) must exhibit *two species of charged quasi-particles* in the bulk, with one species (spin $\uparrow$) related to the other one (spin $\downarrow$) by time-reversal, and each species has two degenerate states per wave vector mimicking a *two-component Dirac fermion* at small energies $\Rightarrow$ quantization of σ !

Experimental situation

Materials of this kind have been produced and studied in the lab of L. Molenkamp in Würzburg.



The experimental data are not very clean, the likely reason being that, due to small magnetic impurities and/or electric fields in the direction $\perp \Omega$, the condition (6.5) is violated, i.e., the $SU(2)$ -gauge field $\vec{W}_\mu$ does **not** only have a non-vanishing 3-component and is genuinely **non-abelian**. In this situation, the spin current is **not** conserved, anymore, (but continues to be covariantly conserved), and T is broken.

The approach to 2D time-reversal invariant topological insulators sketched here can be generalized as follows: Consider a state of matter exhibiting a bulk-spectrum of two species of quasi-particles related to one another by time-reversal.

Generalizations

In order to analyze the transport properties of the state, one should study the **response** of the state when one species is coupled to a (real or virtual, abelian or *non-abelian*) external gauge field⁹ W^+ and the other one to a gauge field W^- related to each other by time-reversal, T , according to

$$(W_0^+)^T = W_0^-, \quad (W_k^+)^T = -W_k^-$$

Assuming again that the leading term in the effective action for the gauge fields W^+ and W^- is given by the sum of two identical Chern-Simons terms, but with **opposite** signs, time-reversal invariance is manifest, and one concludes that there are **two counter-propagating chiral edge currents** generating current (Kac-Moody) algebras (at level 1, for non-interacting electrons) based on a Lie group given by the gauge group corresponding to the gauge fields $W^\pm$. (For non-interacting electrons, this group can be determined from band theory!)

If one gives up the requirement of time-reversal invariance one arrives at a theory of *chiral states of matter*. In particular, if $\vec{W}$ is an $SU(2)$ -gauge field coupling to the spin of electrons (see (6.2) and (6.4)) one finds a framework to describe *chiral spin liquids*; (see Les Houches 1994).

⁹often dubbed “Berry connection”

7. The Chiral Anomaly in Four Dimensions

In Sect. 2 it was claimed that chiral currents carried by particles that are coupled to non-vanishing external gauge fields are *not* conserved. For concreteness, we consider particles of electric charge eQ , $Q \in \mathbb{R}$, coupled to an electromagnetic vector potential $A = \sum_{\mu=0}^3 A_{\mu} dx^{\mu}$ propagating in four-dimensional space-time. Let $\mathcal{J}_{\ell/r}$ denote the left-handed/right-handed chiral current. Then the chiral anomaly says that

$$\partial_{\mu} \mathcal{J}_{\ell/r}^{\mu}(x) = \pm \frac{Q^2}{16\pi^2} \varepsilon^{\mu\nu\rho\lambda} F_{\mu\nu}(x) F_{\rho\lambda}(x), \quad (7.1)$$

where we use units such that $\frac{e^2}{\hbar} = 1$. I will not derive Eq. (7.1); but see Adler, Bell & Jackiw, Fujikawa; and others. Eq. (7.1) permits us to introduce modified chiral currents,

$$\widehat{\mathcal{J}}_{\ell/r}^{\mu} := \mathcal{J}_{\ell/r}^{\mu} \mp \frac{Q^2}{16\pi^2} \varepsilon^{\mu\nu\rho\lambda} A_{\nu} F_{\rho\lambda} \quad (7.2)$$

The second term on the right side is related to the Chern-Simons 3-form, $A \wedge dA$, that we are already familiar with and whose exterior derivative is the dual of the right side of (7.1).

Hamiltonian anomaly

By (7.2), the currents $\hat{\mathcal{J}}_{\ell/r}^\mu$ are (locally well-defined and) conserved, but **not** gauge-invariant. But they give rise to gauge-inv. conserved charges.

I now derive the Hamiltonian anomaly in the form

$$[\mathcal{J}_{\ell/r}^0(\vec{y}, t), \mathcal{J}^0(\vec{x}, t)] = \pm i \frac{Q^2}{8\pi^2} \vec{B}(\vec{y}, t) \cdot \vec{\nabla}_{\vec{y}} \delta(\vec{x} - \vec{y}). \quad (7.3)$$

Let $\mathcal{A}$ denote the affine space of (smooth) em vector potentials, A , corresponding to **time-independent** em fields, $\vec{E}, \vec{B}$. Given a fixed $A \in \mathcal{A}$, let

$\mathcal{F}_A$ = Fock space for a free, massless chiral (e.g., left-handed)
Dirac-Weyl field coupled to A .

The spaces $\mathcal{F}_A, A \in \mathcal{A}$ are all isomorphic to the standard Fock space, $\mathcal{F}$, of a free, massless Dirac-Weyl fermion. Let $\mathcal{H}$ denote the Hilbert bundle with base space $\mathcal{A}$ and fibres $\mathcal{F}_A, A \in \mathcal{A}$, equipped with a flat connection. We can then identify all the fibres $\mathcal{F}_A$ with the standard Fock space $\mathcal{F}$. The bundle $\mathcal{H}$ must carry a **projective representation**, U , of the infinite-dimensional, abelian group, $\mathcal{G}$, of **time-independent** gauge-transfs., g^χ ,

$$g^\chi(x) := e^{i\chi(x)}, \chi \text{ independent of time } t.$$

Projective representation of gauge group on $\mathcal{H}$

Properties of U :

- (i) $U(g^\chi) : \mathcal{F}_A \rightarrow \mathcal{F}_{A+d\chi}$.
- (ii) $U(g^\chi) \psi(x; A) U(g^\chi)^{-1}|_{\mathcal{F}_{A+d\chi}} = e^{i\chi(x)} \psi(x; A + d\chi)|_{\mathcal{F}_{A+d\chi}}$,
and similarly for $\bar{\psi}$. Here $\psi(x; A)$ is the Dirac-Weyl field on $\mathcal{F}_A$.

It follows that

$$U(g^\chi) = \exp G(\chi),$$

where $G(\chi) := \int d^3x \chi(\vec{x}) G(\vec{x})$, and

$$G(\vec{x}) = -i\vec{\nabla} \cdot \frac{\delta}{\delta \vec{A}(\vec{x})} + Q^{-1} \mathcal{J}_\ell^0(\vec{x}; A) \quad (7.4)$$

Locally, the phase factor of the projective representation can be made trivial by replacing $G(\vec{x})$ by

$$\hat{G}(\vec{x}) := -i\vec{\nabla} \cdot \frac{\delta}{\delta \vec{A}(\vec{x})} + Q^{-1} \hat{\mathcal{J}}_\ell^0(\vec{x}; A). \quad (7.6)$$

Anomalous commutators

Then, since the gauge group $\mathcal{G}$ is **abelian**, it follows that

$$[\widehat{G}(\vec{x}), \widehat{G}(\vec{y})] \stackrel{!}{=} 0, \quad (\text{at all times}) \quad (7.7)$$

Since the operator-valued distribution $\mathcal{J}_\ell^0(x; A)$ is *gauge-invariant* (while $\widehat{\mathcal{J}}_\ell^0(x; A)$ is *not*), it follows that

$$[\vec{\nabla} \cdot \frac{\delta}{\delta \vec{A}(\vec{x})}, \mathcal{J}_\ell^0(\vec{y}; A)] = 0.$$

Using this equation, along with (7.6) and (7.2), in (7.7), one finds, after straightforward calculations left to the audience, that

$$[\mathcal{J}_\ell^0(t, \vec{x}), \mathcal{J}_\ell^0(t, \vec{y})] = i \frac{Q^2}{4\pi^2} (\vec{B}(\vec{x}, t) \cdot \vec{\nabla}_{\vec{x}}) \delta(\vec{x} - \vec{y}). \quad (7.8)$$

This implies (7.3). (Further details can be found in the literature.)

5. Chiral Magnetic Effect, Axion Electrodynamics

Let us consider a theory of charged, massless Dirac-Weyl fermions in four space-time dimensions in the presence of a *time-indep.* external electromagnetic field with vector potential A . This theory has a conserved vector current, $\mathcal{J}^\mu$:

$$\partial_\mu \mathcal{J}^\mu = 0.$$

The continuity eq. implies that there exists a vector field, $\vec{\varphi}(x)$, such that

$$\mathcal{J}^0(x) = \frac{Qe}{2\pi} \vec{\nabla} \cdot \vec{\varphi}(x), \quad \vec{\mathcal{J}}(x) = -\frac{Qe}{2\pi} \frac{\partial}{\partial t} \vec{\varphi}(x), \quad (8.1)$$

with Q the electric charge (in units where $\frac{e^2}{\hbar} = 1$). If H denotes the Hamiltonian of the system then (formally)

$$\frac{\partial}{\partial t} \vec{\varphi}(x) = \frac{i}{\hbar} [H, \vec{\varphi}(x)]. \quad (8.2)$$

We define chiral charges

$$N_{\ell/r} := \int d^3x \hat{\mathcal{J}}_{\ell/r}^0(t, \vec{x}), \quad \text{with } \hat{\mathcal{J}}_{\ell/r}^\mu \text{ as in Eq. (7.2)}. \quad (8.3)$$

Thermal equilibrium

Since the fermions are assumed to be massless, these charges are **conserved** and **gauge-invariant**. Let μ_ℓ and μ_r denote *chemical potentials* conjugate to the charges N_ℓ and N_r , respectively; and $\underline{\mu} := (\mu_\ell, \mu_r)$.

We let $\langle(\cdot)\rangle_{\beta, \underline{\mu}}$ denote an equilibrium state of the system at inverse temperature β and chemical potentials $\underline{\mu}$. Our aim is to calculate $\vec{j}(x) = \langle \vec{\mathcal{J}}(x) \rangle_{\beta, \underline{\mu}}$, using arguments reminiscent of those in Sect. 3. By (8.2),

$$\vec{j}(x) = \frac{iQe}{h} \langle [H, \vec{\varphi}(x)] \rangle_{\beta, \underline{\mu}} \quad (8.4)$$

Formally, the right side of this eq. vanishes, because the equilibrium state is time-translation invariant. However, the field $\vec{\varphi}$ turns out to have ill-defined zero-modes, so that we cannot use the identity $[H, \vec{\varphi}(x)] = H\vec{\varphi}(x) - \vec{\varphi}(x)H$. We must regularize the right side of (8.4) by introducing a small mass and then use that

$$\frac{\partial}{\partial t} \vec{\varphi}(x) = \frac{i}{h} [H - \mu_\ell N_\ell - \mu_r N_r, \vec{\varphi}(x)] + \frac{i}{h} [\mu_\ell N_\ell + \mu_r N_r, \vec{\varphi}(x)] \quad (8.5)$$

and that $\langle [H - \mu_\ell N_\ell - \mu_r N_r, \vec{\varphi}(x)] \rangle_{\beta, \underline{\mu}} = 0$.

The chiral magnetic effect

Combining this with (8.4) and (8.5), we find the *“current sum rule”*:

$$\vec{j}(x) = \frac{iQe}{h} \langle [\mu_\ell N_\ell + \mu_r N_r, \vec{\varphi}(x)] \rangle_{\beta, \underline{\mu}}. \quad (8.6)$$

Recalling formula (7.3) for the anomalous current commutators,

$$[\mathcal{J}_{\ell/r}^0(\vec{y}, t), \mathcal{J}^0(\vec{x}, t)] = \pm i \frac{(Qe)^2}{4\pi} \vec{B}(\vec{y}, t) \cdot \vec{\nabla}_{\vec{y}} \delta(\vec{x} - \vec{y}),$$

and (8.1), we conclude that

$$[\hat{\mathcal{J}}_{\ell/r}^0(\vec{y}, t), \vec{\varphi}(\vec{x}, t)] = \mp i \frac{Qe}{4\pi} \vec{B}(\vec{y}, t) \delta(\vec{x} - \vec{y}) + \vec{\nabla} \wedge \vec{\Pi}_{\ell/r}(\vec{x} - \vec{y}, t) \quad (8.7)$$

Using (8.3) and (8.6), we find¹⁰

$$\vec{j}(x) = -\frac{(Qe)^2}{4\pi h} (\mu_\ell - \mu_r) \vec{B}(x). \quad (8.8)$$

Chiral Magnetic Effect

¹⁰See also: A. Vilenkin, Phys. Rev. D **22**, 3080 (1980); A. Alekseev et al., Phys. Rev. Letters **81**, 3503 (1998)

5D QHE

Note that, as in *Symanzik's* proof of the [Goldstone theorem](#), one can show that, at $T = 0$, if $\vec{j}(x) \neq 0$ then there must exist **massless modes**!

In our derivation (see (8.5), (8.6)), it has been important to assume that the external electromagnetic field is *time-independent*. This is usually not the case, and in applications to cosmology and condensed-matter physics, it is **unrealistic** to assume that $\mu_5 := \mu_\ell - \mu_r$ is (space-)time-independent! It turns out that a *dynamical cousin* of μ_5 has been known in particle physics under the name of *"axion"*. The most natural way of introducing axions is to study an analogue of the *quantum Hall effect in 5D*: Imagine that space-time is a five-dimensional slab, $\Lambda = \Omega \times [0, L]$, with two four-dim. boundary branes, $\partial_\pm \Lambda \simeq \Omega$. The bulk is assumed to be filled, e.g., with massive four-component Dirac fermions coupled to the 5D em vector potential, $\hat{A}$. Integrating out the Dirac fermions ($\nearrow$ Sect. 5!), we find the effective action for $\hat{A}$:

$$\begin{aligned} S_\Lambda(\hat{A}) &= \frac{1}{4LQ^2} \int_\Lambda d^5x \, \hat{F}_{MN}(x) \hat{F}^{MN}(x) + CS_\Lambda(\hat{A}) \\ &+ \Gamma_\ell(\hat{A}|_{\partial_+ \Lambda}) + \Gamma_r(\hat{A}|_{\partial_- \Lambda}) + \dots, \end{aligned} \tag{8.9}$$

Dimensional reduction and axions

where L is the width of the 5D slab,

$$CS_{\Lambda}(\hat{A}) := \frac{1}{24\pi^2} \int_{\Lambda} \hat{A} \wedge \hat{F} \wedge \hat{F} \quad (8.10)$$

is the *5D Chern-Simons action*, and $\Gamma_{\ell/r}$ is the anomalous action of left-handed/right-handed Dirac-Weyl fermions located on the boundary branes, $\partial_{\pm}\Lambda$, (canceling the anomaly of $CS_{\Lambda}(\hat{A})$!). The action (8.9), with (8.10), describes the *electrodynamics of the 5D QHE*.

Dimensional reduction to 4D, assuming that the components $\hat{F}_{MN}$ are independent of x^4 , $\forall M, N$: We define

$$\varphi(x) := \int_{\gamma_x} \hat{A},$$

where γ_x is a path connecting $\partial_-\Lambda$ to $\partial_+\Lambda$ at constant values of $x = (x^0, x^1, x^2, x^3)$. Then, for $\Lambda = \Omega \times [0, L]$, the action (8.9) becomes

$$\begin{aligned} S_{\Omega}(A; \varphi) &= \frac{1}{2Q^2} \int_{\Omega} d^4x [F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{L^2} \partial_{\mu}\varphi(x) \partial^{\mu}\varphi(x)] \\ &+ \frac{1}{8\pi^2} \int_{\Omega} \varphi (F \wedge F) + \Gamma_{\Omega}(A) + \dots \end{aligned} \quad (8.11)$$

Axion electrodynamics

Here $\Gamma_\Omega(A) = \Gamma_\ell(A) + \Gamma_r(A)$ is **not anomalous** and is ignored in the following. Expression (8.11) shows that the pseudo-scalar field φ can be interpreted as an **axion**. One can add a self-interaction term $U(\varphi)$ to the Lagrangian density in (8.11), requiring that $U(\varphi)$ be periodic in φ . From (8.11) we derive the equations of motion for $F_{\mu\nu}$ and φ :

$$\partial_\mu F^{\mu\nu} = \frac{Q^2}{8\pi^2} \partial_\mu (\varphi \tilde{F}^{\mu\nu}), \quad L^{-2} \square \varphi = \frac{Q^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{\delta U(\varphi)}{\delta \varphi},$$

where $\tilde{F}^{\mu\nu}$ is the dual field tensor, and the homogeneous Maxwell eqs. read $\partial_\mu \tilde{F}^{\mu\nu} = 0$. In terms of the electric and magnetic fields, these equations become:

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0, \quad \vec{\nabla} \wedge \vec{E} + \dot{\vec{B}} = 0, \\ \vec{\nabla} \cdot \vec{E} &= \frac{Q^2}{8\pi^2} (\vec{\nabla} \varphi) \cdot \vec{B}, \\ \vec{\nabla} \wedge \vec{B} &= \dot{\vec{E}} - \frac{Q^2}{8\pi^2} \{ \dot{\varphi} \vec{B} + \vec{\nabla} \varphi \wedge \vec{E} \}. \end{aligned} \quad (8.12)$$

A generalized chiral magnetic effect

The equation of motion for φ is as shown above. If φ only depends on time then $\vec{\nabla}\varphi = 0$, and, comparing the right side of (8.12) with Eq. (8.8) and re-installing $\frac{e^2}{\hbar}$, we find that¹¹

$$\dot{\varphi} = \mu_\ell - \mu_r \equiv \mu_5 \quad (8.13)$$

In condensed-matter theory, the equation of motion for $\mu_5 \equiv \dot{\varphi}$ may take the form of a *diffusion equation*, including a term, $\tau^{-1}\mu_5$, describing *dissipation* of the asymmetry between left- and right-handedness:

$$\dot{\mu}_5 + \tau^{-1}\mu_5 - D \triangle \mu_5 = L^2 \frac{e^2}{2\pi\hbar} \vec{E} \cdot \vec{B}, \quad (8.14)$$

($Q = 1$) where τ is a relaxation time, D is a diffusion constant, and it is assumed that $U(\varphi) \equiv 0$. As time t tends to ∞ (assuming that D is very small), μ_5 approaches

$$\mu_5 \simeq \frac{\tau(Le)^2}{2\pi\hbar} \vec{E} \cdot \vec{B}. \quad (8.15)$$

¹¹ ↗ F-Pedrini (2000), Hehl et al. (2008), ..., S.-C. Zhang et al. (2010). 

Manifestation of the chiral magnetic effect in the conductivity tensor of Weyl semi-metals

This expression for μ_5 can be plugged into equation (8.8) for the current, which then yields an expression for a conductivity tensor in the presence of an external magnetic field:

$$\sigma_{k\ell} = \frac{\tau(L\alpha)^2}{4\pi^2} B_k B_\ell \quad (8.16)$$

This expression is relevant in the study of transport properties of Weyl semi-metals (to mention one example), as discussed in the next section.

Axion electrodynamics may have interesting applications not only in cond-mat physics, but also in the theory of heavy-ion collisions, in astrophysics, and in cosmology, where it may explain the growth of tiny, but highly uniform cosmic magnetic fields extending over intergalactic distances. But that's another story!

Additional remarks about dimensional reduction

For some purposes, it is of interest to assume that one boundary brane, e.g., $\partial_- \Lambda$ (located at $x^4 = 0$), does not carry any dynamical degrees of freedom, and that $\widehat{A}|_{\partial_- \Lambda} = 0$, while $\widehat{A}|_{\partial_+ \Lambda} =: A$ is arbitrary. We then set

$$\widehat{A}_M(x, x^4) := \frac{x^4}{L} A(x)_\mu, \quad M \equiv \mu = 0, 1, 2, 3, \quad \widehat{A}_4(x, x^4) =: \frac{1}{L} \varphi(x).$$

The "axion" φ then transforms under em gauge transformations like an **angle**. From the action (8.9) of 5D Chern-Simons electrodynamics we then derive the *gauge-invariant* action in 4D

$$\begin{aligned} S_\Omega(A, \varphi) &:= \frac{1}{4Q^2} \int_\Omega d^4x \left[\frac{1}{3} F_{\mu\nu}(x) F^{\mu\nu}(x) + \right. \\ &+ L^{-2} (\partial_\mu \varphi(x) - A_\mu(x)) \cdot (\partial^\mu \varphi(x) - A^\mu(x)) \Big] \\ &+ \frac{1}{8\pi^2} \int_\Omega \varphi (F \wedge F) + \cdots + \Gamma_\ell(A). \end{aligned} \quad (8.17)$$

This is an anomaly-free 4D theory of chiral fermions coupled to electromagnetism and an "axion"-like (*not* gauge-invariant) field φ .

9. 3D Topological Insulators and Weyl Semi-Metals

In this section, we study 3D systems, representing topological insulators and Weyl semi-metals, on a sample space-time $\Lambda := \Omega \times \mathbb{R}$, with $\partial\Omega \neq \emptyset$. We are interested in the general form of the effective action describing the response of the systems to turning on an external em field. Until the mid nineties, the effective action of a *3D insulator* was thought to be given by

$$S_\Lambda(A) = \frac{1}{2} \int_\Lambda dt d^3x \{ \vec{E} \cdot \varepsilon \vec{E} - \vec{B} \cdot \mu^{-1} \vec{B} \} + \text{“irrelevant” terms}, \quad (9.1)$$

where ε is the tensor of dielectric constants and μ is the magnetic permeability tensor. The action (9.1) is dimensionless. In the seventies, particle theorists taught us that one could add another dimensionless term:

$$S_\Lambda(A) \rightarrow S_\Lambda^{(\theta)}(A) := S_\Lambda(A) + \theta I_\Lambda(A), \quad (9.2)$$

where I_Λ is a “*topological term*”, the “instanton number”, given by

$$\begin{aligned} I_\Lambda(A) &= \frac{1}{4\pi^2} \int_\Lambda dt d^3x \vec{E}(\vec{x}, t) \cdot \vec{B}(\vec{x}, t) = \\ &= \frac{1}{8\pi^2} \int_\Lambda F \wedge F \stackrel{\text{Stokes}}{=} \frac{1}{8\pi^2} \int_{\partial\Lambda} A \wedge dA \end{aligned} \quad (9.3)$$

“Vacuum angle” and surface degrees of freedom

In particle physics, the parameter θ is called “vacuum (or ground-state) angle”. The partition function of an insulator (after having integrated over all matter degrees of freedom) is given by

$$\Xi_{\Lambda}^{(\theta)}(A) = \exp(iS_{\Lambda}^{(\theta)}(A)) ,$$

with $S_{\Lambda}^{(\theta)}$ as in (9.2), (9.3). In the thermodynamic limit, $\Omega \nearrow \mathbb{R}^3$, $\Xi_{\Lambda}^{(\theta)}(A)$ is periodic in θ with period 2π and invariant under time reversal iff

$$\theta = 0, \pi$$

For $\theta = \pi$, $\Xi_{\Lambda}^{(\theta)}(A)$ contains a factor only depending on $A|_{\partial\Lambda}$:

$$\exp\left(\pm \frac{i}{8\pi} \int_{\partial\Lambda} A \wedge dA\right) , \quad (9.4)$$

This is the partition function of a Hall insulator on $\partial\Lambda$ with a Hall conductivity

$$\sigma_H = \pm \frac{1}{2} \cdot \frac{e^2}{h} \quad (9.5)$$

Promoting the vacuum angle θ to an “axion”

We have encountered the “boundary partition function” (9.4) (with (9.5)) in Sect. 5; see formulae (5.2), (5.3): Up to further, less relevant terms in the exponent, it is the partition function of one species of 2-component Dirac fermions coupled to $A|_{\partial\Lambda}$. *Gapless quasi-particles with spin $\frac{1}{2}$ located at $\partial\Lambda$ could mimic such Dirac fermions* and give rise to (9.4).

One may now argue that the vacuum angle θ could be the ground-state expectation of a dynamical field, φ , an “*axion*”, and replace the topological term $\theta I_\Lambda(A)$ by

$$I_\Lambda(A, \varphi) := \frac{1}{8\pi^2} \int_\Lambda \varphi F \wedge F + S_0(\varphi), \quad (9.6)$$

where $S_0(\varphi)$ is invariant under shifts $\varphi \mapsto \varphi + n\pi$, $n \in \mathbb{Z}$. We then enter the realm of *axion-electrodynamics*, as reviewed in Sect. 8! Recalling the equations of motion (8.12), we find the equation for *Halperin's “3D quantum Hall effect”*:

A 3D quantum Hall effect in axionic topological insulators

From Eq. (8.12) we infer a formula for the current $\vec{j}$ generated in an electromagnetic field:

$$\vec{j} = -\frac{e^2}{4\pi h} \left(\dot{\varphi} \cdot \vec{B} + \vec{\nabla} \varphi \times \vec{E} \right) \quad (9.7)$$

Let us consider a 3D spatially periodic (Crystalline) system with an axion φ . We suppose that φ is time-independent, i.e., $\mu_5 = 0$. Taking into account the periodicity of $\exp(i l_{\Lambda}(A, \varphi))$ under shifts, $\varphi \mapsto \varphi + 2n\pi$, $n \in \mathbb{Z}$, invariance under lattice translations implies that

$$\varphi(\vec{x}) = 2\pi (\vec{K} \cdot \vec{x}) + \phi(\vec{x}), \quad (9.7)$$

where $\vec{K}$ belongs to the *dual lattice*, and ϕ is invariant under lattice translations. Neglecting ϕ , we find that

$$\vec{\nabla} \varphi = 2\pi \vec{K} \text{ is "quantized"}.^{12}$$

¹²This last point was brought to my attention by Greg Moore. 

Why there might be axions in condensed-matter physics

It has been argued that **axions** may emerge as **effective degrees of freedom** in:

- certain 3D topological insulators with anti-ferromagnetic short-range order, (magnetic fluctuations playing the role of a dyn. axion)¹³ ; and in
- crystalline 3D Weyl semi-metals,

i.e., in systems with two energy bands exhibiting two (or, more generally, an even number¹⁴ of) double-cones in “frequency-quasi-momentum space” corresponding to **chiral** quasi-particle states, assuming that the Fermi energy is close to the apices of those double-cones. At low frequencies, namely near the apices of those double-cones, the quasi-particle states of such systems satisfy the **Weyl equation** of left- or right-handed Weyl fermions, respectively. In these systems, the time-derivative, $\mu_5 \equiv \dot{\varphi}$ of the “axion”, φ , really has the meaning of a (time-dependent) *difference of chemical potentials of left-handed and right-handed quasi-particles*.

It satisfies an equation of motion of the kind described in (8.14):

¹³a conjecture proposed by S.-C- Zhang (inspired by our work in cosmology)

¹⁴This follows from the celebrated Nielsen-Ninomiya theorem 

How one might discover “axions” in Weyl semi-metals

$$\dot{\mu}_5 + \tau^{-1} \mu_5 - D \triangle \mu_5 = L^2 \frac{e^2}{2\pi h} \vec{E} \cdot \vec{B}, \quad (9.8)$$

A non-vanishing initial value of the chemical potential μ_5 may be triggered by strain applied to the system, leading to a slightly $\ell \leftrightarrow r$ - asymmetric population of the Fermi sea. Due to “*inter-valley*” *scattering processes*, a non-vanishing μ_5 will then relax towards 0, with a relaxation time corresp. to the parameter τ in Eq. (9.8). Applying an electric field $\vec{E}$ and a magnetic induction $\vec{B}$ to the system, with the property that $\vec{E} \cdot \vec{B} \neq 0$, one finds from (9.8) that the potential μ_5 relaxes towards $\mu_5 \simeq \frac{\tau(Le)^2}{2\pi h} \vec{E} \cdot \vec{B}$. Thus, the conductivity tensor, $\sigma = (\sigma_{k\ell})_{k,\ell=1,2,3}$, is given by

$$\sigma_{k\ell} = \sigma_{k\ell}^{(0)} + \frac{\tau(L\alpha)^2}{4\pi^2} B_k B_\ell,$$

where the first term on the right side is the standard Ohmic conductivity (due to phonon- and impurity scattering), and the second term is a manifestation of the *chiral magnetic effect*. (Alas, this term may be too small to be detected in actual measurements.)

And how one might discover “axionic insulators”

People¹⁵ have described various other Gedanken experiments serving to discover effects due to axions in Weyl semi-metals; but we won't review their ideas here. Instead, we describe some axionic effects in topological insulators with an effective action given by – see (9.1) and (9.6) –

$$S_{\Lambda}(A, \varphi) = S_{\Lambda}(A) + \frac{1}{8\pi^2} \int_{\Lambda} \varphi F \wedge F + S_0(\varphi), \quad (9.9)$$

where $S_0(\varphi)$ is invariant under shifts $\varphi \mapsto \varphi + n\pi$, $n \in \mathbb{Z}$. It is compatible with time-reversal invariance that $S_0(\varphi)$ has minima at $\varphi = n\pi$. Then the material described by (9.9) is *not* an ordinary insulator, and it may exhibit a *Mott transition* at a positive temperature: The bulk of such a material will be filled with *domain walls* across which φ jumps by (an integer multiple of) π . Applying the insight described after (9.4) and (9.5), we predict that such domain walls may carry gapless two-component Dirac-type fermions. At sufficiently high temperatures, domain walls can be expected to become macroscopic, and this would then give rise to a *non-vanishing conductivity*.¹⁶

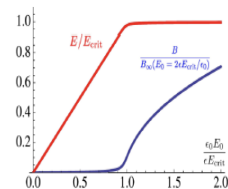
¹⁵e.g., theorists in Würzburg including J. Erdmenger

¹⁶↗ F-Werner (2014)

Instabilities in axionic topological insulators

It has been pointed out by Pedrini and myself in 2000 that the presence of a dynamical axion φ with $\mu_5 \equiv \dot{\varphi} = \text{a const. or a periodic function of time, } t$, will give rise to the growth of a *helical em field*; modes of the magnetic induction $\vec{B}$ at wave vectors of size $\leq c \mu_5$ will be unstable and exhibit unlimited growth. This growth is stopped by the relaxation of μ_5 to 0. (Our mechanism has first been applied in cosmology.)

Another, albeit related instability has been pointed out by Ooguri and Oshikawa: Assuming that $\vec{E}$ and $\vec{B}$ are essentially time-independent, an external electric field $\vec{E}$ applied to an axionic magnetic material is screened once its strength $|\vec{E}|$ exceeds a certain critical value E_c , the excess energy giving rise to a magnetic field, as shown in the following diagram taken from the paper Phys. Rev. Lett. **108**, 161803 (2012):



Courtesy Ooguri & Oshikawa

10. Summary, Open Problems

1. Apparently, concepts and methods from (relativistic) quantum field theory can be used to study general features of (*interacting*) systems of cond-mat physics; e.g., to exhibit various examples of “topological states of matter” that cannot be characterized by local order parameters. This has been illustrated in my lectures by showing how concepts from gauge theory, in particular, the chiral anomaly, the chiral magn. effect and axion electrodynamics yield rather surprising insights into properties of such states of matter.
2. What's missing in my lectures is an account of the *bare-hands analysis* of spectral properties of many-body Hamiltonians descr. “topological states of matter” at energies quite close to the ground-state energy and to derive properties of quasi-particles, using tools, such as renormalization group methods. Colleagues who have devoted serious efforts extending over many years towards reaching results in this direction are: T. Banjan, J. Feldman, G. Gallavotti, A. Giuliani, H. Knörrer, V. Mastropietro, M. Porta, E. Trubowitz, and some others. I recommend their work to the attention of this audience! Of course, many questions remain open. ...

“Survivre et Vivre” – almost half a Century later

To conclude, here is something more important to think about:

“... depuis fin juillet 1970 je consacre la plus grande partie de mon temps en militant pour le mouvement *Survivre*, fondé en juillet à **Montréal**. Son but est la lutte pour la survie de l'espèce humaine, et même de la vie tout court menacée par le déséquilibre écologique croissant causé par une utilisation indiscriminée de la science et de la technologie et par des **mécanismes sociaux suicidaires**, et menacée également par des conflits militaires liés à la prolifération des appareils militaires et des industries d'armements. ...”

Alexandre Grothendieck

“Réveillez-vous, indignez-vous!”

(Stéphane Hessel)