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PAULI CENTER
for Theoretical Studies

Workshop, ETH Zürich, 3rd-6th September 2018

Recent progress in mathematics of topological insulators

Workshop, ETH Zürich, 3rd-6th September 2018

Organizers

Clément Tauber (ETH Zürich)
Gian Michele Graf (ETH Zürich)

# *Bulk-edge corresponding revisited*

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*Univ. Tsukuba*

*Sep.3, 2018*

科研費
KAKENHI

筑波大学
University of Tsukuba

# Plan

- Topological phases and bulk-edge correspondence
  - Hofstadter problem to TKNN
  - Edge states
  - Bulk-edge correspondence
- QHE a bit more (old story)
  - Laughlin argument
  - Edge states (Halperin '82)
  - Bulk-edge correspondence '93
- Topological pumping
  - Bulk-edge correspondence '16
  - Stability for randomness '18

The same model
but
different physics

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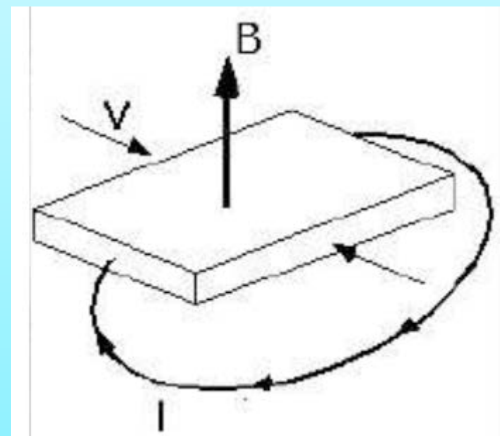
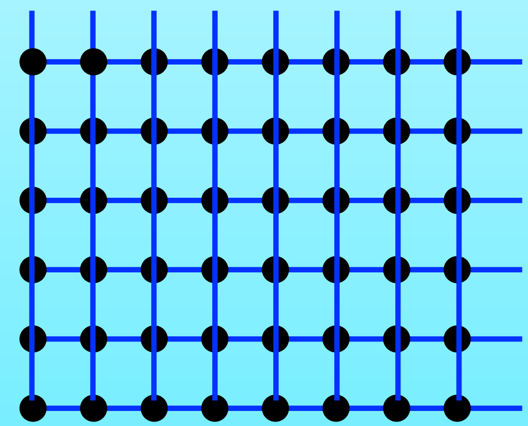
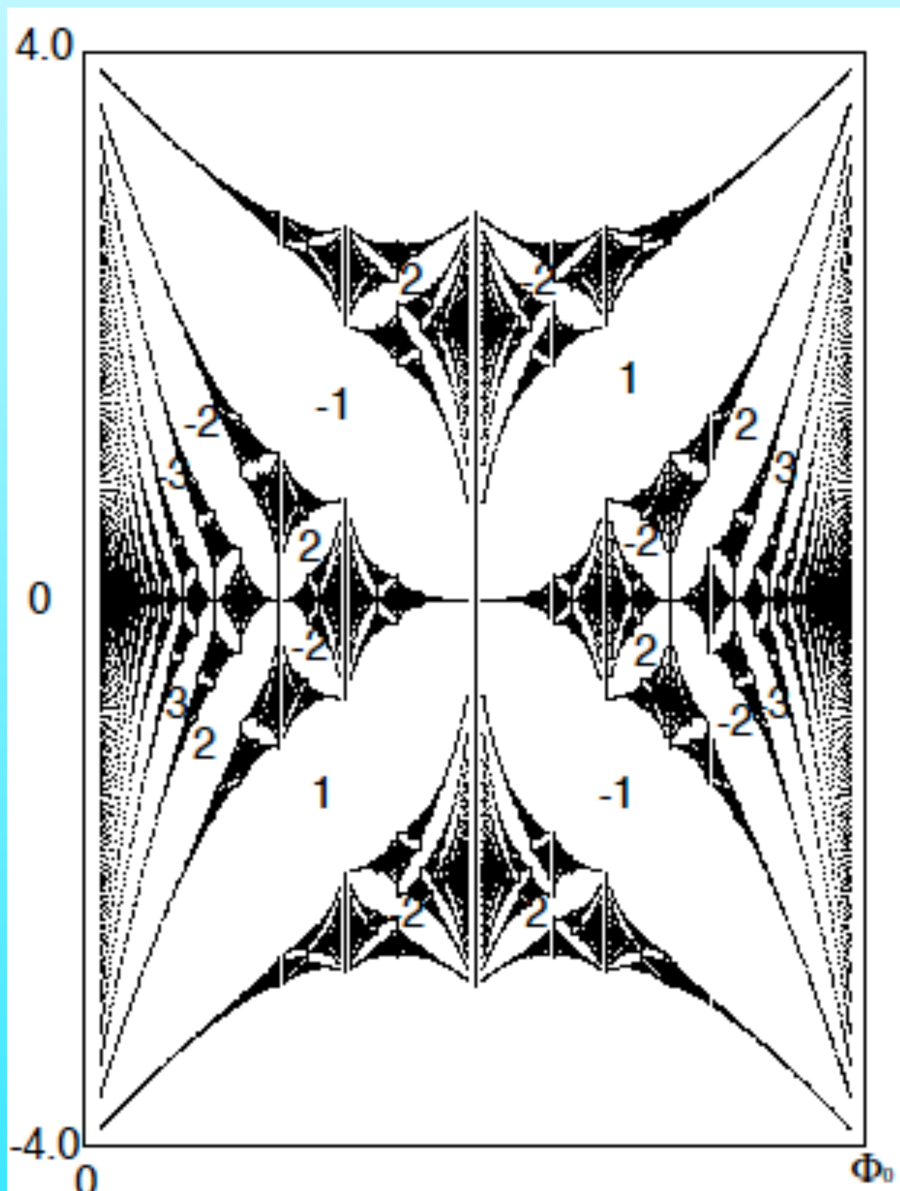
# Hofstadter's problem

2-dimensional electrons in a magnetic filed
with periodic potential
(on a lattice)



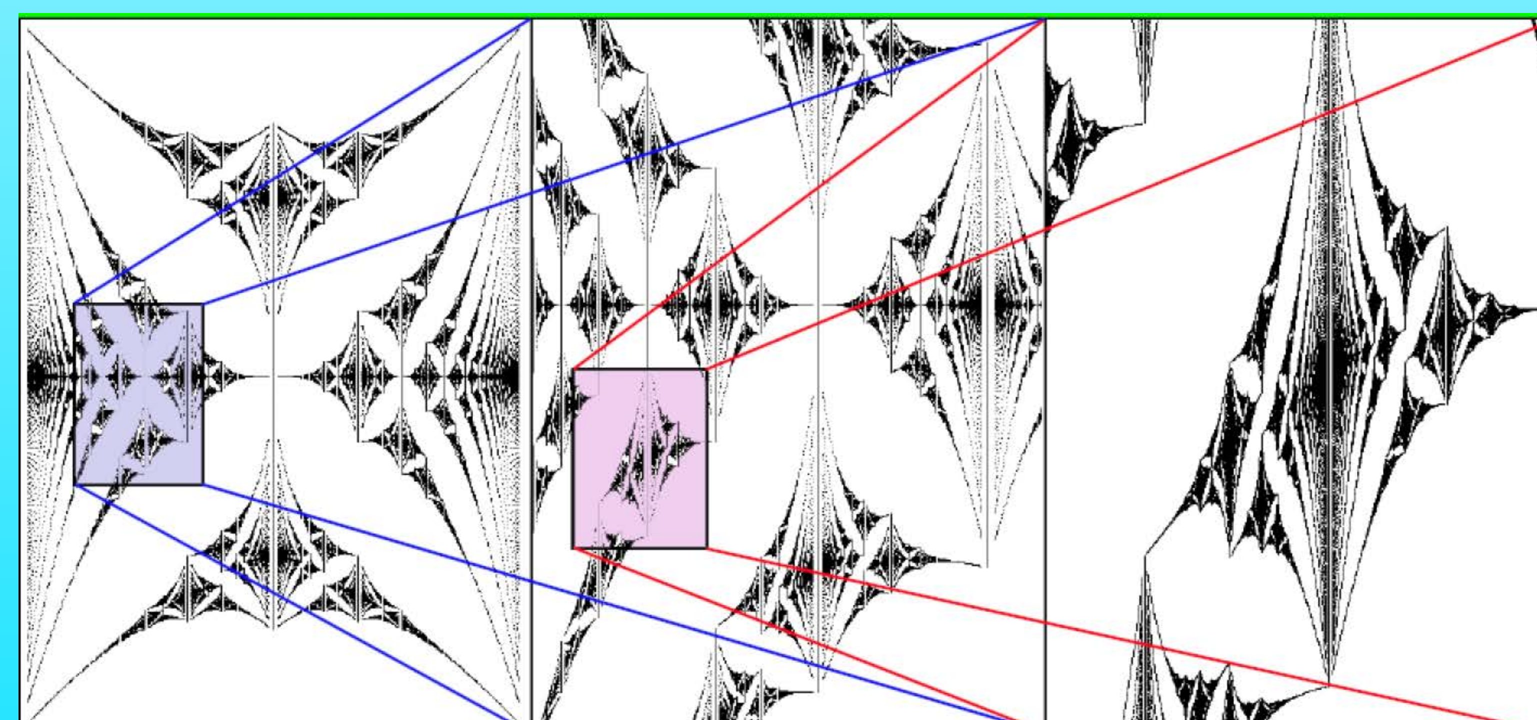






Energy

magnetic flux

Self-similar & fractal



Hofstadter '76

# Hofstadter's problem

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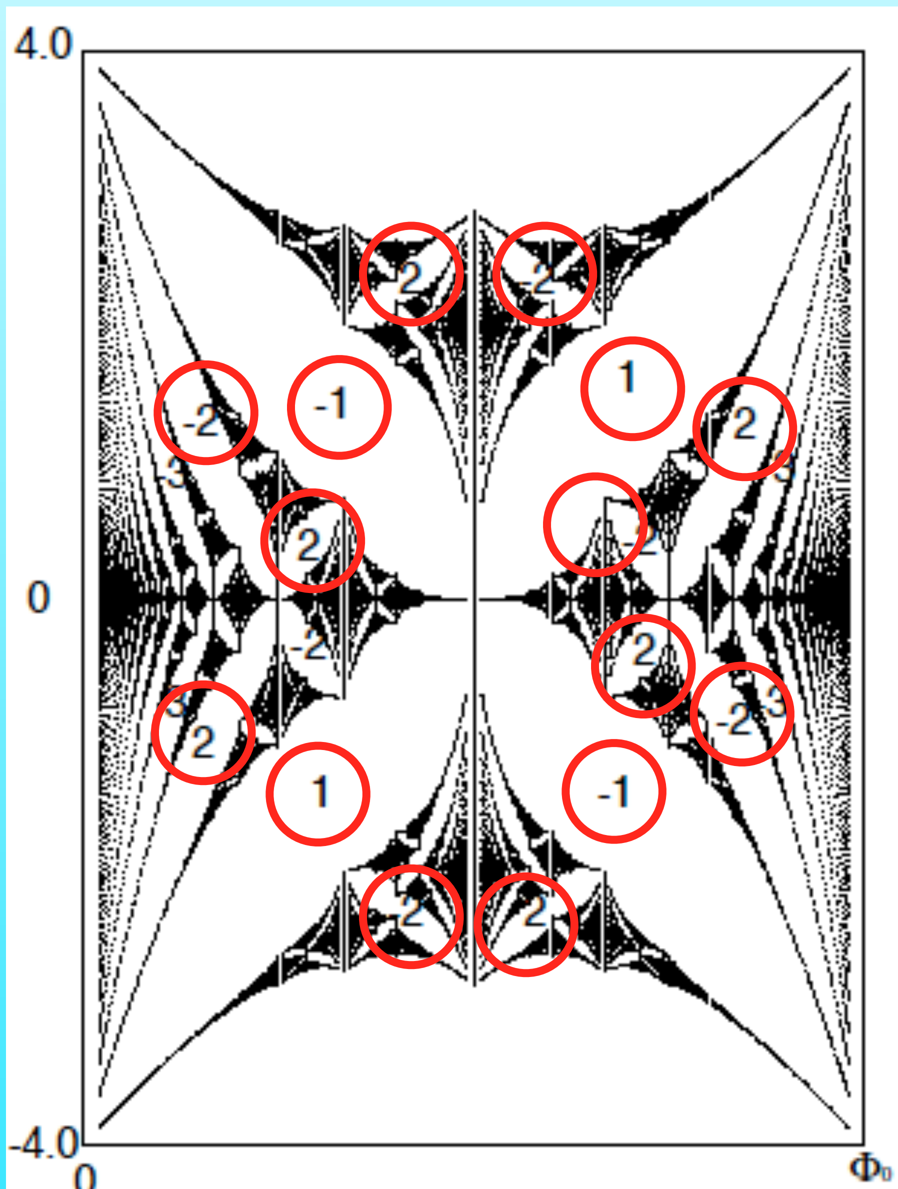
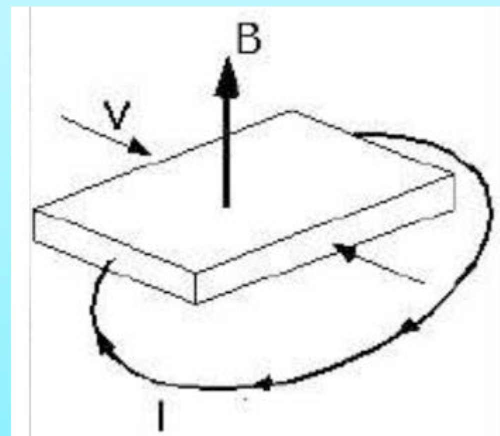



Energy

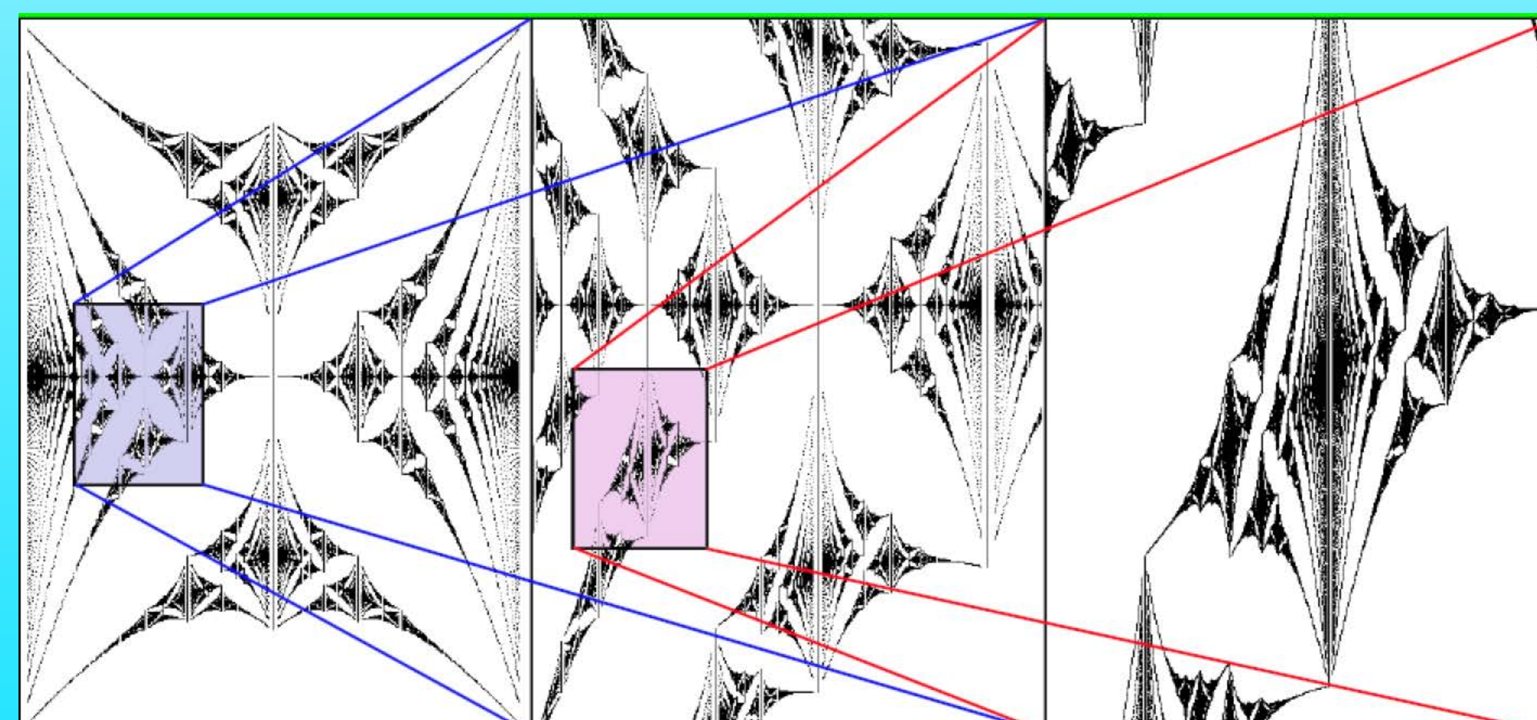
magnetic flux




Integers !

(many-body) Chern numbers

Self-similar & fractal



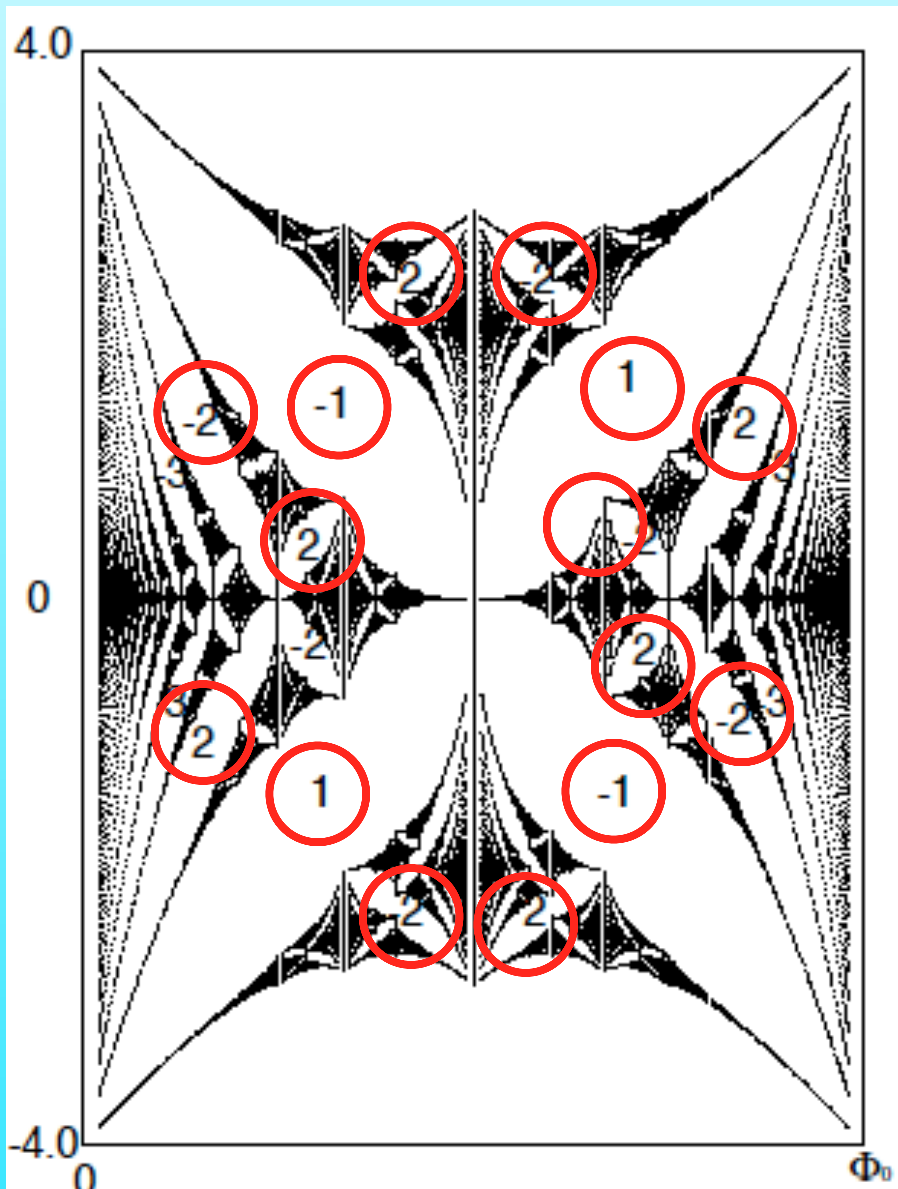
Hofstadter '76

# Hofstadter's problem

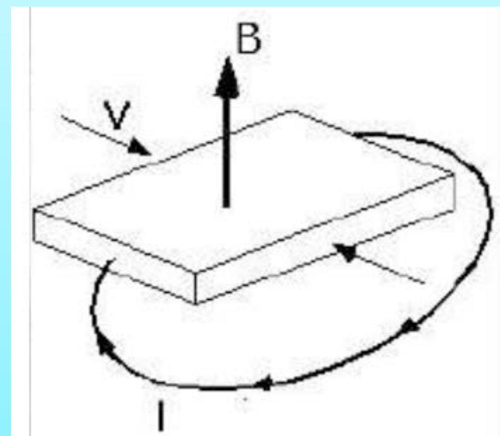
'85

Integer quantum Hall effect '80, K.v.Klitzing et al.

Energy




magnetic flux




$$\sigma_{xy} = \frac{e^2}{h} \times C$$

$$C = C_1 + \cdots + C_j$$

Thouless-Kohmoto-Nightingale-den Nijs (TKNN) '82
Avron-Seiler-Simons '83
Kohmoto '85
Niu-Thouless-Wu '85

$C_k$: Chern number of the k-th band
TKNN number

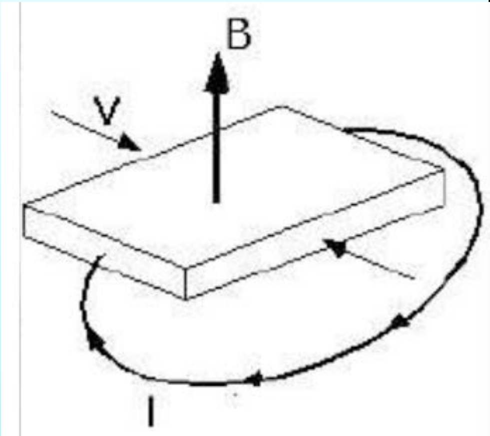
discovery of "topological phases"

Thouless-Kosterlitz-Haldane '16

Hofstadter '76

# (Integer/fractional) quantum Hall effects

## Typical examples of “topological phases”

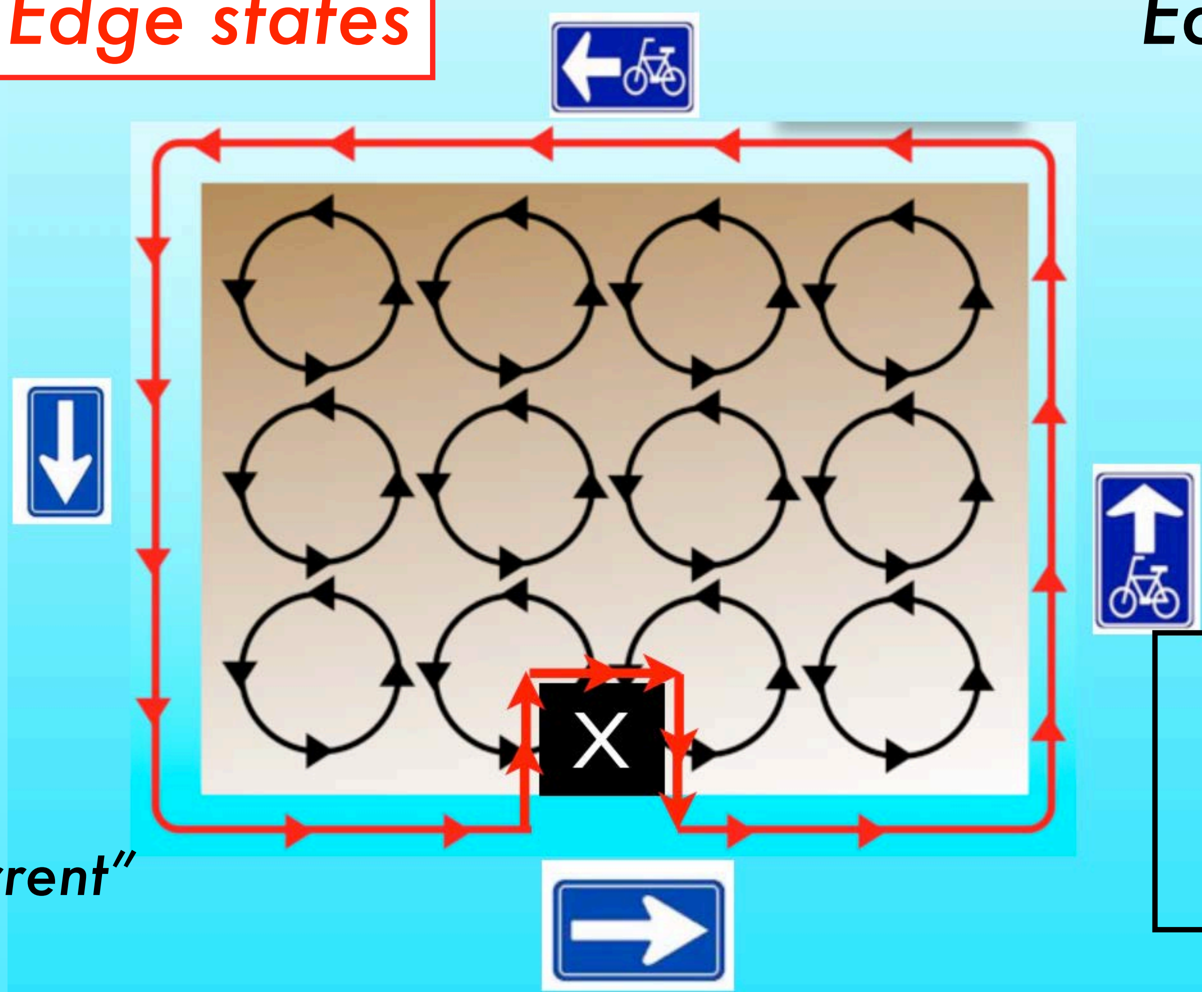



Cyclotron motion
due to Lorentz force

$$F = -e v \times B$$

Current is canceled
in the bulk
but
remains “a boundary current”

Edge states




Edge states are chiral

One way going !!

Cannot stop !


No back scattering

Stable for impurities !!

“Topological” stability
of
Chiral edge states

## Edge states are fundamental

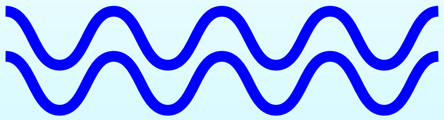
in“topological phases”

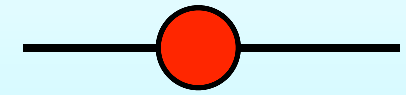
Halperin ’82 : Laughlin argument

X. G. Wen ‘90 : effective theory

YH ’93 : Hofstadter’ problem

# Bulk-Edge Correspondence (BEC)



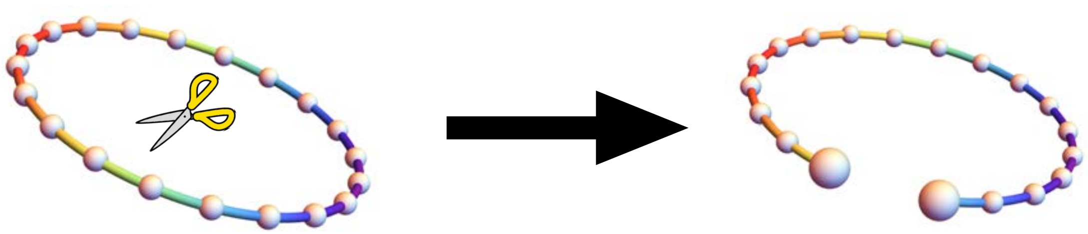


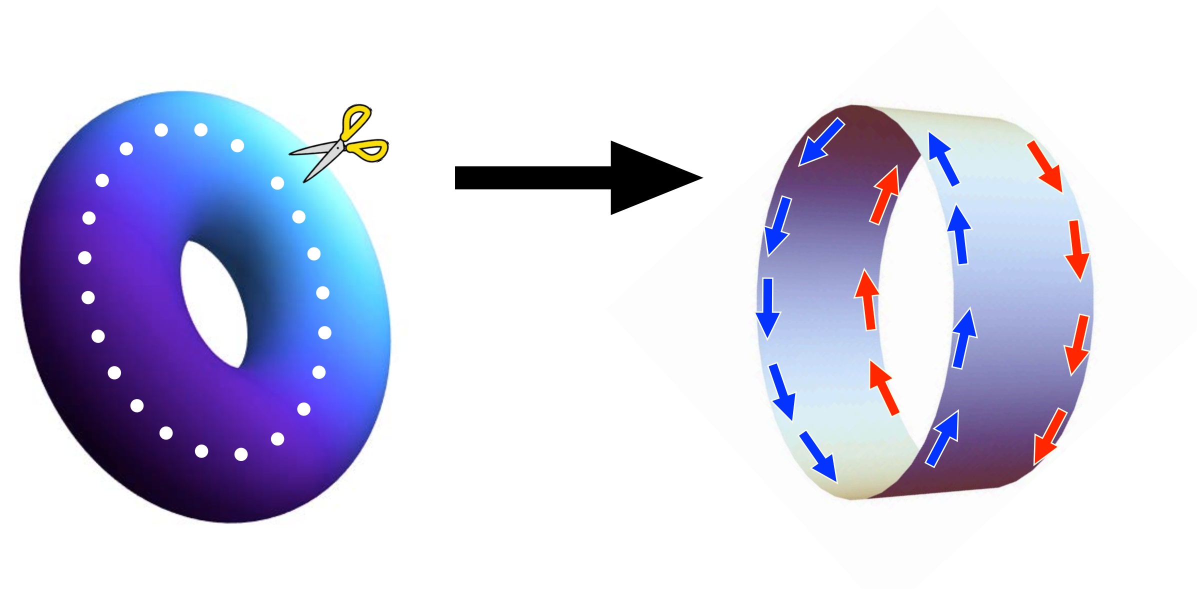
Bulk state
(scattering state)
Bulk Gap
Non trivial Vacuum

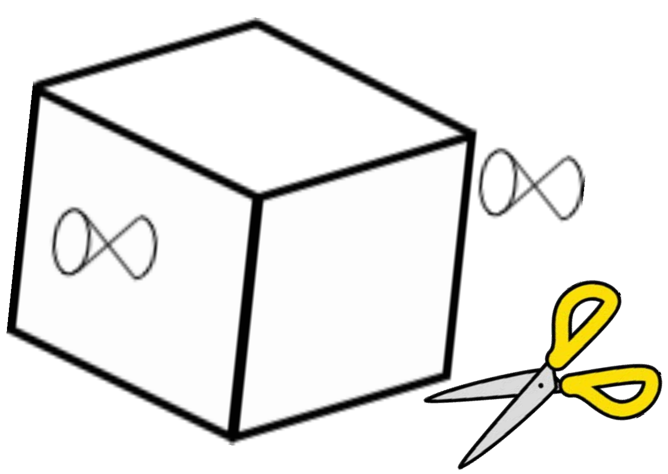
Control
with
each other

Edge state
(Bound state)
Particles in the gap

can not be independent

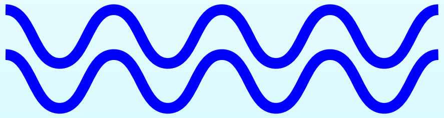


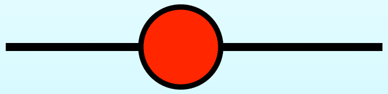




Consider bulk with edges
or
edge states from bulk

# Bulk-Edge Correspondence (BEC)





Bulk state
(scattering state)
Bulk Gap
Non trivial Vacuum

Control
with
each other

Edge state
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Particles in the gap

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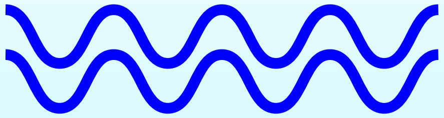
*Halperin '82: Laughlin argument*
*X.G.Wen '90 : Effective theory (gauge invariance)*
*YH '93 : BEC of IQHE (Hofstadter)*
*Kitaev '01 Majorana boundary states of superconductors*
*Ryu-YH '02 BEC of graphene, d-wave superconductivity*
*Qi-Wu-Zhang '06, BEC : general theorem*
*... more & more...*

*Also historical examples:*
*Levinson's theorem in QM, Friedel sum rule, ...*

Quantum effects ?

# Bulk-Edge Correspondence ("BEC")



Bulk state
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Bulk Gap
Non trivial Vacuum

Control
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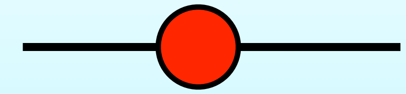


Edge state
(Bound state)
Particles in the gap

"Discovery in this century"

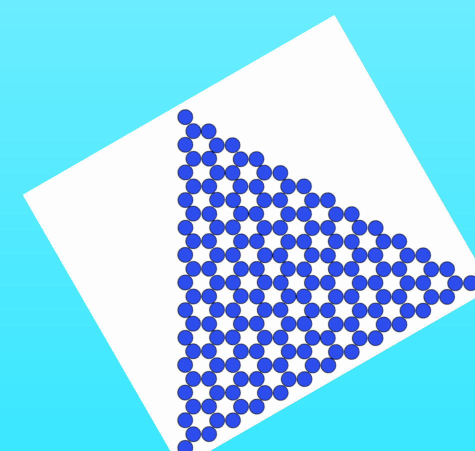
Maxwell eq.
"BEC" in photonics

Haldane-Raghu '08

Wang-Chong-Joannopoulos-Solijacic '08

Newton eq.
"BEC" in classical mechanics

Kane-Lubensky '13



Kariyado-YH '15

Do NOT need quantum mechanics : universal

# Hofstadter's problem

bulk $$C_j = I_j - I_{j-1}$$ edge

YH '93

## Quantum Hall Effect 2D

bulk

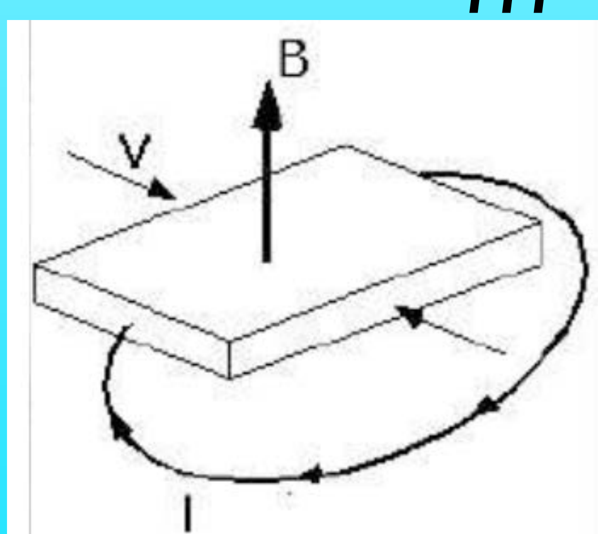
$C_j$ : Chern number
(Hall conductance)

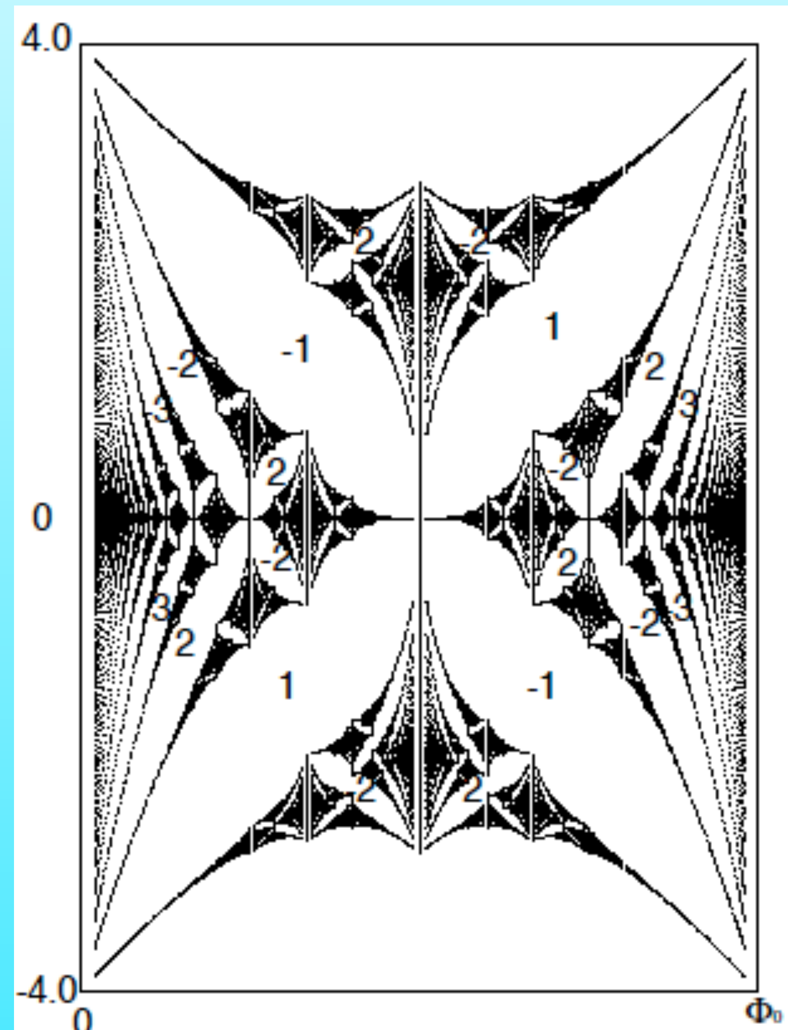
TKNN '82

edge

$I_j$ : Winding number

YH '93




The same model
but
different physics

time as a synthetic dimension

## Topological pump 1+1D

bulk

$C_j$ : Chern number
(pumped charge)

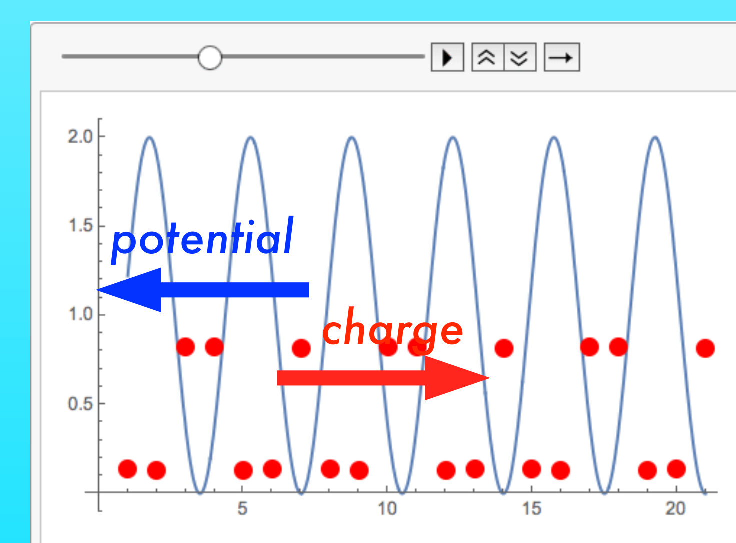
Thouless '83

experiments '15

edge

$I_j$

missing ?



# Hofstadter's problem

$$C_j = I_j - I_{j-1}$$

YH '93

time as a synthetic dimension

## Quantum Hall Effect 2D

bulk

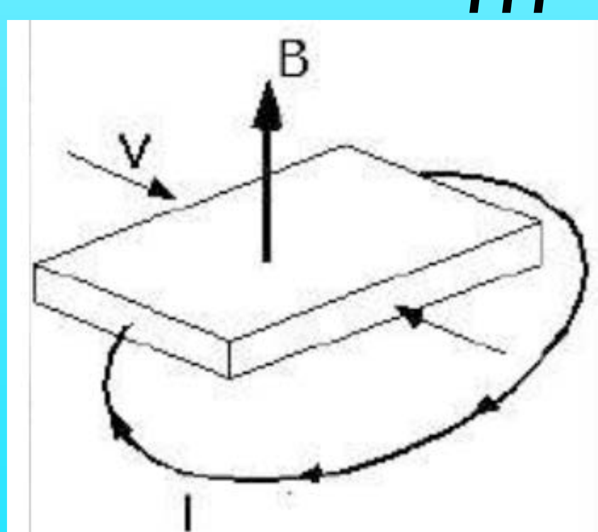
$C_j$ : Chern number (Hall conductance)

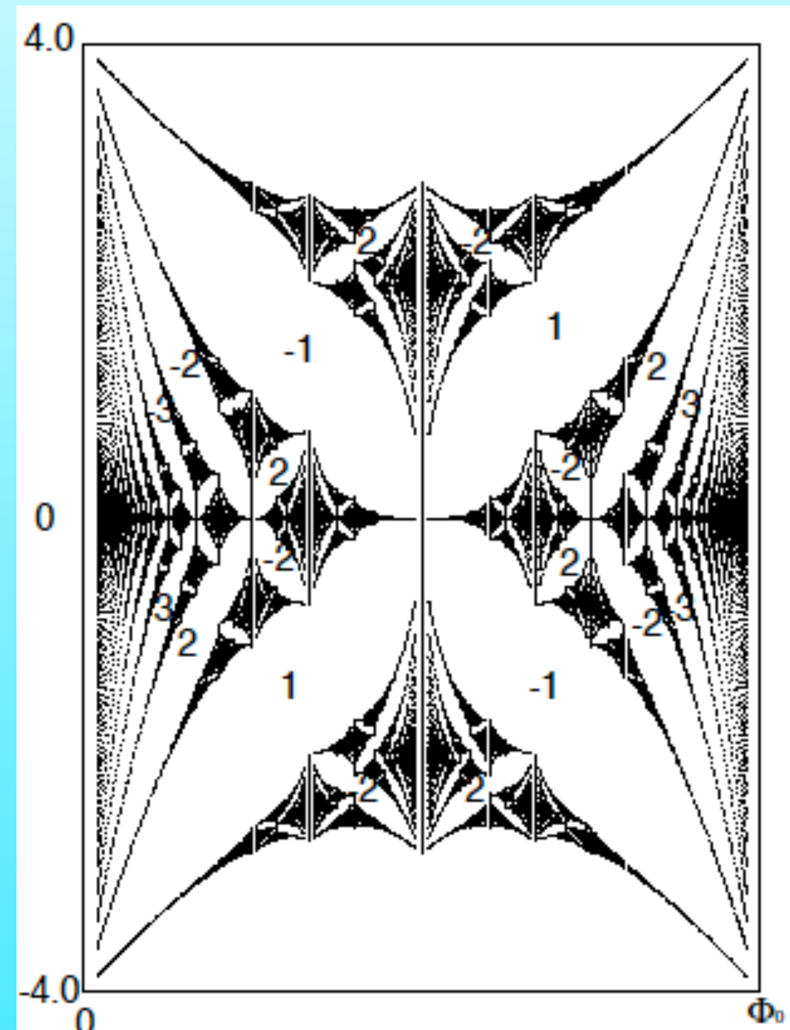
TKNN '82

edge

$I_j$ : Winding number

YH '93




The same model
but
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## Topological pump 1+1D

bulk

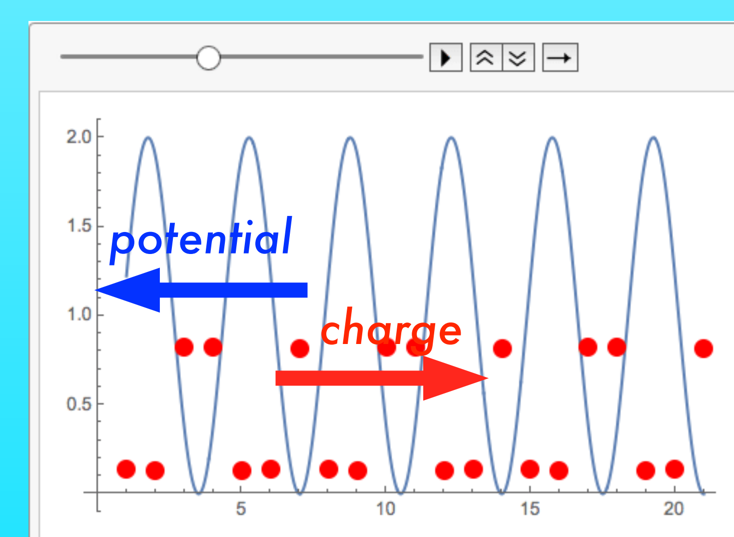
$C_j$ : Chern number (pumped charge)

Thouless '83

experiments '15

edge

$I_j$ YH-Fukui '16
new topological inv.



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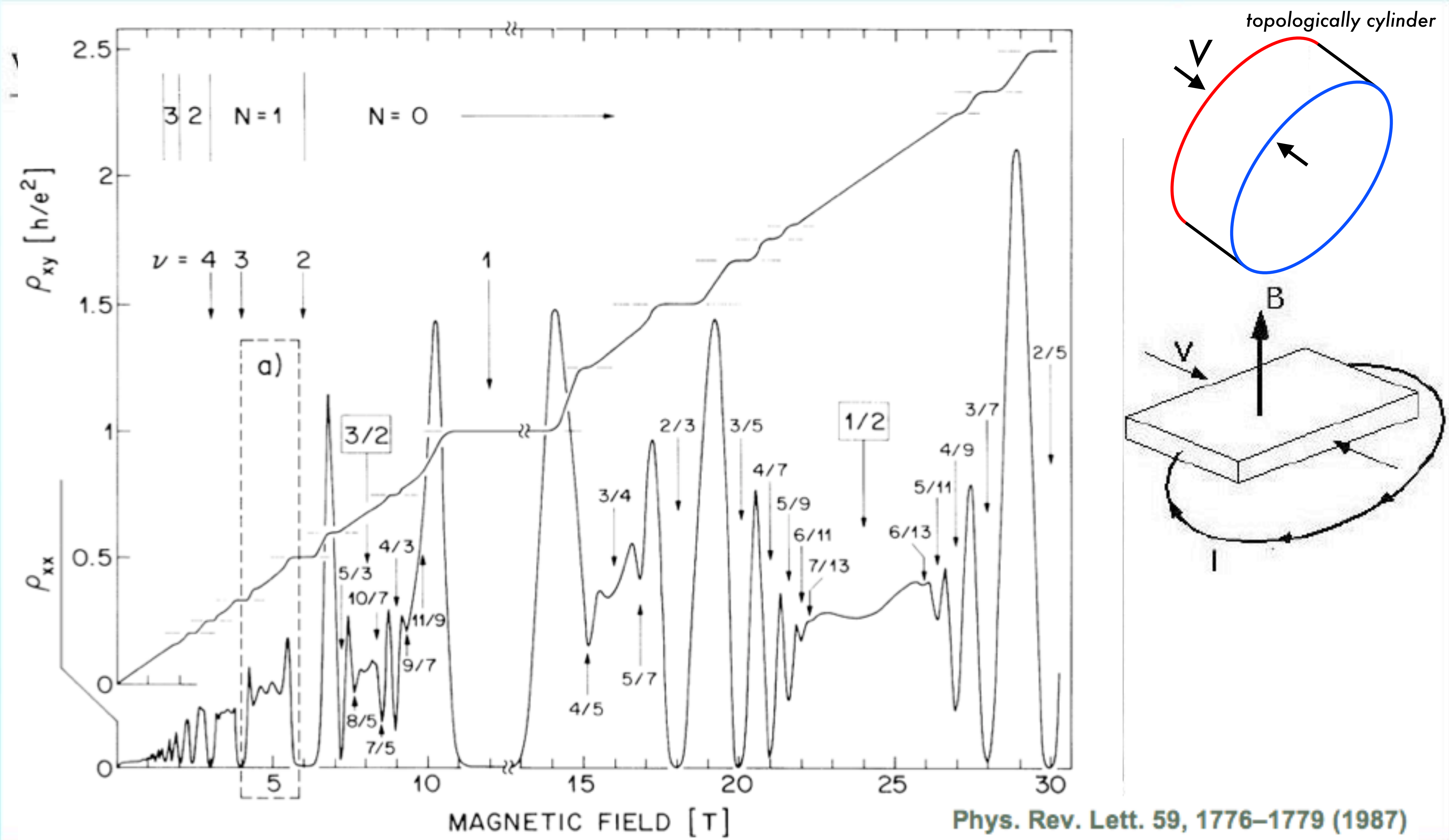
# Quantum Hall Effect

*'80, K.v.Klitzing et al.*

The Nobel Prize in Physics 1985
Klaus von Klitzing
Nobelprize.org

Quantization of the Hall conductance $\sigma_{xy}$ with anomalous accuracy: $I = \sigma_{xy} V$



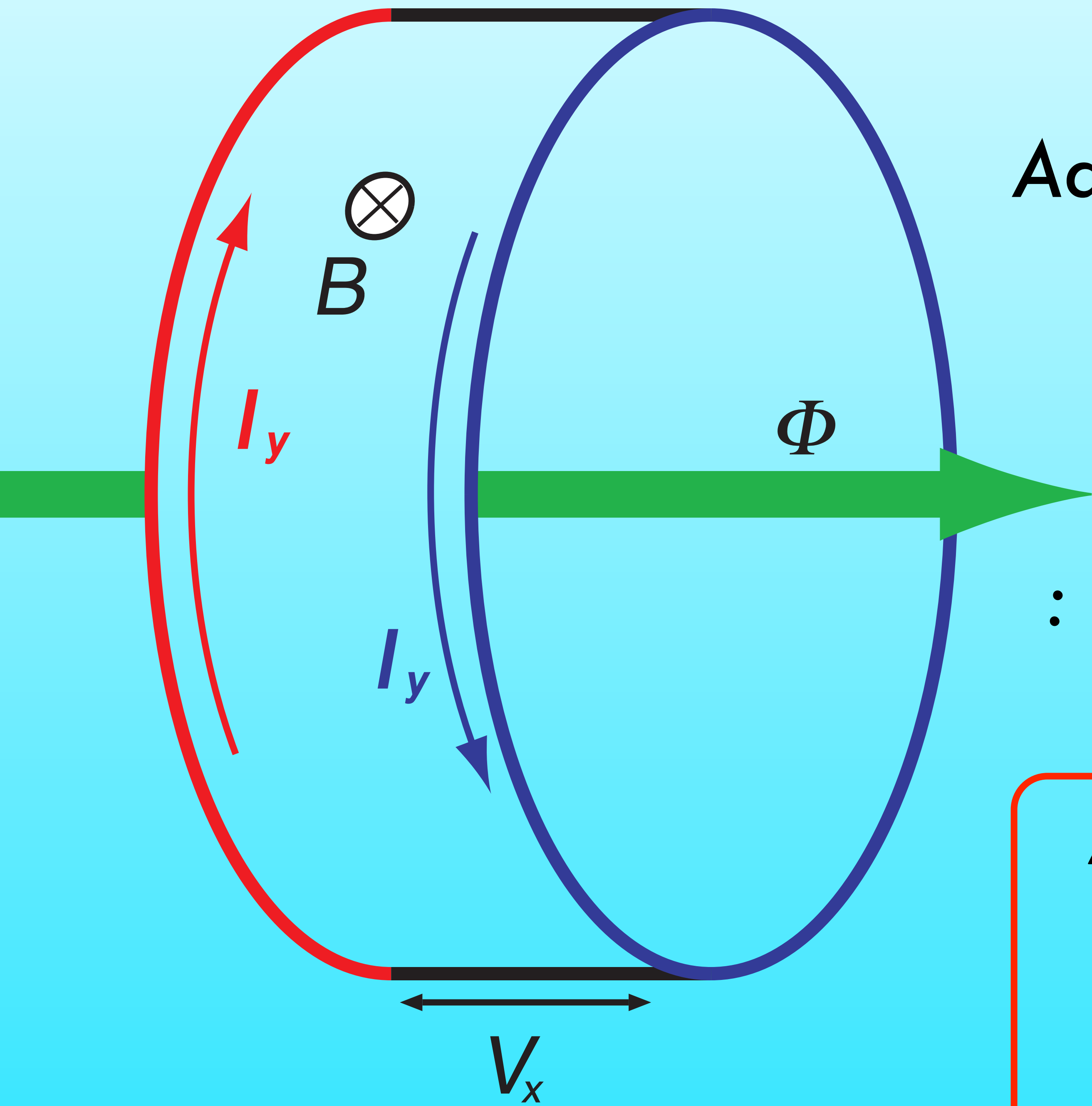

Phys. Rev. Lett. 59, 1776–1779 (1987)

*R. Willett et al.*

# Laughlin's argument

Laughlin '81

☆ Gauge Invariance




Byers-Yang

$$I_y = \frac{\Delta E}{\Delta \Phi} = \sigma_{xy} V_x$$

flux quantum

Adiabatic increase by $\Delta\Phi = \Phi_0 = \frac{h}{e}$

→ Insulating system goes back to the original state

: assuming "n" electrons are carried from the left to the right

$$\Delta E = neV_x$$

$$\sigma_{xy} = \frac{e^2}{h} n$$

n is an integer but "unknown"

Quantization of $\sigma_{xy}$ by Edge states

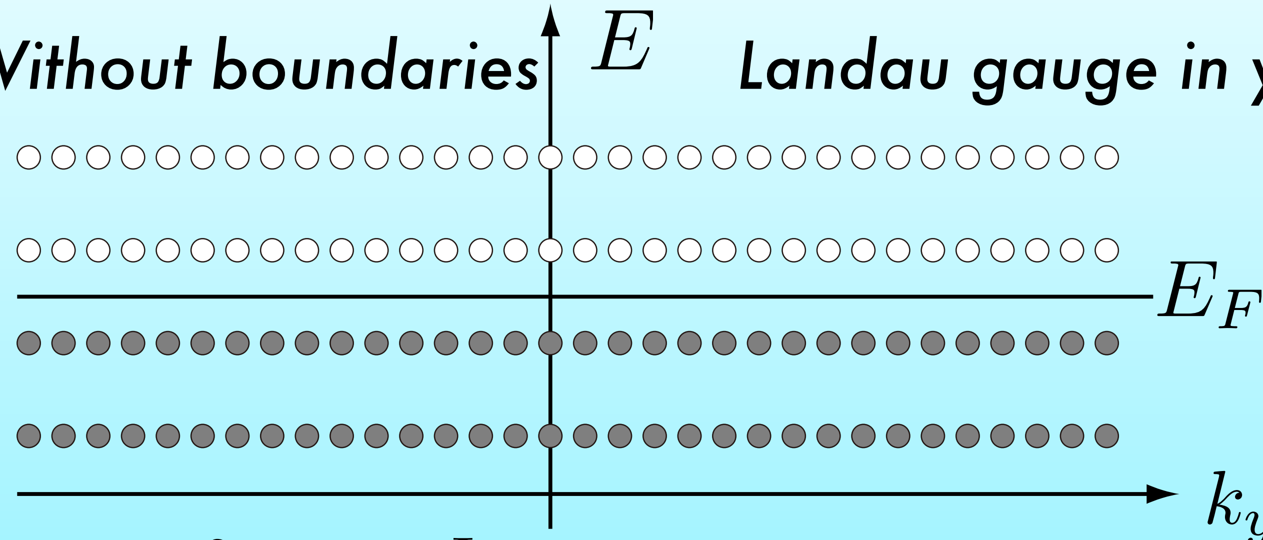
# ☆ Edge states and Hall conductance

Halperin '82

Continuum space

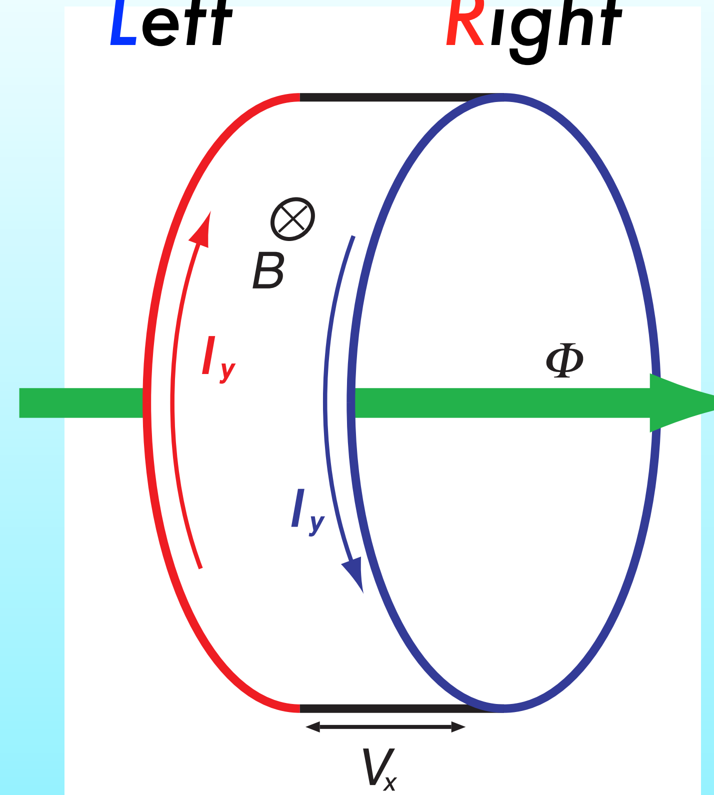
Without boundaries

Landau gauge in y




$$k_y = \frac{2\pi}{L_y}\left(n_y + \frac{\Phi}{\Phi_0}\right) \quad n_y = 0, \pm 1, \pm 2, \cdots$$

$$\ell_B = \sqrt{h/eB}$$

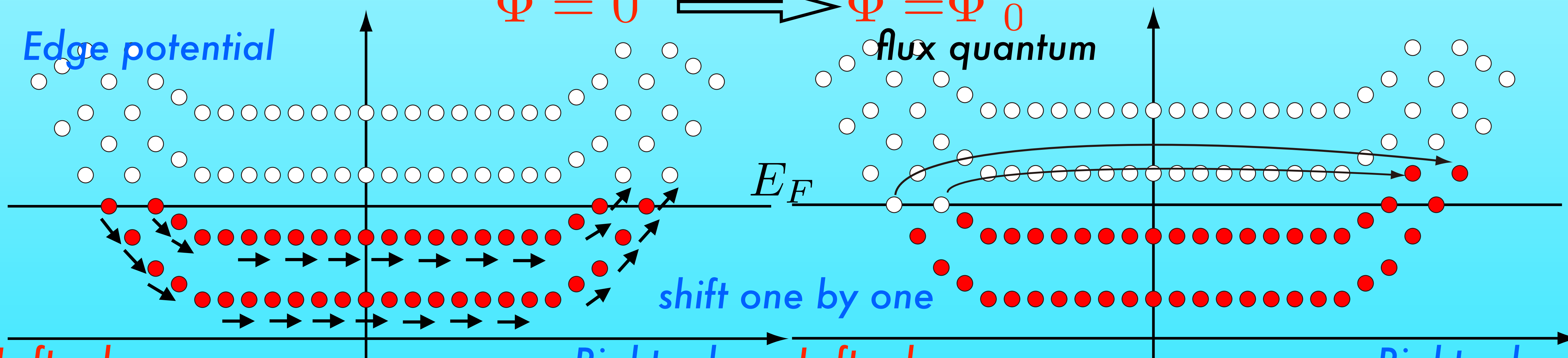



With boundaries

$\Phi = 0 \Longrightarrow \Phi = \Phi_0$

flux quantum

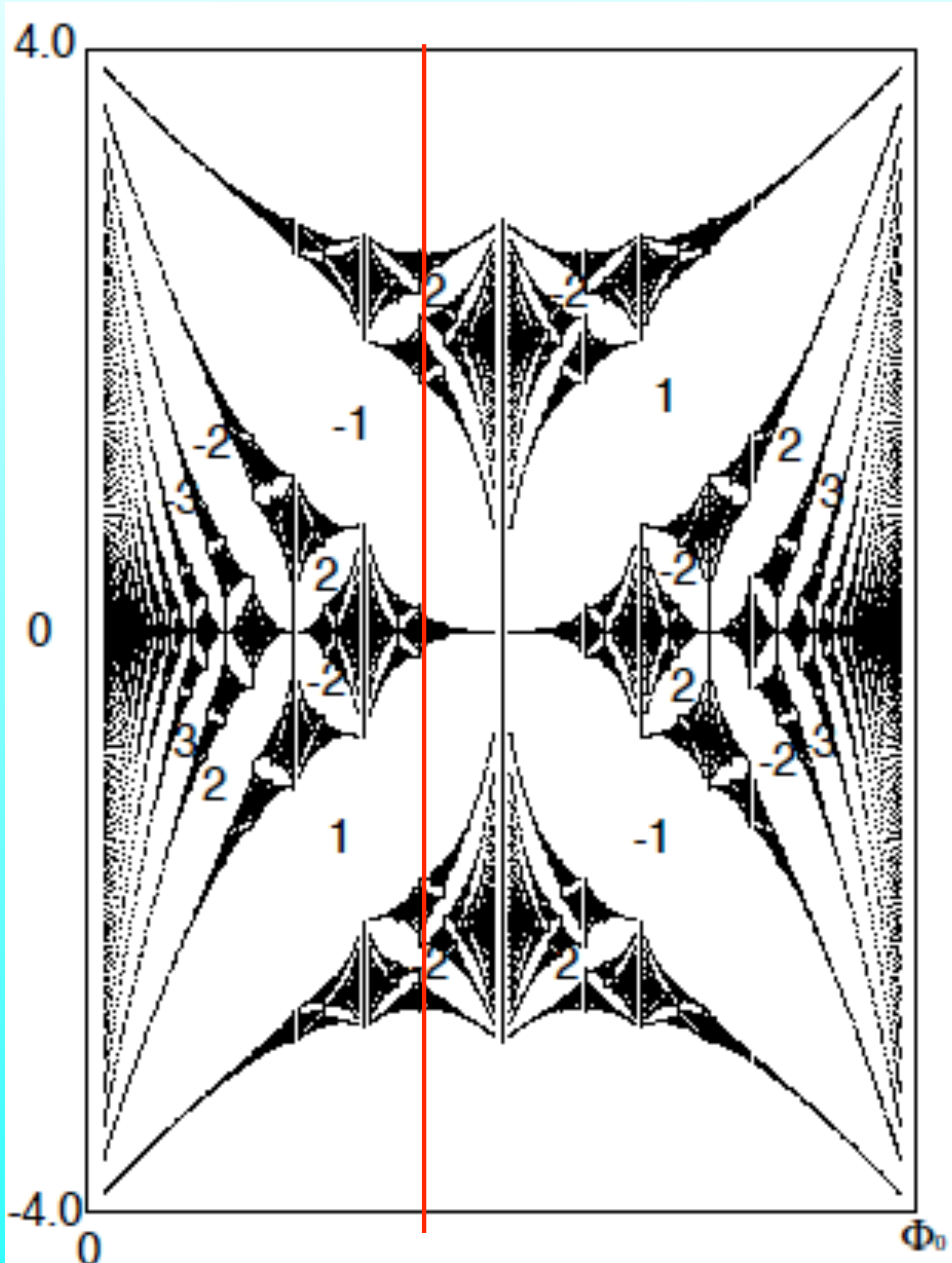
2 states are carried from L to R

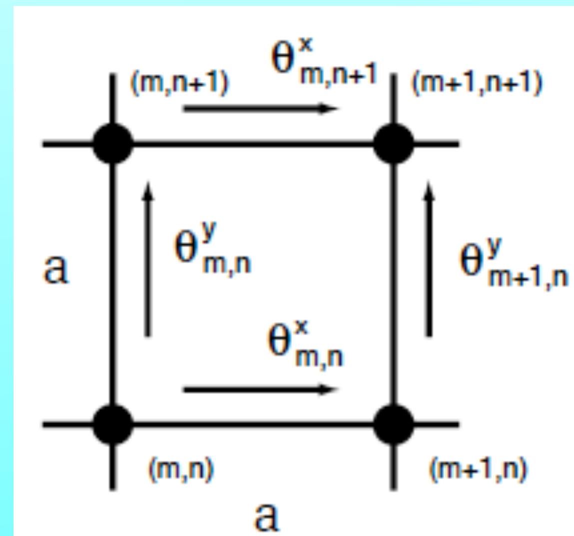



## Unknown " $n$ " is a # of Landau levels below $E_F$

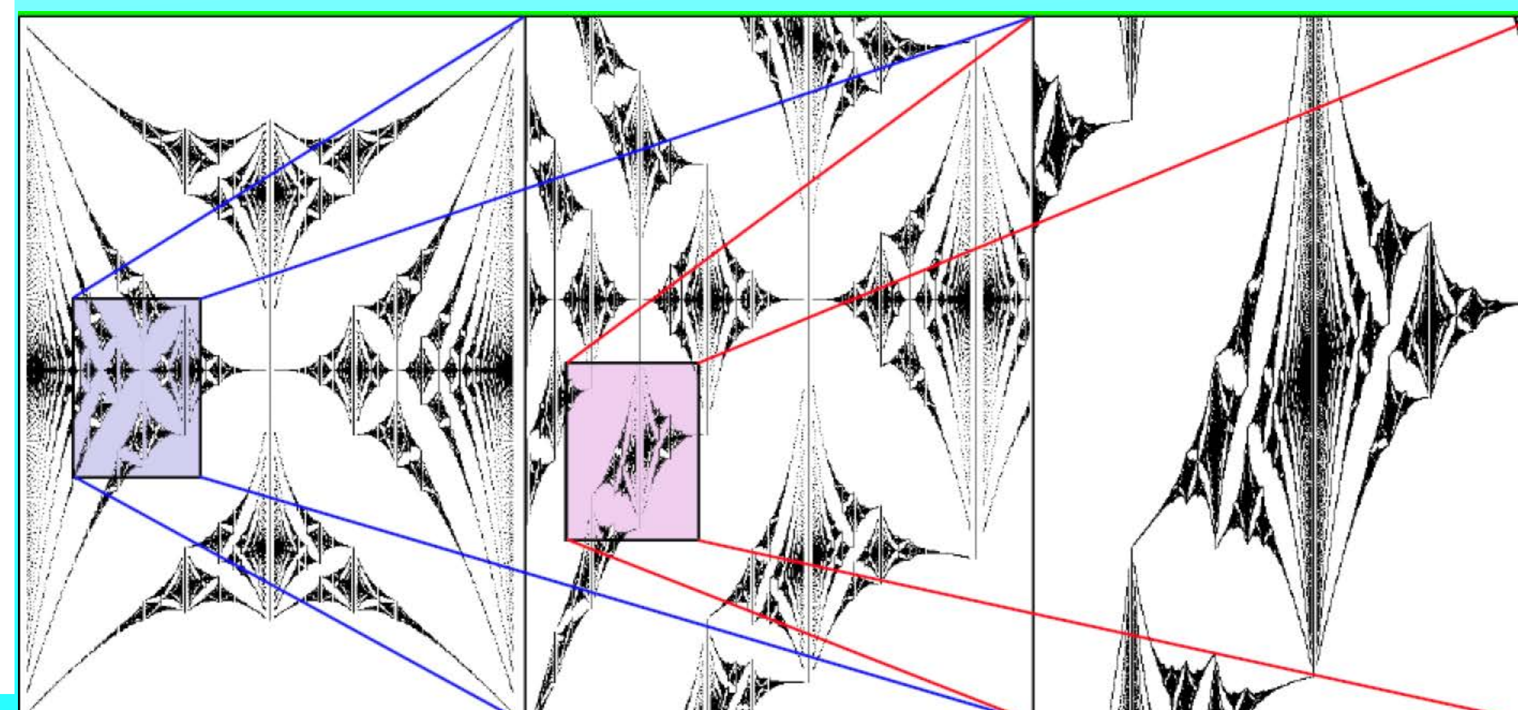
Bulk-edge correspondence (implicit)

# QHE (Hofstadter's)




$$\sum_{j\in\langle ij\rangle} \theta_{ij}=2\pi\phi_i = 2\pi\phi$$



$$\phi = 2/5$$

*5 bands*

# Topological meaning of the Hall Conductance

Kubo formula

☆ TKNN formula: $\sigma_{xy}$ as a topological invariant

$$\sigma_{xy} = \frac{e^2}{h} C$$

$$C = C_1 + \cdots + C_j$$

sum over the filled states

Thouless-Kohmoto-Nightingale-den Nijs '82
Avron-Seiler-Simons '83
Kohmoto '85
Niu-Thouless-Wu '85

$$C_\ell = \frac{1}{2\pi i} \int_{T^2:\mathrm{BZ}} F_\ell$$

:First Chern number of the $\ell$-th Band
TKNN number

$$F_\ell = dA_\ell = \langle d\psi_\ell | d\psi_\ell \rangle$$

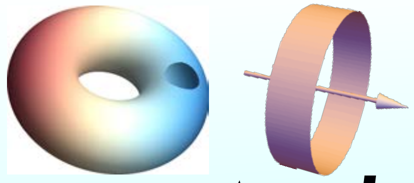
$$A_\ell = \langle \psi_\ell | d\psi_\ell \rangle$$

:Berry connection

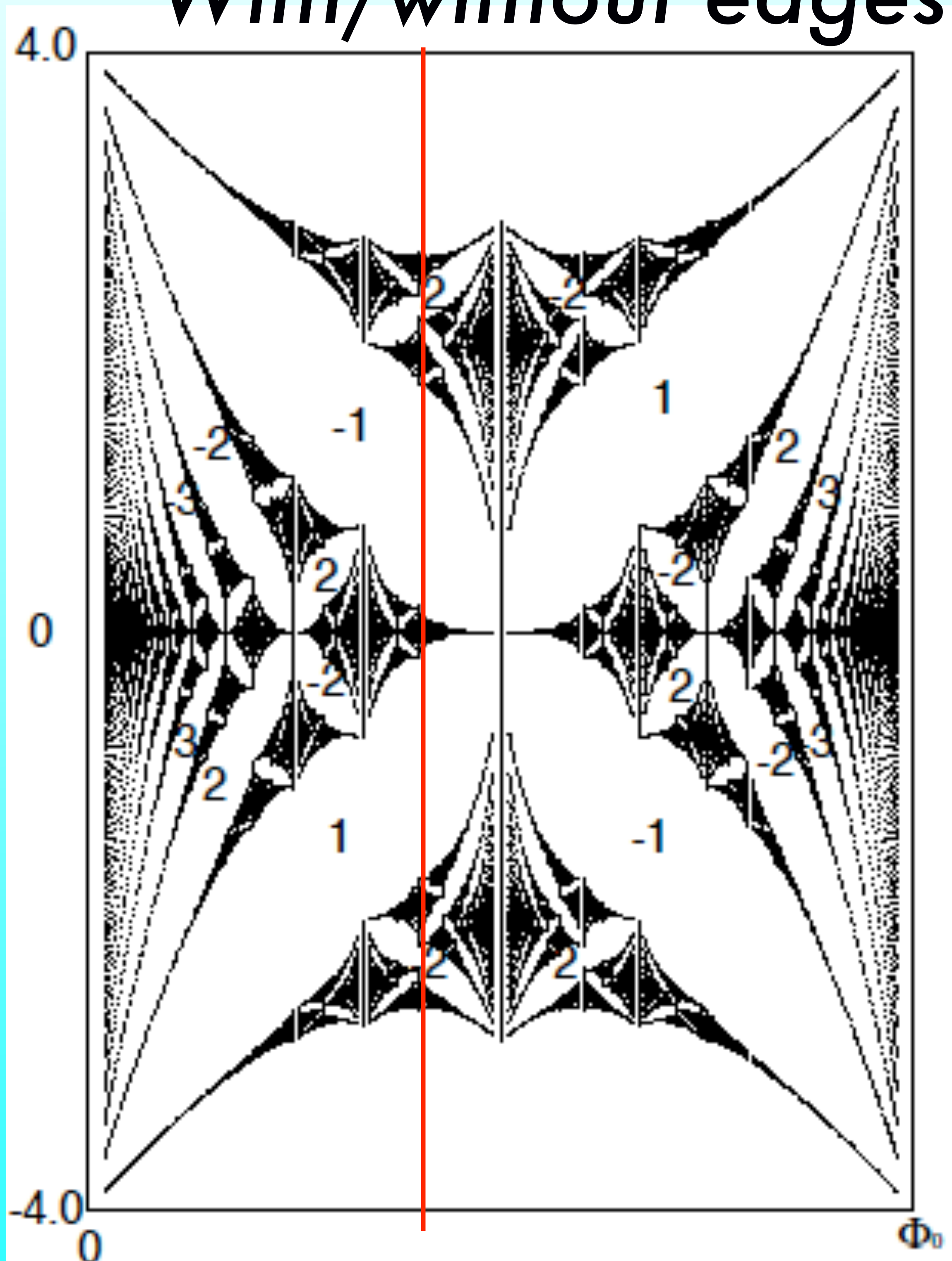
$$H(k)|\psi_\ell(k)\rangle = \epsilon(k)|\psi_\ell(k)\rangle$$

:Bloch state

## Without boundaries : "BULK"

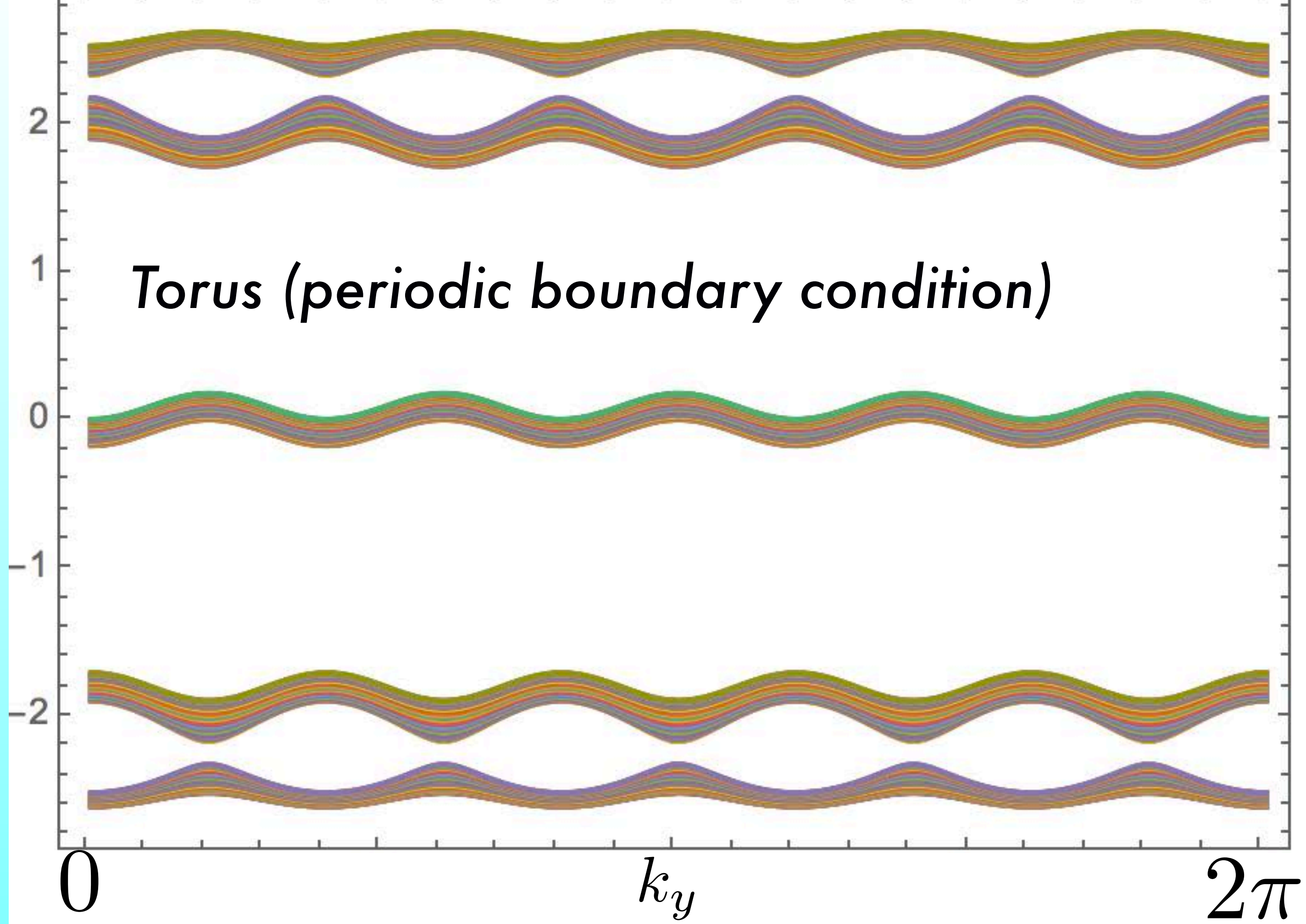


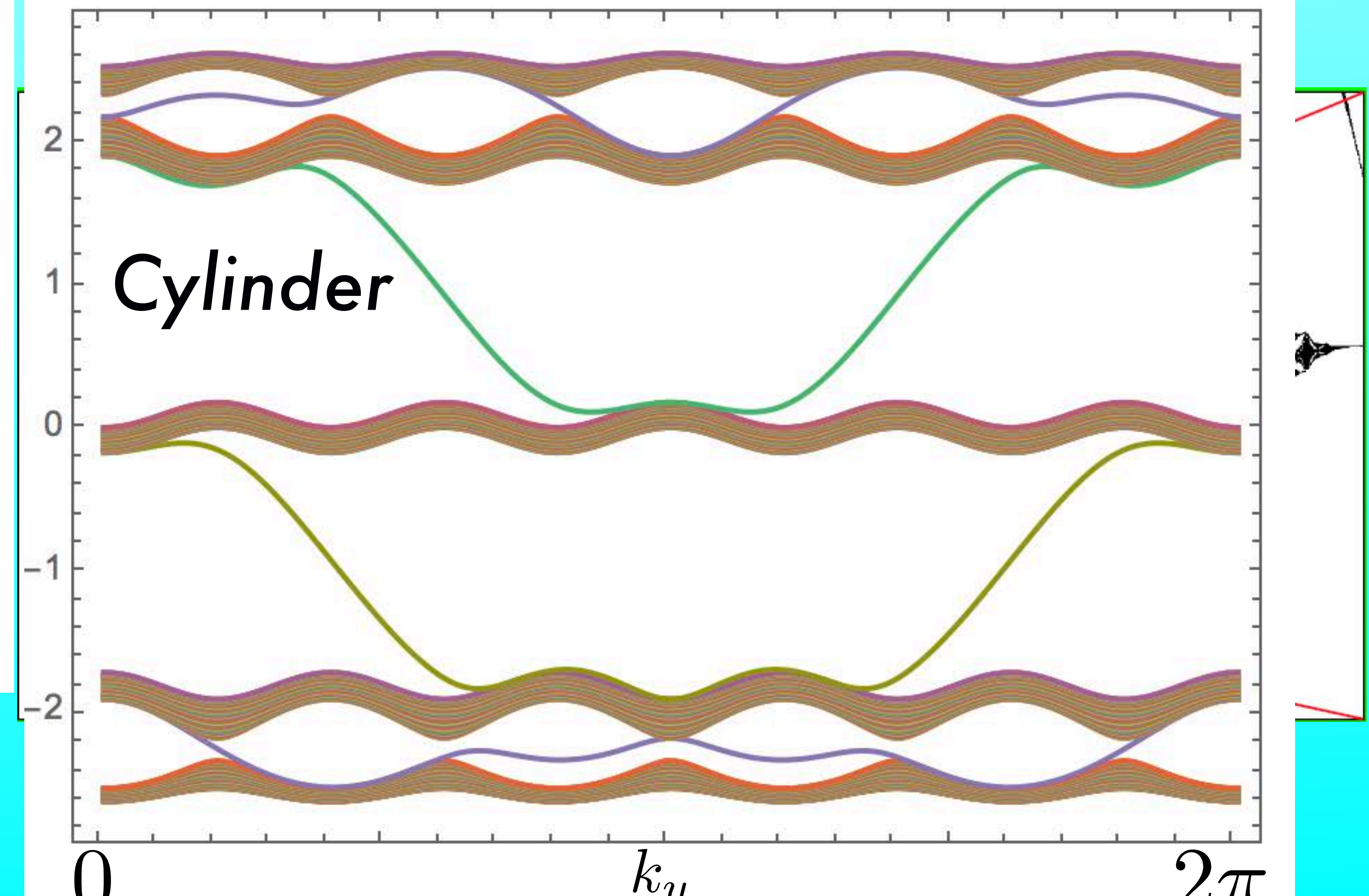
# With/without edges




$\phi = 2/5$

*5 bands*

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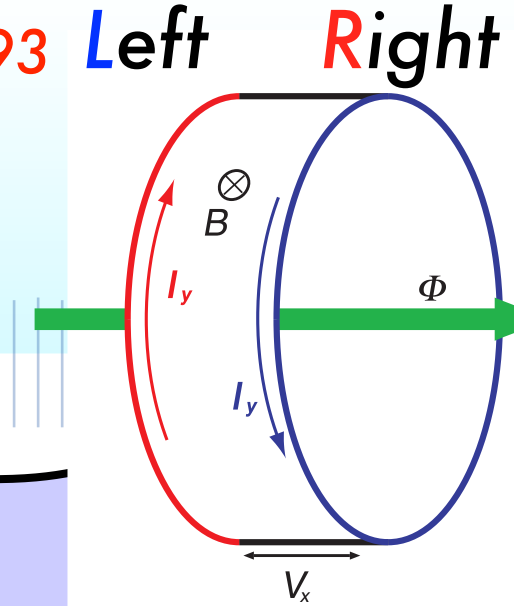
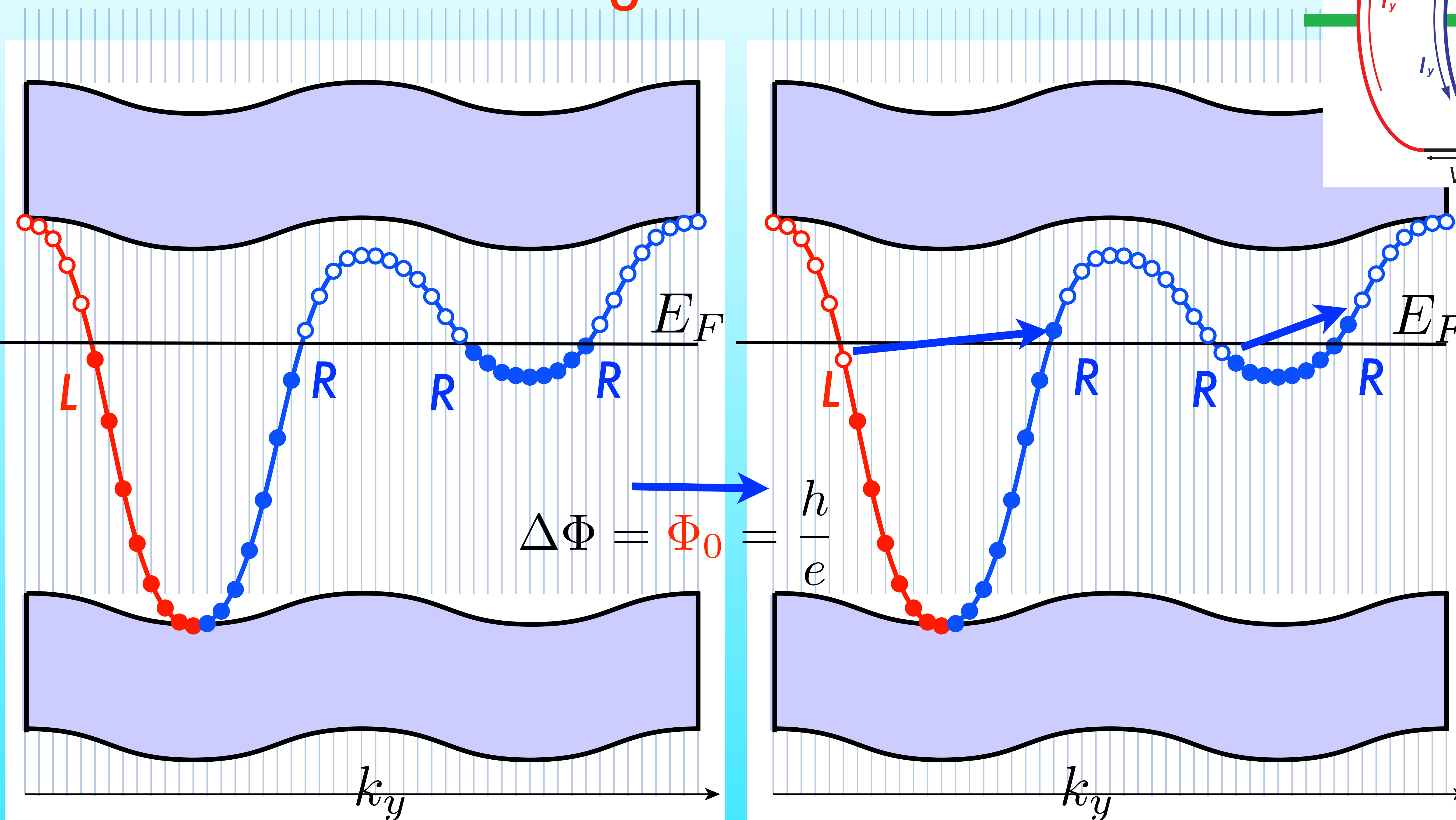
# Unknown “$n$” is a # of edge state connecting the bands

YH ’93

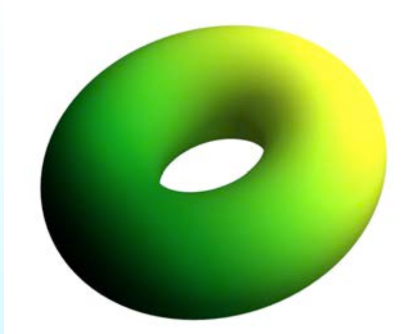



$$k_y = 2\pi \frac{n + \frac{\Phi}{\Phi_0}}{L_y}, \quad n: \ \text{integers}$$

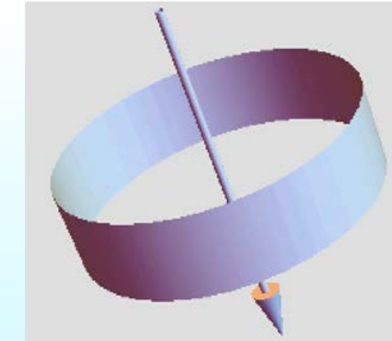
1 Electron is carried from the Left to the right in this case

$$\sigma_{xy} = \frac{e^2}{h} \cdot 1$$



# BEC of IQHE YH '93

## physically simple



bulk
$\sigma_{xy}$
Hall conductance of the bulk

=

edge
$\sigma_{xy}$
Hall conductance of the cylinder
(with edges)

(many-body)
**Chern number**

=

**Number of edge states**

Sum of TKNN integers

Number of edge states

$$C_1 + C_2 + \cdots + C_{j-1} + C_j = I_j$$

$$C_1 + C_2 + \cdots + C_{j-1} = I_{j-1}$$

winding number of edge states

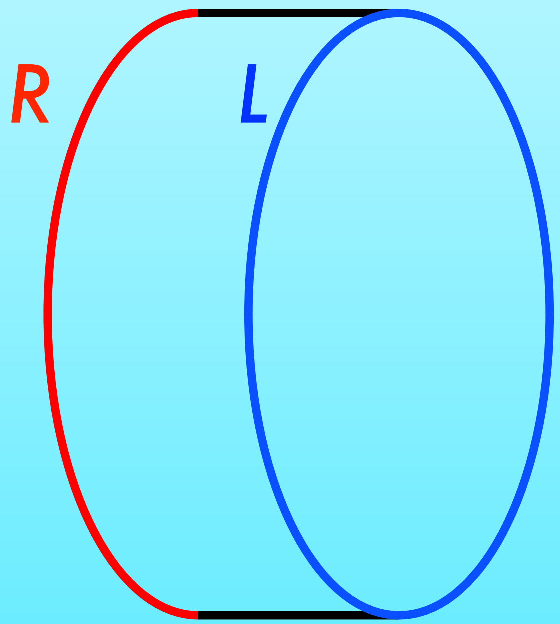
$$C_j = I_j - I_{j-1}$$

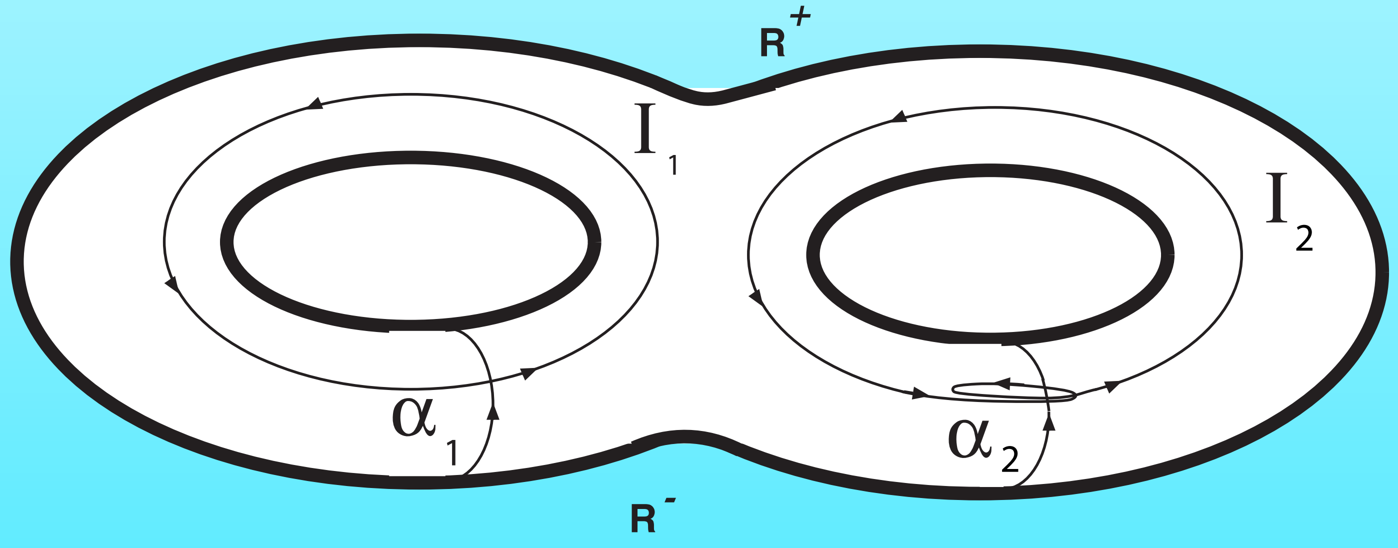
TKNN#
Chern number of the band

Number of the edge states

# Analytic continuation
# Bloch states & Edge states

Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)




Unknown "$n$" is a winding # of the edge states

# Edge state and Bloch state

- Bloch electrons, 2D = $\sum$(1D Harper problem with parameter $k_y$ )

  c.f. Landau level 2D = $\sum$ (1D harmonic oscillator with parameter $k_y$ )

  guiding center

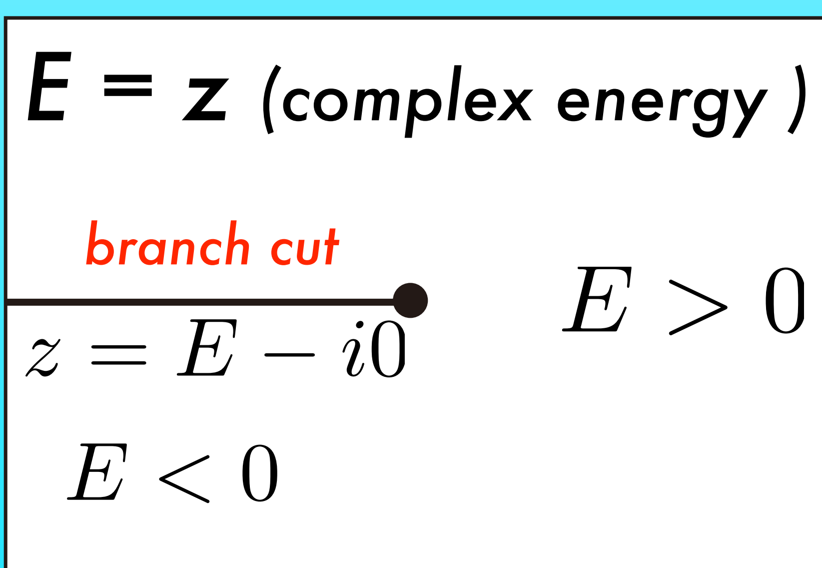
  - Edge state : bound state
  - Bloch state: scattering state

## These two can be treated in a unified way by considering complex energy

standard quantum mechanics

- bound state
- scattering state $\psi \sim e^{ikx}$

$$E = \frac{\hbar^2 k^2}{2m} \begin{cases} < 0 & k = i\kappa, \quad \psi \sim e^{-\kappa x} \\ > 0 & k \in \mathbb{R}, \quad \psi \sim e^{ikx} \end{cases}$$




unified description

$$\psi \sim e^{i\sqrt{2mE}\,x/\hbar}$$

energy of the bound state is in the gap region E<0

# Analytic continuation of the Harper eq.

- The edge state is obtained from the Bloch State by analytical continuation

Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

  - Energy of the Bloch state $\psi_B$ is in the band
  - Energy of the edge state $\psi_E$ is in the gap

- Complex energy surface : genus Q Riemann surface

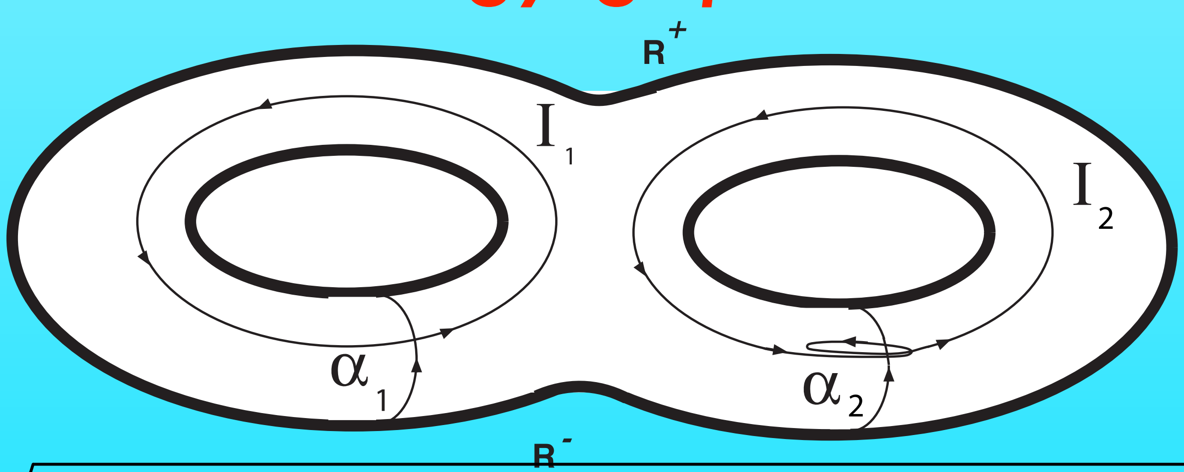
$\psi_B$ & $\psi_E$ :Unified on complex energy surface

  - Energy bands : branch cuts, 2 Riemann sheets required
  - Q branch cuts
  - genus (number of holes) g=Q-1 Riemann surface
  - g : number of the energy gaps

$$\phi = P/Q$$

Flux per plaquette

Complex energy surface of the Harper eq.




$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

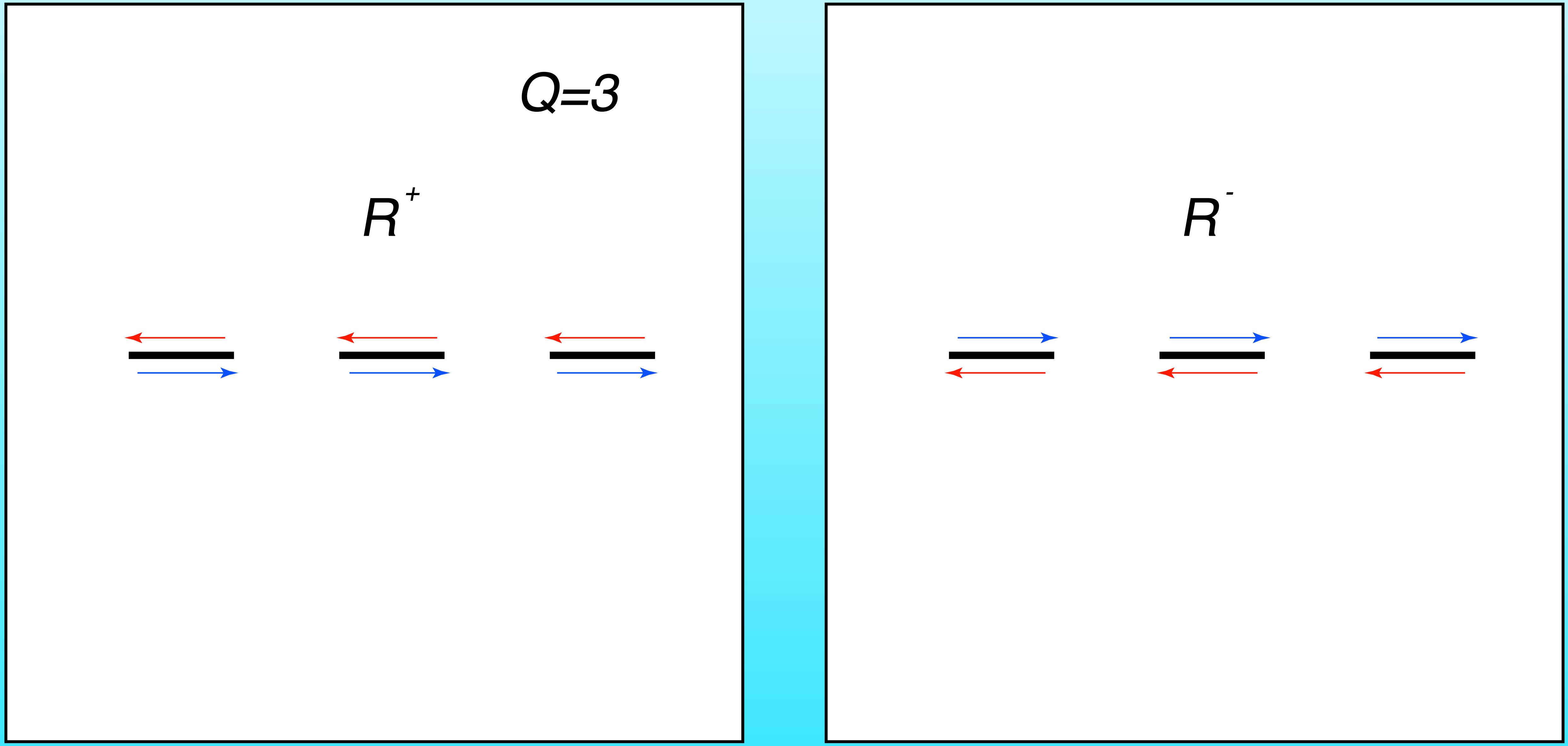
# Construction of the Riemann surface

$\phi = 1/3$

$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

- Glue 2 complex planes

Q=3 energy bands: Q=3 branch cuts
Q-1=2 energy gaps



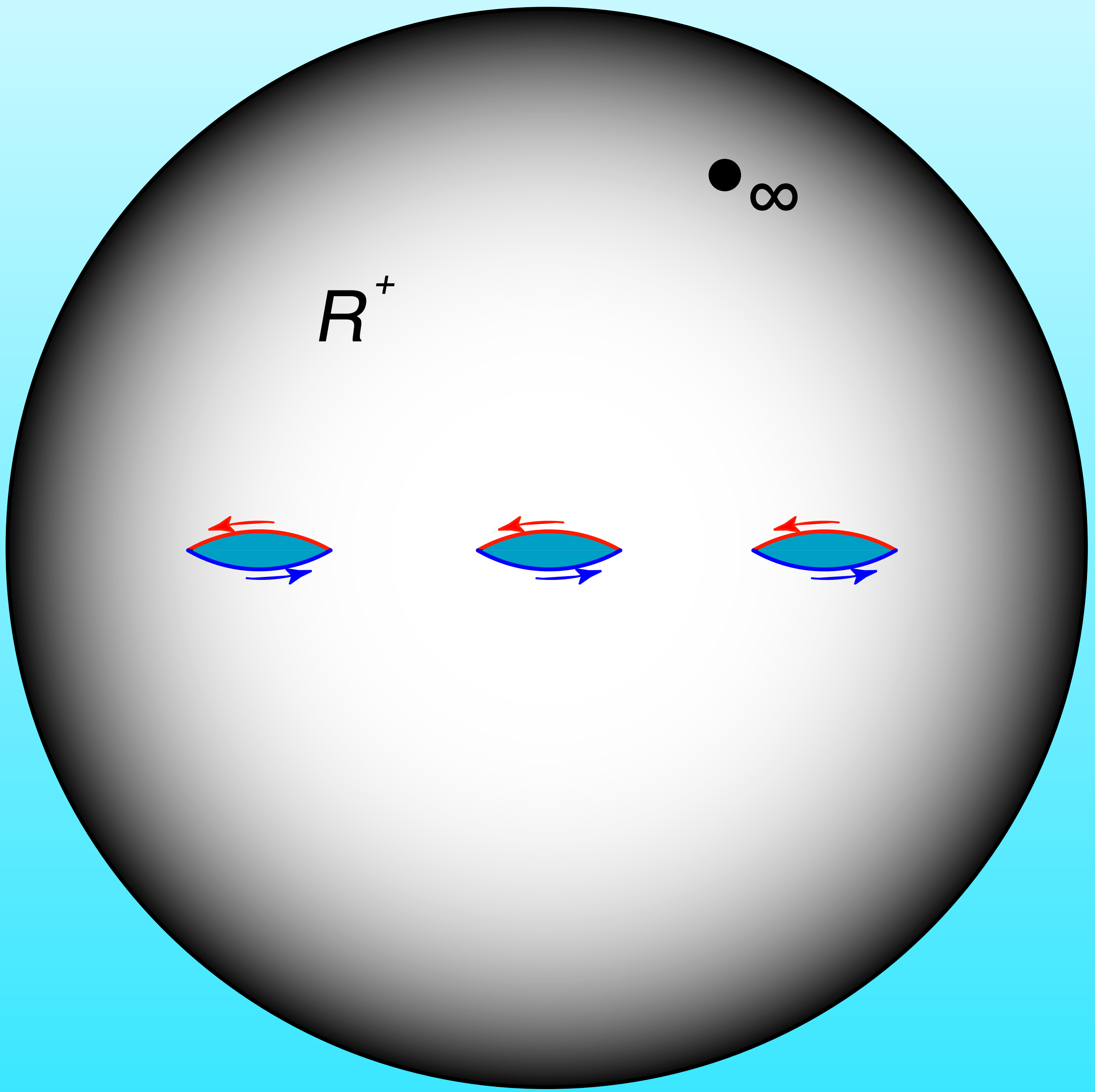
# Construction of the Riemann surface

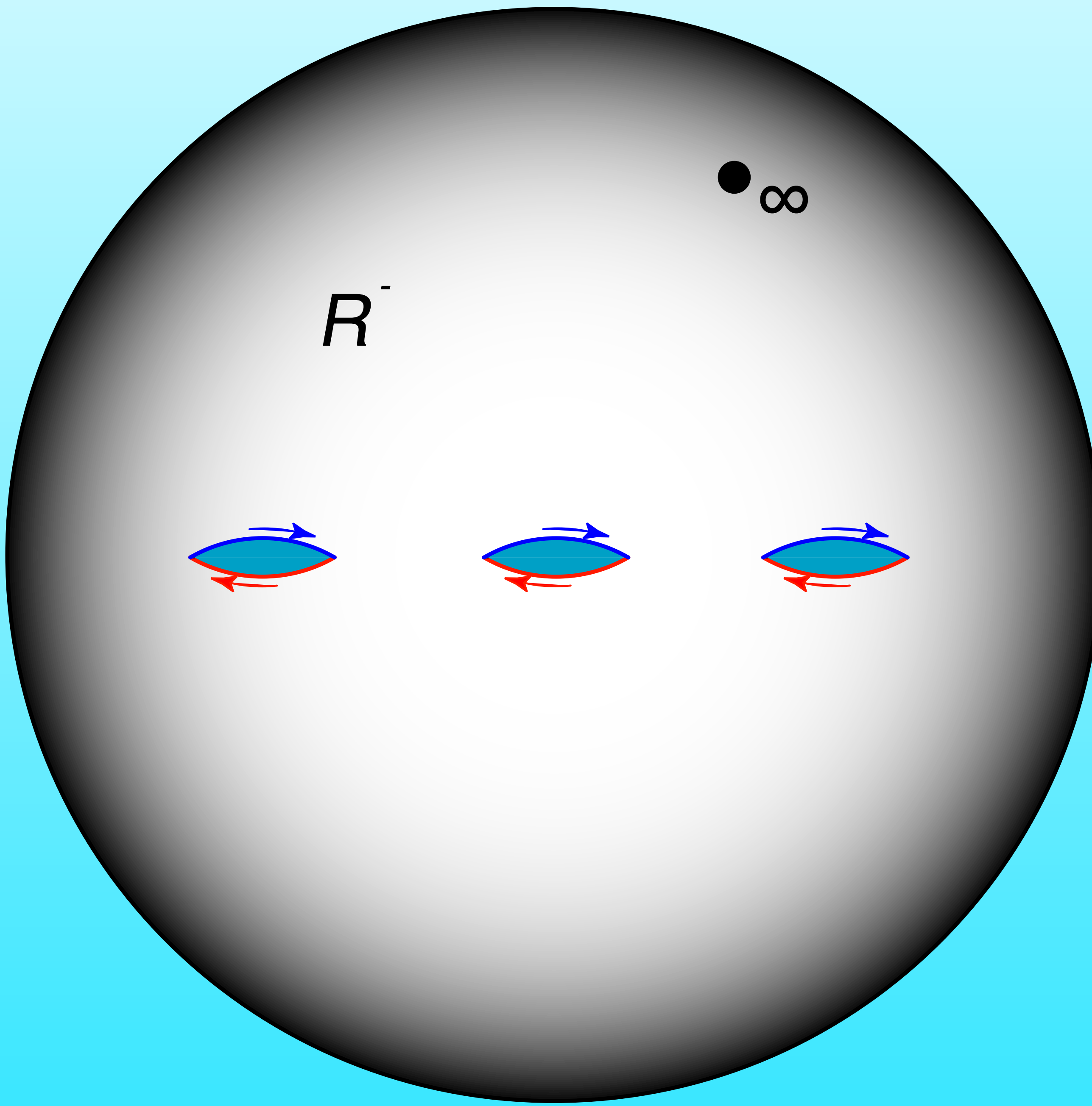
$\Phi = P/Q, \quad Q = 3$

$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$

- Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts




g=Q-1 holes

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$\Phi = P/Q, \quad Q = 3$

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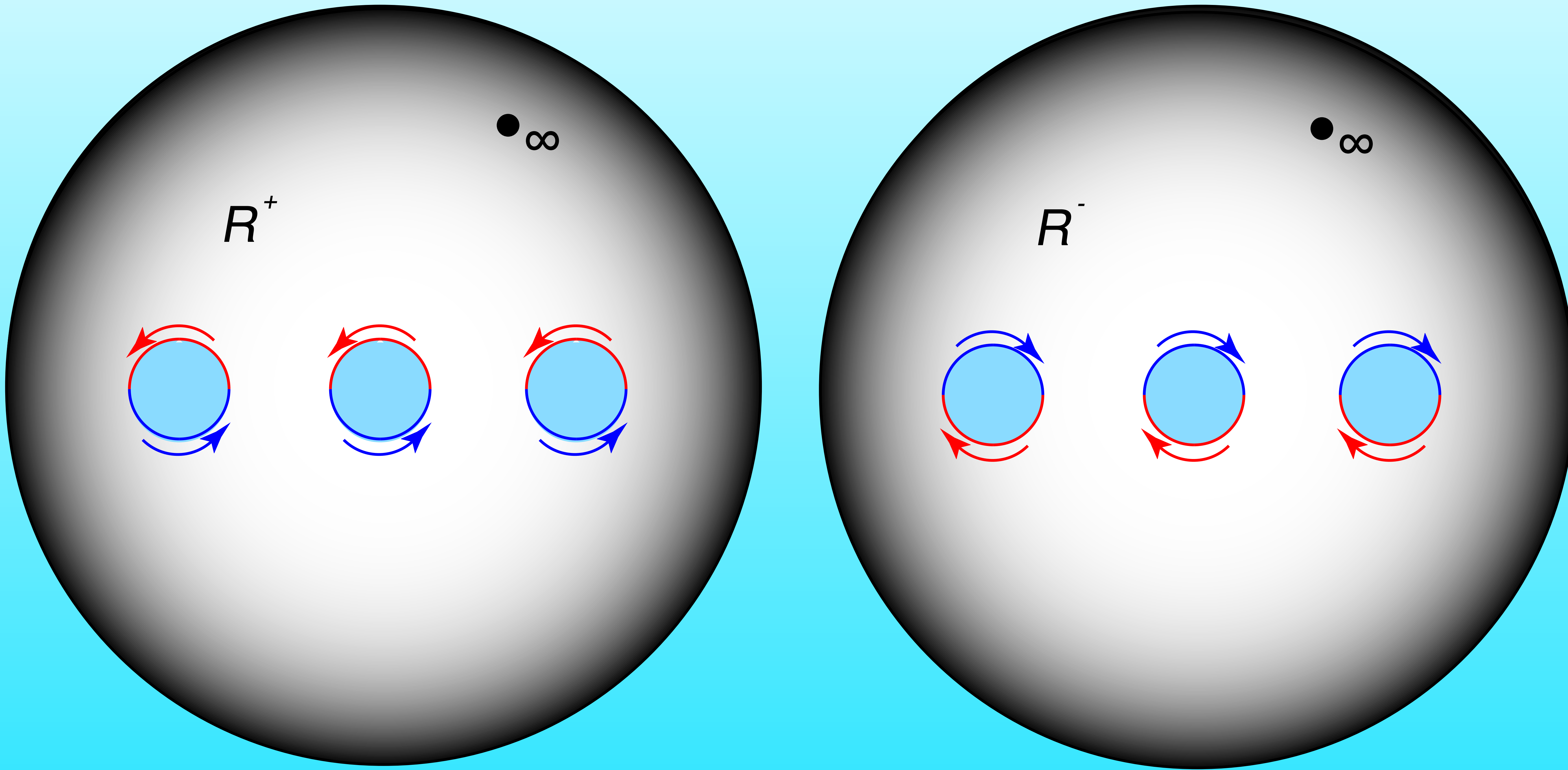


g=Q-1 holes

# Construction of the Riemann surface

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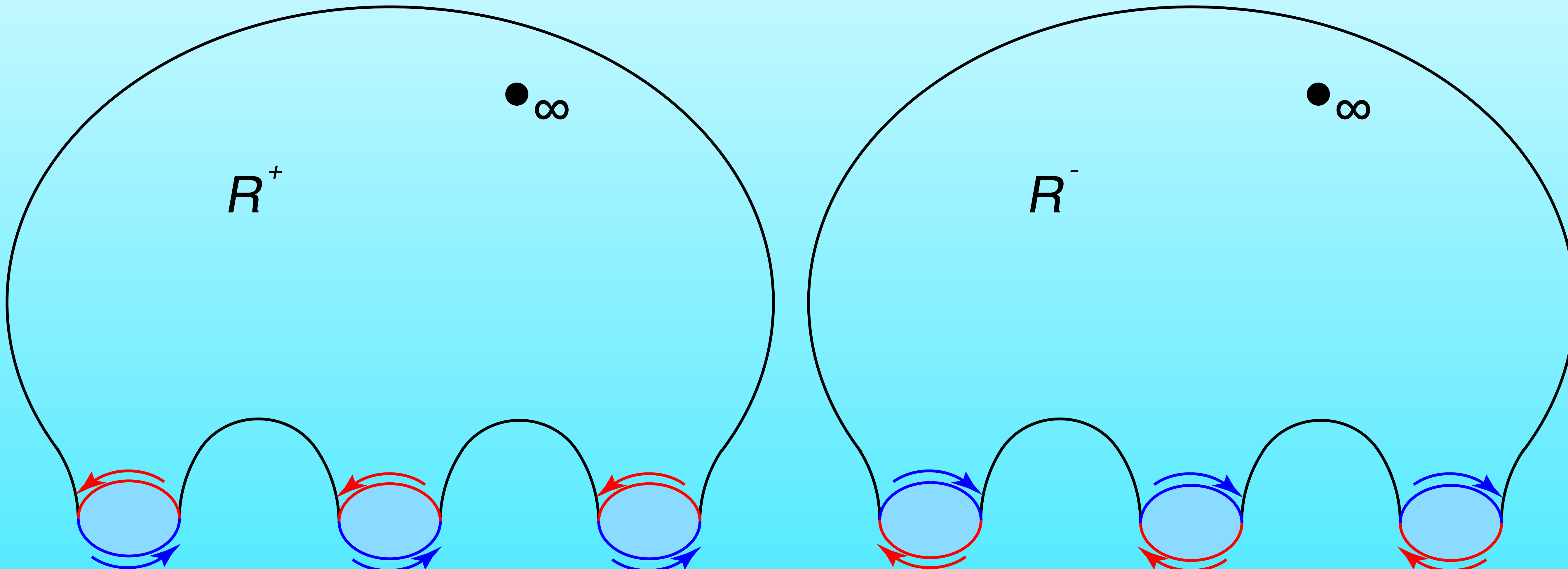



g=Q-1 holes

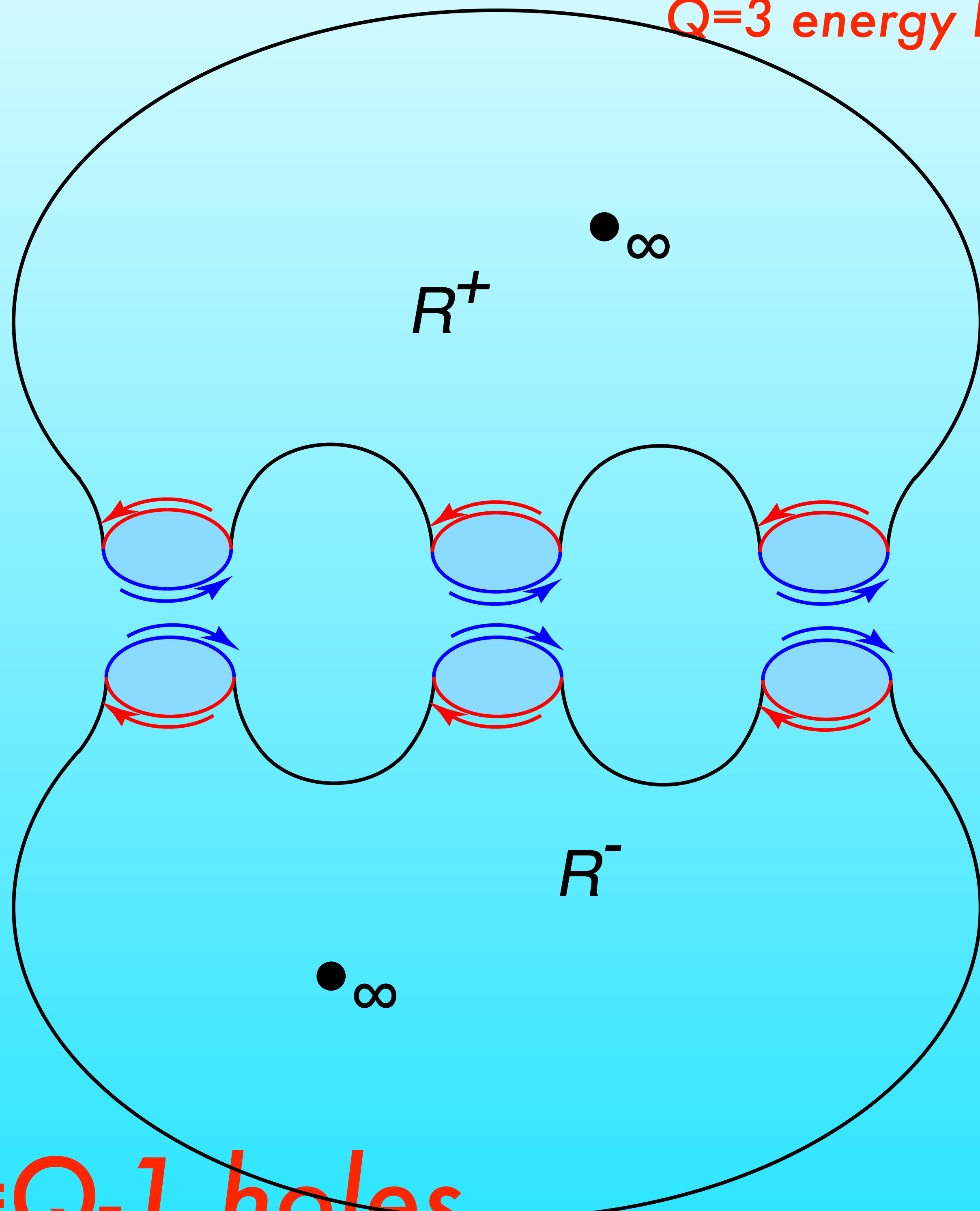
# Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$

$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

- ☆ Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts




g=Q-1 holes

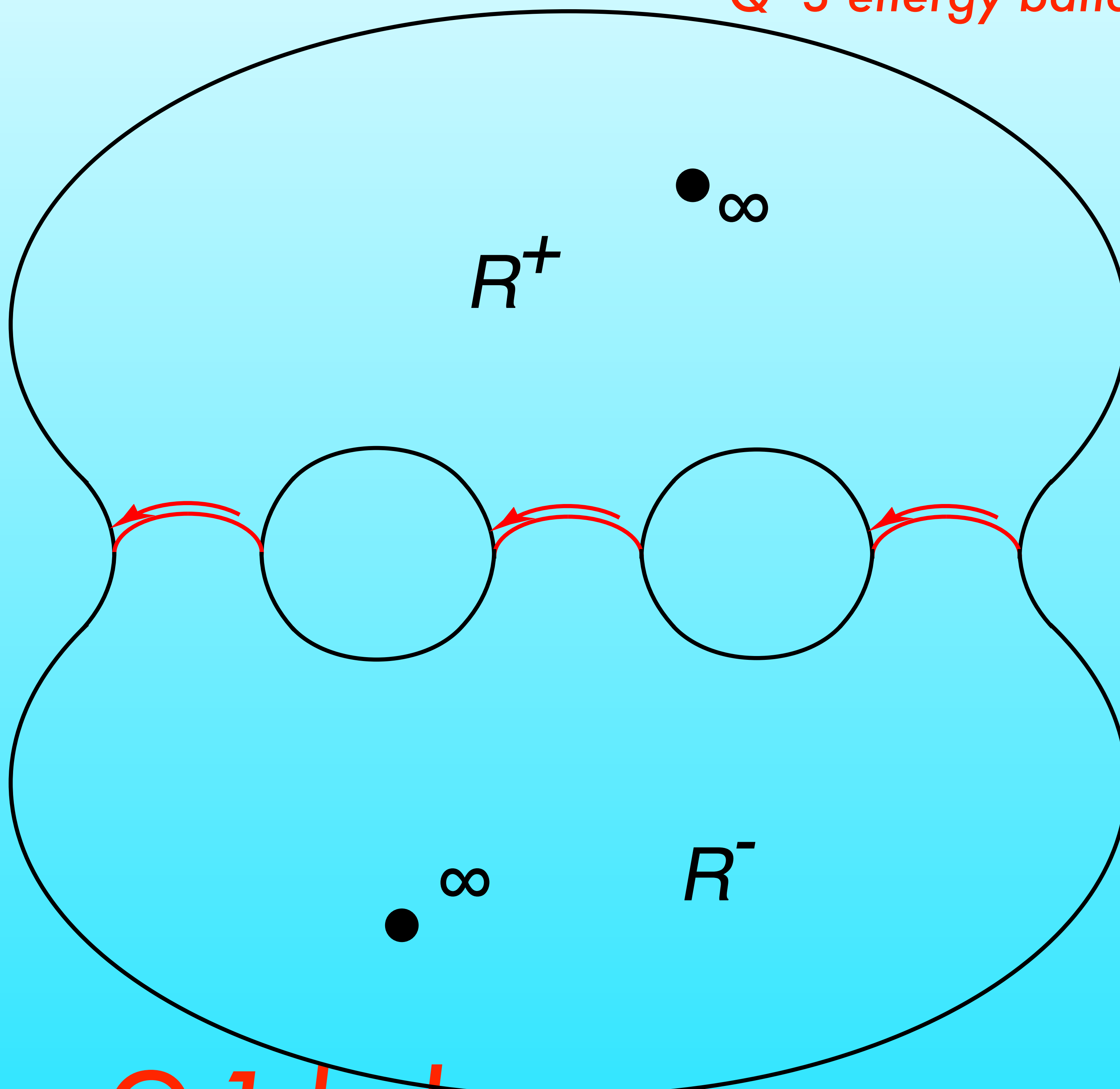
# Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$

$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

- Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts



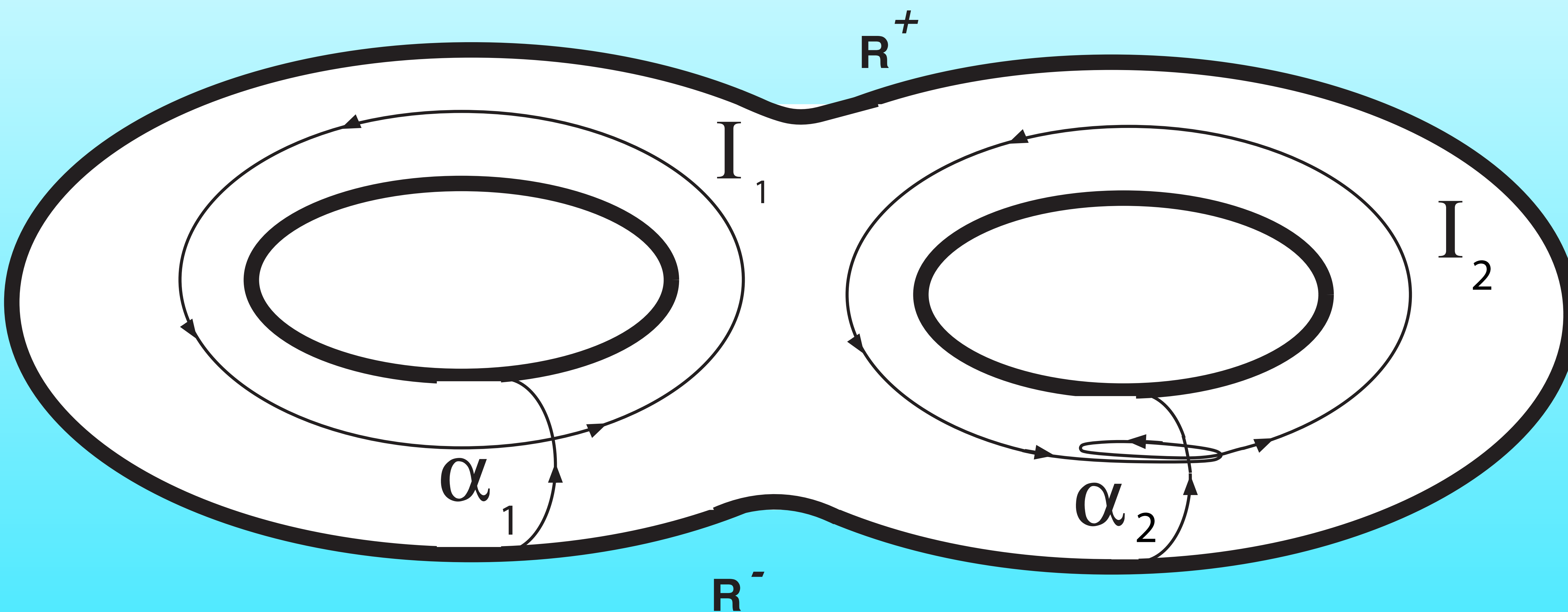

g=Q-1 holes

# Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$ $\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$

- Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts




g=Q-1=2 holes

Complex Energy surface of Harper eq.

# Wave function & Riemann Surface $\Sigma_g$

As for fixed $k_y$ of the 1D-Harper systems

- Zeros of the Bloch fn. defines the Edge State Energies

Energy bands ⟷ Branch cuts

Energy gaps ⟷ Holes

$\phi = P/Q$
$g = Q - 1$

W. fn. is localized at ⟷ The zero of the Bloch fn. is on

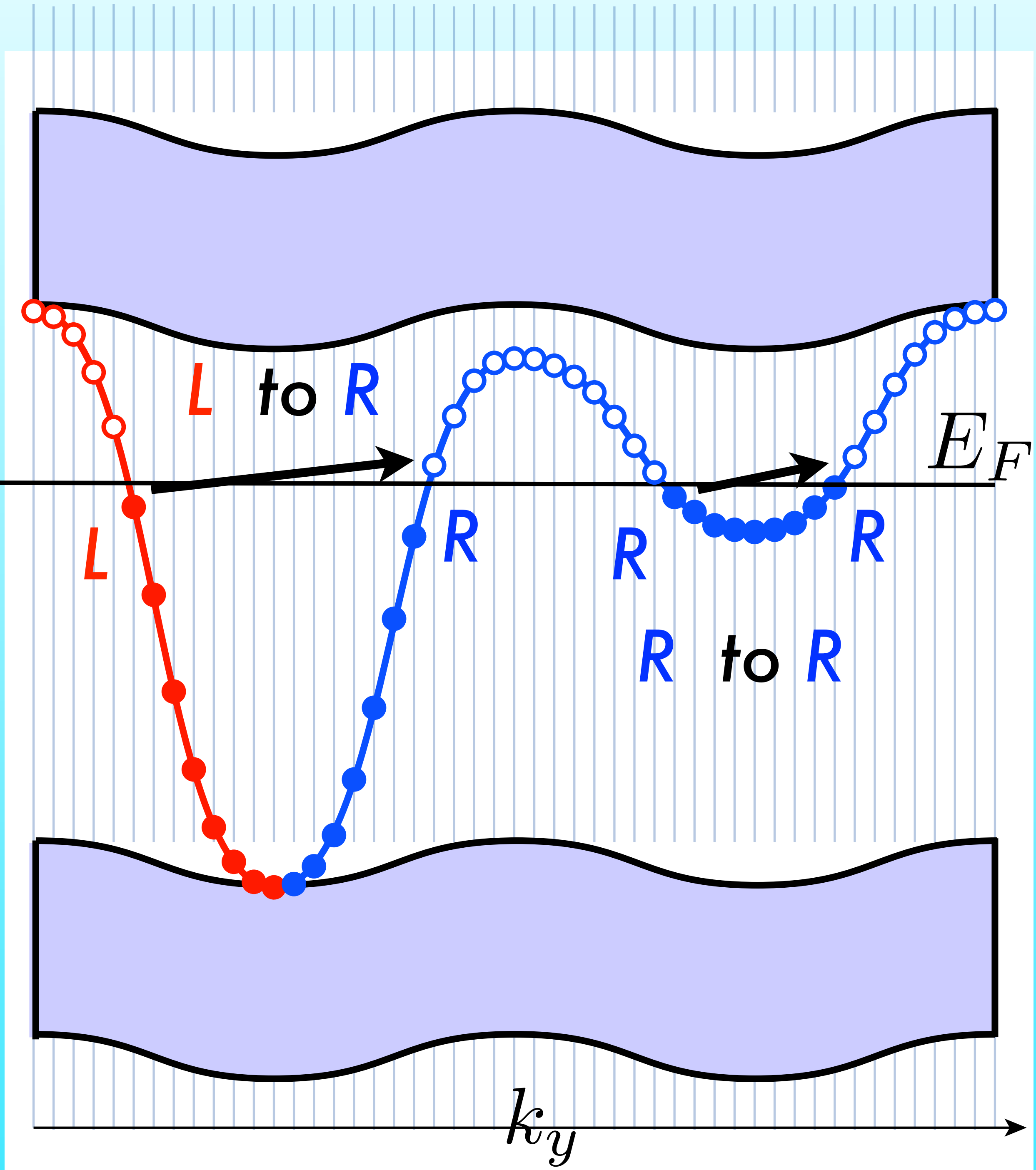
the left edge ⟷ the upper Riemann Surface $R^+$

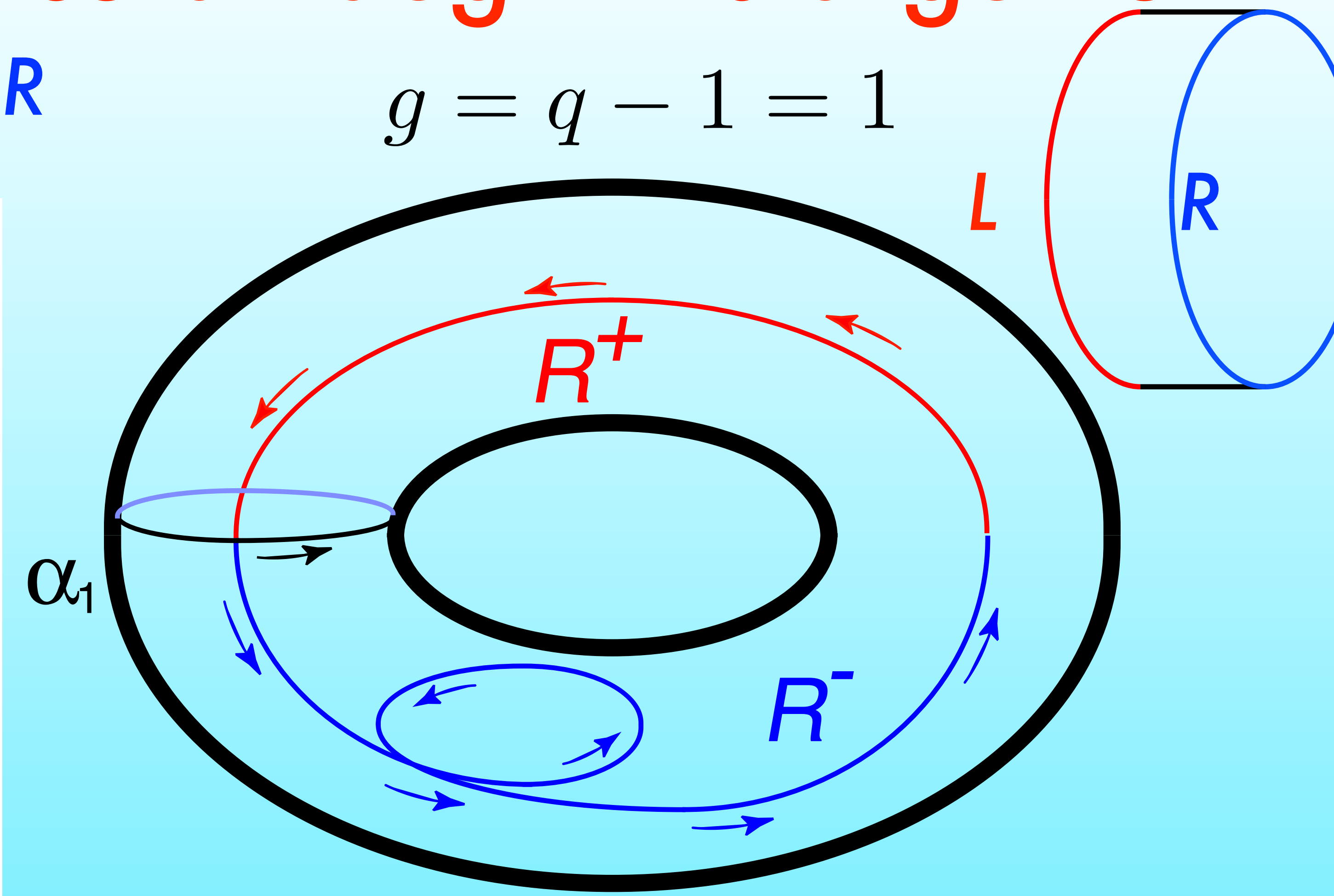
the right edge ⟷ the lower Riemann Surface $R^-$

- Changing $k_y \in [0, 2\pi]$ , the zero of the Bloch function in the j-th gap makes a closed loop on $\Sigma_g$

# Complex energy surface & Laughlin's argument

1 state is carried from the L to R



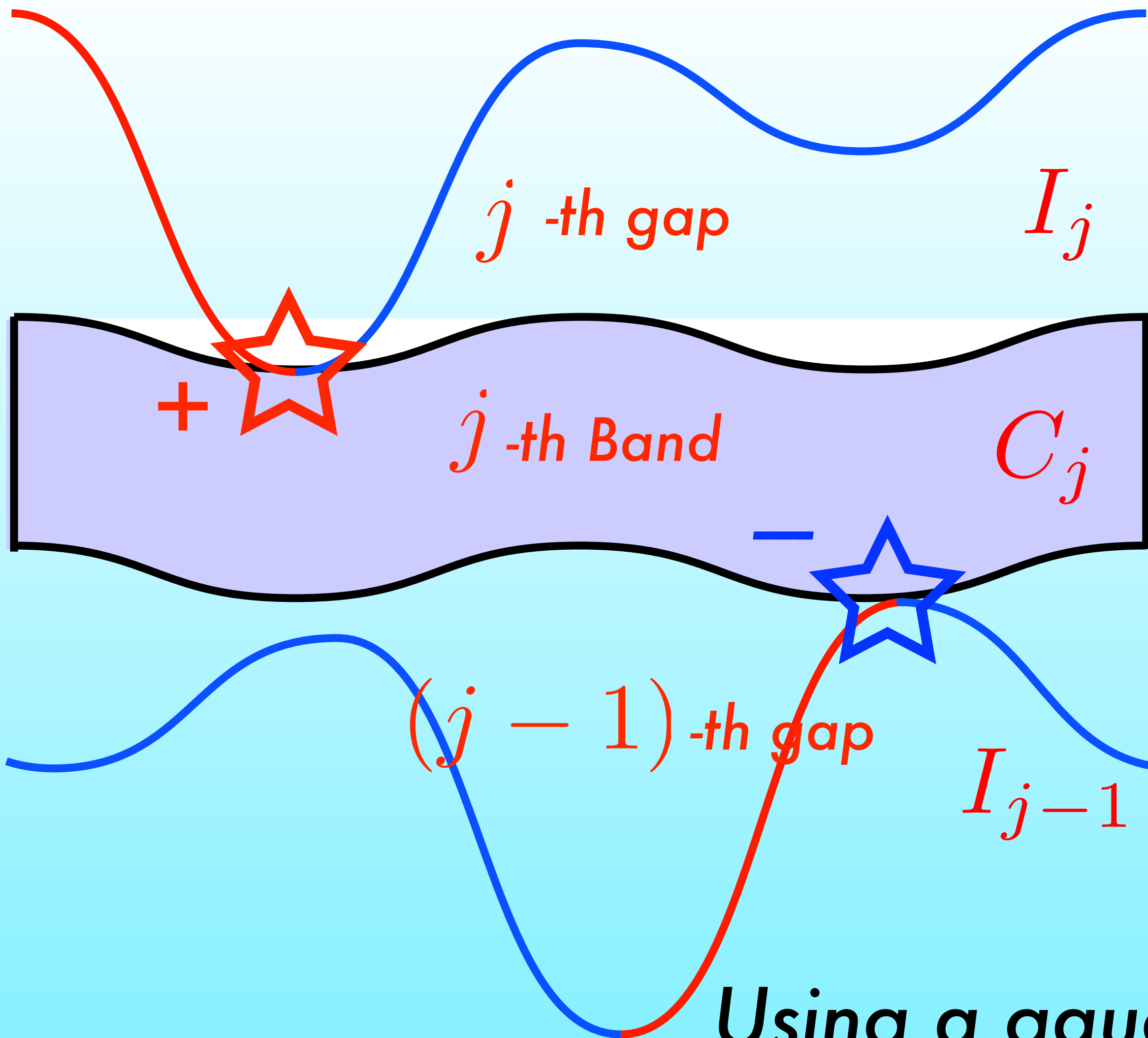

# of edge state connecting the bands

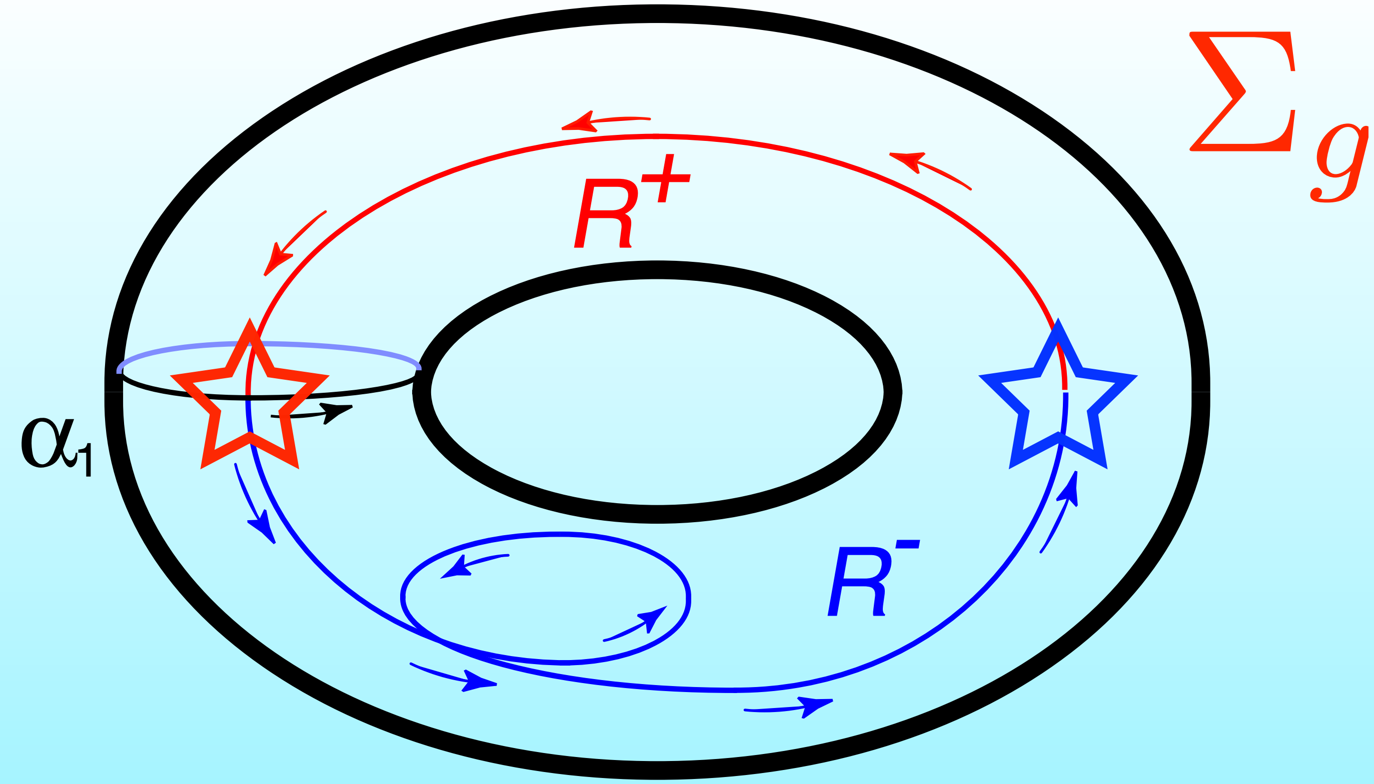
: winding number

$$I_1 = 1$$

Y.H., Phys. Rev. B 48, 11851 (1993)

Unknown " $n$ " is a winding # of edge states




Y.H., Phys. Rev. Lett. 71, 3697 (1993)

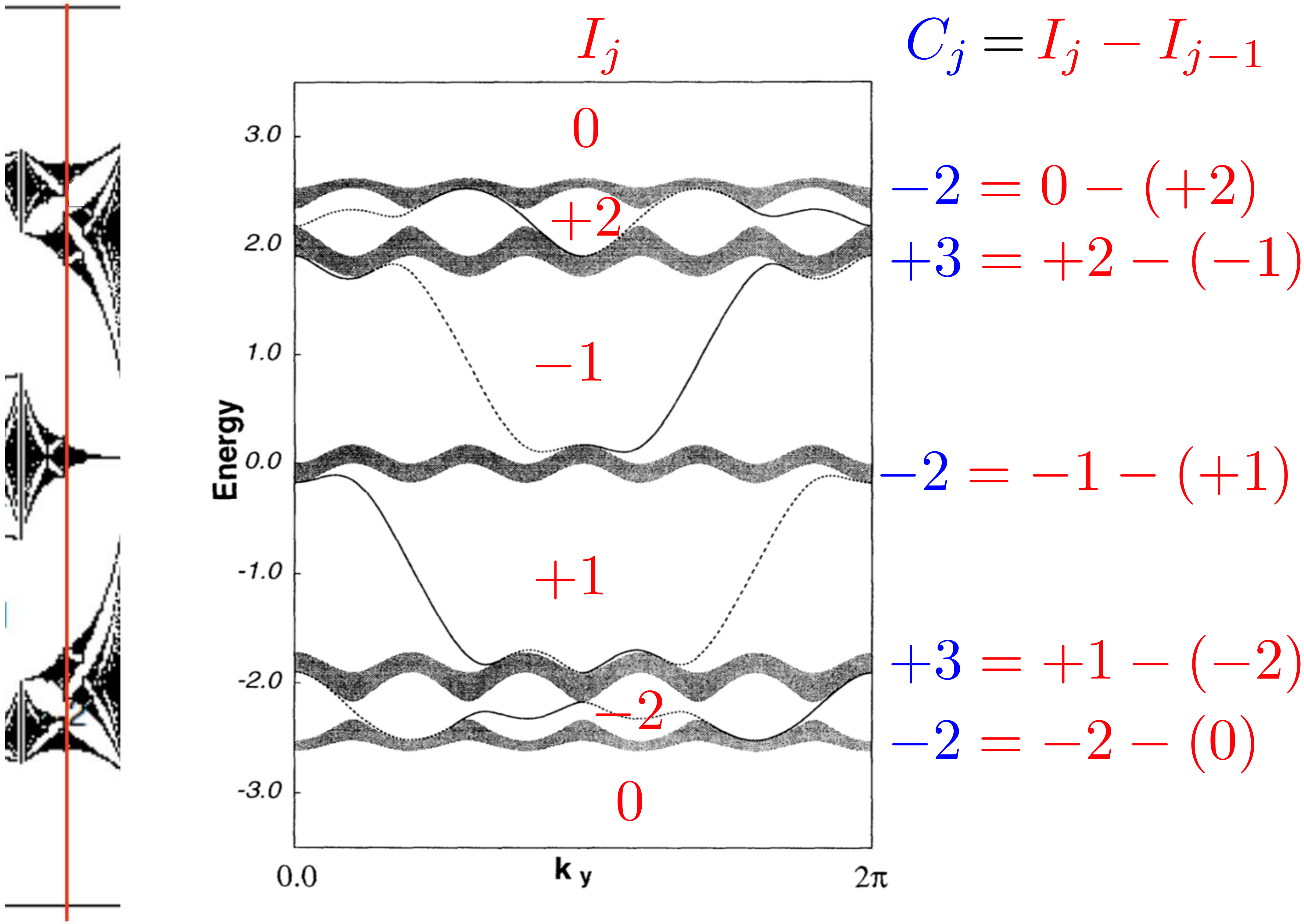
Using a gauge fixing condition of the Berry connection

$$C_j = I_j - I_{j-1}$$

$$\sum_{j=1,\cdots\ell} C_j = I_\ell, \ (\because I_0 = 0)$$

$$\boxed{{\sigma_{xy}}^{\text{bulk}} = {\sigma_{xy}}^{\text{edge}}}$$

Bulk-Edge Correspondence in their topological numbers



$I_j$
$C_j = I_j - I_{j-1}$
0
$-2 = 0 - (+2)$
+2
$+3 = +2 - (-1)$
−1
$-2 = -1 - (+1)$
+1
$+3 = +1 - (-2)$
−2
$-2 = -2 - (0)$
0
Energy
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
k y
0.0
2π

# Plan

- Topological phases and bulk-edge correspondence
  - Hofstadter problem to TKNN
  - Edge states
  - Bulk-edge correspondence
- QHE a bit more (old)
  - Laughlin argument
  - Edge states (Halperin '82)
  - Bulk-edge correspondence '93
- Topological pumping
  - Bulk-edge correspondence '16
  - Stability for randomness '18

"Bulk-edge correspondence in a topological pumping",
Y.Hatsugai & T. Fukui, Phys. Rev. B 94, 041102(R), (2016)

- ## Topological pumping
  - Back to Thouless
  - Time as a synthetic dimension of QHE
  - Experimental realizations after 30 years
  - Edge states ?
- ## Pumped charge & Berry connection
  - Pumped charge, Berry connection & Temporal gauge
- ## Pumped charge & edge states
  - Temporal gauge & center of mass (CM)
  - Singular motion of CM
  - The Chern number & Bulk-Edge-Correspondence
- ## Observations
  - Adiabatic for a bulk & sudden approximation for edges

# Adiabatic pump (Thouless '83)

## Periodically driven1D charge transport

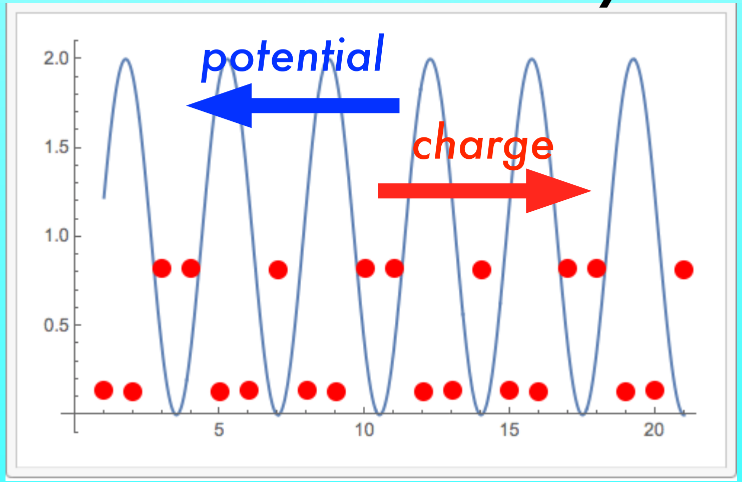
Many-body but non-interacting as IQHE

$$i\hbar\partial_t|G(t)\rangle = H(t)|G(t)\rangle \quad |G(t)\rangle = Te^{-(i/\hbar)\int_{t_0}^{t} d\tau H(\tau)}|G(t_0)\rangle$$

$$H(t) = \sum_{j}^{L}\left[-t_x c_{j+1}^{\dagger}c_j + h.c. + \underline{v_j(t)}c_j^{\dagger}c_j\right]$$

free fermion

manybody




$$v_j(t+T) = v_j(t)$$ period $T$

**ex.** $$v_j(t) = -2t_y\cos(2\pi\frac{t}{T} - 2\pi\phi j)$$

$$\phi = p/q$$

Adiabatic : ground state is gapped & slow pumping

$$\boxed{\Delta E \gg \hbar/T}$$

Topological !

Pumped charge is quantized as an integer

Workshop, ETH Zürich, 3rd-6th September 2018

# Adiabatic pump (Thouless '83)

## Periodically driven1D charge transport

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$$\mathrm{i}\hbar\partial_t|G(t)\rangle = H(t)|G(t)\rangle \quad |G(t)\rangle = Te^{-(\mathrm{i}/\hbar)\int_{t_0}^{t} d\tau H(\tau)}|G(t_0)\rangle$$

$L$

$H(t$ … $+ v_j(t)c_j^\dagger c_j]$

free fermion

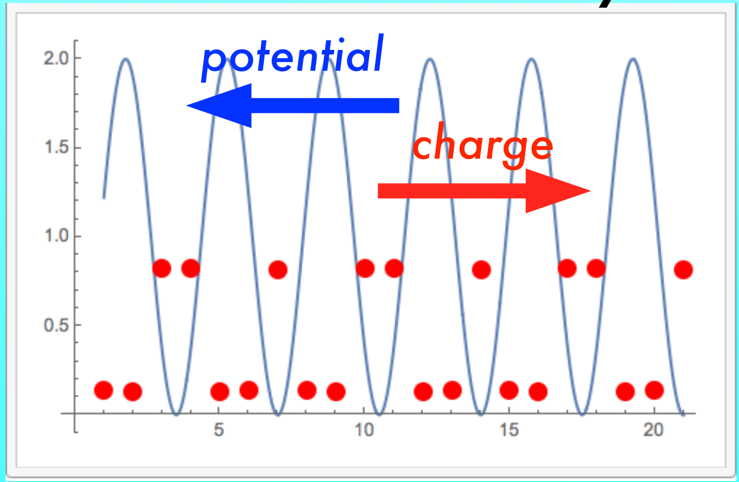
manybody

$v_j(t$ …

ex.

$q$

a bit cheating :
flipbook style animation
by the ground state of
"snapshot" hamiltonian
is justified

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# Back to Thouless '83

Time dependent 1D charge transport

PHYSICAL REVIEW B VOLUME 27, NUMBER 10 15 MAY 1983

1+1=2

Time as a synthetic dimension

Quantization of particle transport

D. J. Thouless

*Department of Physics, FM-15, University of Washington, Seattle, Washington 98195*

(Received 4 February 1983)

2D Integer quantum Hall effect

TKNN '82 Hall conductance by the Chern number

Brouwer '98

...

Wang-Troyer-Dai '13

Marra-Citro-Ortix '15

**Experimentally realized in cold atoms after 30+ years in '15**

Y.Takahashi, Kyoto

*Nakajima et al.,*

*Nature Phys. 12, 296 (2016)*

**Topological Thouless Pumping of Ultracold Fermions**

Shuta Nakajima, Takafumi Tomita, Shintaro Taie, Tomohiro Ichinose, Hideki Ozawa, Lei Wang, Matthias Troyer, Yoshiro Takahashi

I. Bloch, Munich

*Lohse et al.,*

*Nature Phys. 12, 350 (2016)*

**A Thouless Quantum Pump with Ultracold Bosonic Atoms in an Optical Superlattice**

Michael Lohse, Christian Schweizer, Oded Zilberberg, Monika Aidelsburger, Immanuel Bloch

motivation of our work : revisit the problem (edge state?)

# QHE (Hofstadter's) on 2D cylinder & TP

$$H(t) = \sum_{j}^{L} [ -t_x c_{j+1}^\dagger c_j + h.c. + v_j(t) c_j^\dagger c_j ]$$

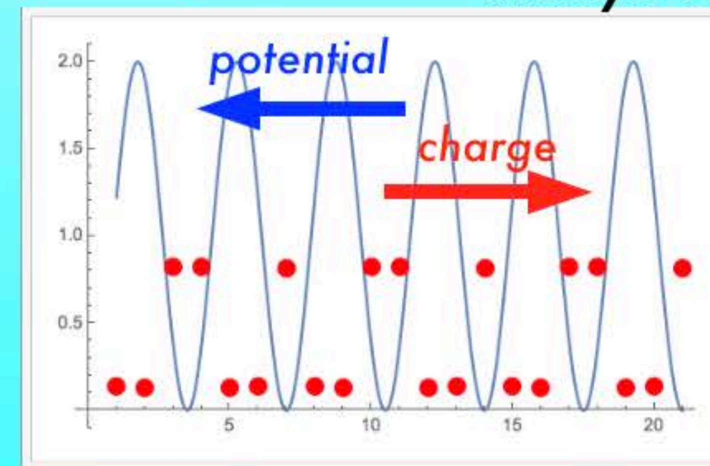
free fermion

manybody

$v_j(t+T) = v_j(t)$ period $T$

ex. $v_j(t) = -2t_y \cos(2\pi \frac{t}{T} - 2\pi\phi j)$

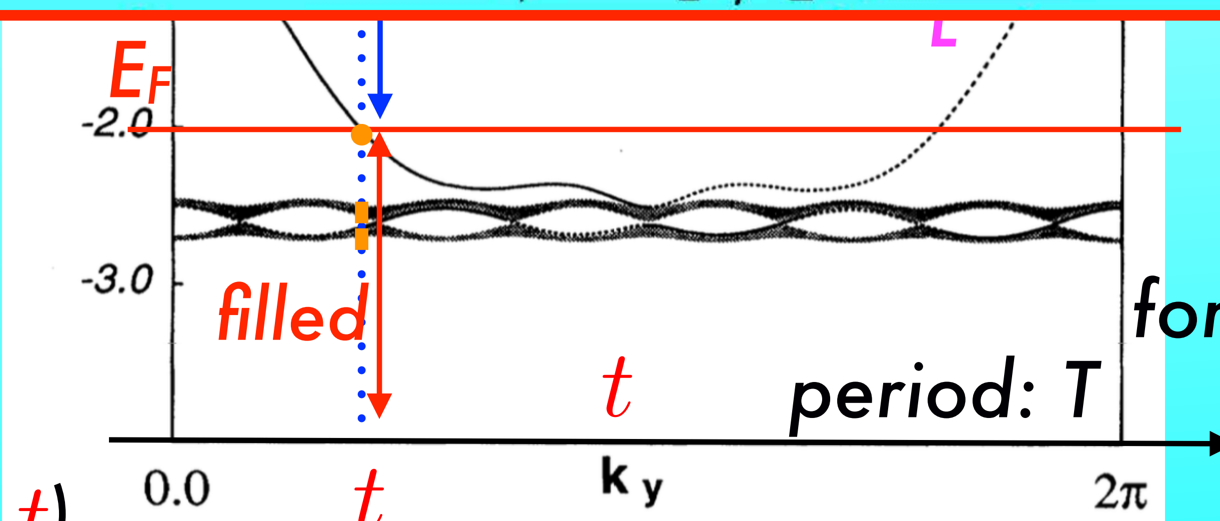
$\phi = p/q$




Periodic in

$$\boxed{k_y \to 2\pi \frac{t}{T}}$$

periodic pumping in 1D




Filling states below $E_F$ for the snapshot Ham.

Harper eq. (1D for each $t$)

$$-t_x(\psi_{m+1} + \psi_{m-1}) - 2t_y \cos(k_y - 2\pi \frac{p}{q} m)\psi_m = E\psi_m$$

$$-t_x(\psi_{m+1} + \psi_{m-1}) - 2t_y \cos(2\pi \frac{t}{T} - 2\pi \frac{p}{q} m)\psi_m = E\psi_m$$

# The pumping is topological ! (Thouless)

$\mathcal{O}(N^0)$ charge is pumped for an insulator

with $N$ particles

Macroscopic

Large 1D system

Pumped charge is quantized if gapped

Independent of the parameters

# Adiabatic pump (Thouless '83)

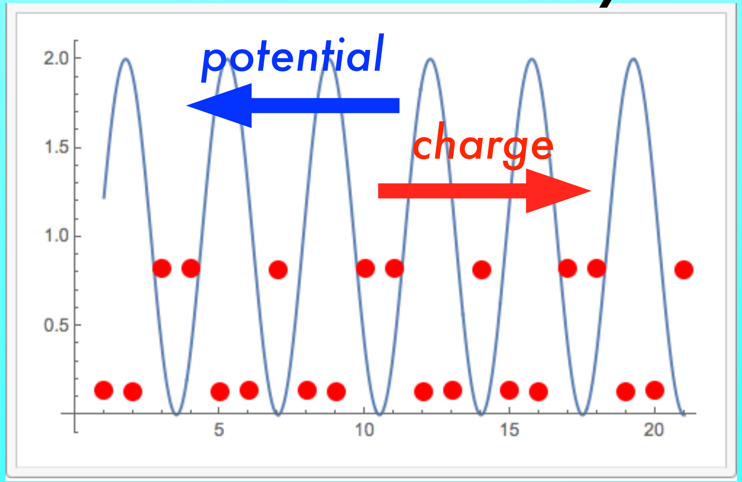
## Periodically driven1D charge transport

$$\mathrm{i}\hbar\partial_t|G(t)\rangle = H(t)|G(t)\rangle \quad |G(t)\rangle = Te^{-(\mathrm{i}/\hbar)\int_{t_0}^{t} d\tau H(\tau)}|G(t_0)\rangle$$

$$H(t) \quad = \quad \sum_{j}^{L}\left[-t_x c_{j+1}^\dagger c_j + h.c. + \underline{v_j}(t)c_j^\dagger c_j\right]$$

free fermion

manybody

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Adiabatic : ground state is gapped & slow pumping

$$\boxed{\Delta E \gg \hbar/T}$$

Topological !

Pumped charge is quantized as an integer

# Pumped charge by adiabatic approximation

Thouless '83: (Periodic in 1D)  The same with boundaries

$$j = \langle G|J|G\rangle \qquad H(\theta,t) = \sum_j^L \left[ -t_x e^{-\mathrm{i}\frac{\theta}{L_x}} c_{j+1}^\dagger c_j + h.c. + v_j(t) c_j^\dagger c_j \right]$$

$$J = \frac{1}{L_x}\left(+\mathrm{i}\frac{t_x}{\hbar} e^{-\mathrm{i}\theta/L_x}\right)\sum_j c_{j+1}^\dagger c_j + h.c$$

twist $k_x \sim \theta/L_x$ if periodic

$$= +\hbar^{-1}\partial_\theta H(\theta)$$

$|\alpha(t)\rangle$ : Snapshot eigen state

$$H(t)|\alpha(t)\rangle = E_\alpha(t)|\alpha(t)\rangle, \quad \langle\alpha|\beta\rangle = \delta_{\alpha\beta}.$$

$$|G\rangle = e^{-(\mathrm{i}/\hbar)\int_0^t dt' E_g(t')} e^{\mathrm{i}\gamma(t)} \left[ |g\rangle + \mathrm{i}\hbar \sum_{\alpha\neq g} \frac{|\alpha\rangle\langle\alpha|\partial_t g\rangle}{E_\alpha - E_g} \right]$$

$$|g(t)\rangle : \quad H(t)|g(t)\rangle = E(t)|g(t)\rangle$$

snapshot eigen state

$$\boxed{\delta j_x = \langle G|J|G\rangle - \langle g|J|g\rangle = -\mathrm{i}B}$$

$B$ is invariant for the phase choice of $|g\rangle$ : gauge freedom

$$B = \partial_\theta A_t - \partial_t A_\theta, \; A_\mu = \langle g|\partial_\mu g\rangle, \quad \mu = \theta, t.$$

Berry connection

# Pumped charge & Berry connection

Thouless '83

**Pumped charge in $T$** Adiabatic approximation

$$\boxed{\Delta Q = \int_0^T dt\, \delta j_x = -\mathrm{i}\int_0^T dt\, B}$$

$$B = \partial_\theta A_t - \partial_t A_\theta$$

Berry connection $A_\mu = \langle g|\partial_\mu g\rangle, \quad \mu = t, \theta$

twist

$$\boxed{t_x \to t_x e^{-\mathrm{i}\theta/L}}$$

$$|g(t)\rangle : \quad H(t)|g(t)\rangle = E(t)|g(t)\rangle$$

snapshot ground state

$B$ is invariant for the phase choice of $|g\rangle$ : gauge freedom

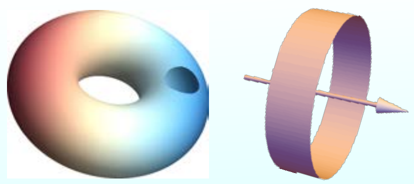
$$\boxed{\text{Temporal gauge: } A_t^{(t)} = 0}$$

$$B = \cancel{\partial_\theta A_t} - \partial_t A_\theta$$

$$\Delta Q = \mathrm{i}\int_0^T dt\, \partial_t A_\theta^{(t)} = \mathrm{i}\left[A_\theta^{(t)}(T) - A_\theta^{(t)}(0)\right]$$

Physical observable

Berry connection (gauge fixed)





# Temporal gauge

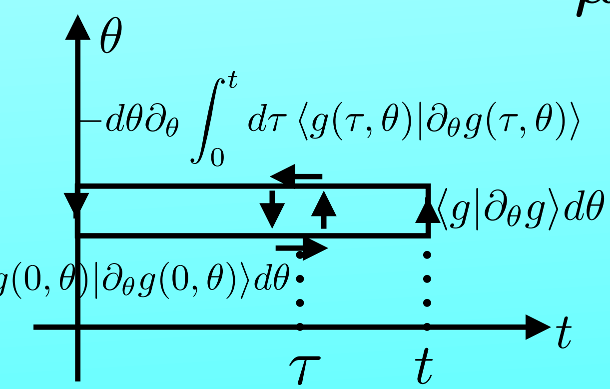
$\boxed{\text{Temporal gauge: } A_t^{(t)} = 0}$ $\quad B = \cancel{\partial_\theta A_t} - \partial_t A_\theta$

Gauge transformation $\langle g'|\partial_\mu g'\rangle = \langle g|\partial_\mu g\rangle + i\partial_\mu \chi$, $|g'\rangle = |g\rangle e^{i\chi}$

temporal general

$$A_\mu^{(t)}(t,\theta) = A_\mu(t,\theta) + \mathrm{i}\partial_\mu \chi(t,\theta)$$

$C : (0,0) \to (0,\theta) \to (t,\theta)$



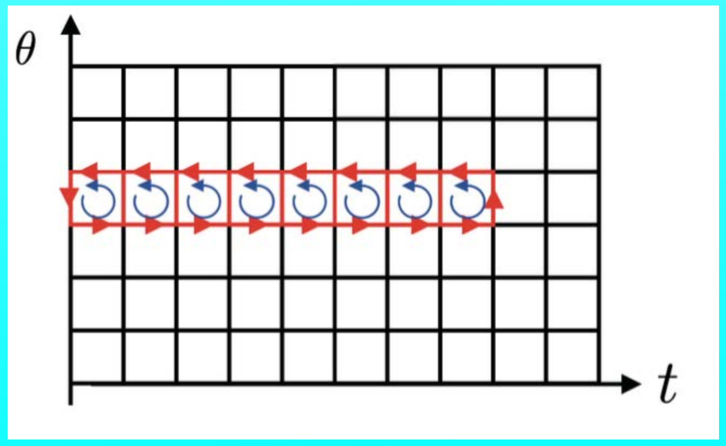

$$\chi(t,\theta) = \mathrm{i}\int_0^t d\tau \, A_t(\tau,\theta) + \mathrm{i}\int_0^\theta d\vartheta \, A_\theta(0,\vartheta)$$

one can always take a temporal gauge

$$\because A_t^{(t)}(t,\theta) = A_t(t,\theta) + \mathrm{i}\partial_t \chi(t,\theta) = 0$$

$$A_\theta^{(t)}(t,\theta) = A_\theta(t,\theta) + \mathrm{i}\partial_\theta \chi(t,\theta)$$

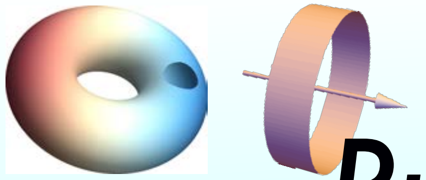
$$= A_\theta(t,\theta) - A_\theta(0,\theta) - \partial_\theta \int_0^t d\tau \, A_t(\tau,\theta)$$




$$A_\theta^{(t)}(t,\theta) \neq A_\theta^{(t)}(t+T,\theta)$$

non periodic gauge fixing

$$H(t,\theta) = H(t+T,\theta)$$





# Pumped charge & Center of mass (CM)

$$H(\theta,t) \quad = \quad \sum_{j}^{L} \left[ -t_x e^{-\mathrm{i}\frac{\theta}{L_x}} c_{j+1}^\dagger c_j + h.c. + v_j(t) c_j^\dagger c_j \right]$$

$$\mathcal{U} c_j \mathcal{U}^\dagger = e^{+\mathrm{i}\theta j/L_x} c_j$$
$$\mathcal{U} c_j^\dagger \mathcal{U}^\dagger = c_j^\dagger e^{-\mathrm{i}\theta j/L_x}$$

twist $\theta$ : is gauged out for an open system (with edges)

$$\boxed{H(\theta,t) = \mathcal{U} H(0,t) \mathcal{U}^\dagger}$$

$$H(\theta,t)|g(\theta)\rangle = |g(\theta)\rangle E$$
$$H(0,t)|g(0)\rangle = |g(0)\rangle E$$

$$\boxed{|g(\theta)\rangle \quad = \quad \mathcal{U}(\theta)|g(0)\rangle}$$

$$\mathcal{U}(\theta) = \prod_{j=1} e^{-\mathrm{i}\theta n_j (j-j_0)/L_x} = \prod_{j=1} e^{-\mathrm{i}\theta x_j n_j}$$

$$\boxed{x_j = \frac{j-j_0}{L_x} \in [-\frac{1}{2}, \frac{1}{2}]}$$
$$j = 1, \cdots, L_x, \ j_0 = \frac{L_x}{2}$$

Large gauge transformation

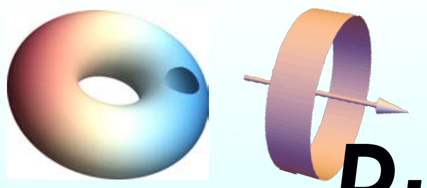
macroscopic system is rescaled to [ -1/2,+1/2]

$$A_\theta \quad = \quad \langle g(\theta)|\partial_\theta g(\theta)\rangle = \langle g(0)|\mathcal{U}^\dagger \partial_\theta \mathcal{U}|g(0)\rangle$$
$$= \quad -\mathrm{i}\sum_j x_j \rho_j = -\mathrm{i}P(t) \quad \theta\text{-independent}$$

$$\rho_j = \langle g(t)|n_j|g(t)\rangle$$

$$P(t) = \sum_j x_j \rho_j$$ : center of mass (CM)

$$\sum_j \rho_j = N$$ number of particles





# Pumped charge & Center of mass (CM)

$$H(\theta,t) \;=\; \sum_j^L \left[-t_x e^{-\mathrm{i}\frac{\theta}{L_x}} c_{j+1}^\dagger c_j + h.c. + v_j(t) c_j^\dagger c_j\right]$$

twist $\theta$ : gauged out for an open system (<u>with edges</u>)

$$A_\theta = -\mathrm{i}P(t) \qquad P(t) = \sum_j x_j \rho_j$$

$\theta$-independent

Center of mass (CM)

$$\sum_j \rho_j = N$$

to the temporal gauge

$P(t) = \mathcal{O}(N^0)$ insulator

$$\begin{aligned} A_\theta^{(t)}(t,\theta) &= A_\theta(t,\theta) - A_\theta(0,\theta) - \cancel{\partial_\theta \int_0^t d\tau A_t(\tau,\theta)} \\ &= -\mathrm{i}[P(t) - P(0)] \end{aligned}$$

$\theta$-independent

$$\Delta Q \;=\; \mathrm{i}\int_0^T dt\, \partial_t A_\theta^{(t)} = \mathrm{i}\left[A_\theta^{(t)}(T) - A_\theta^{(t)}(0)\right]$$

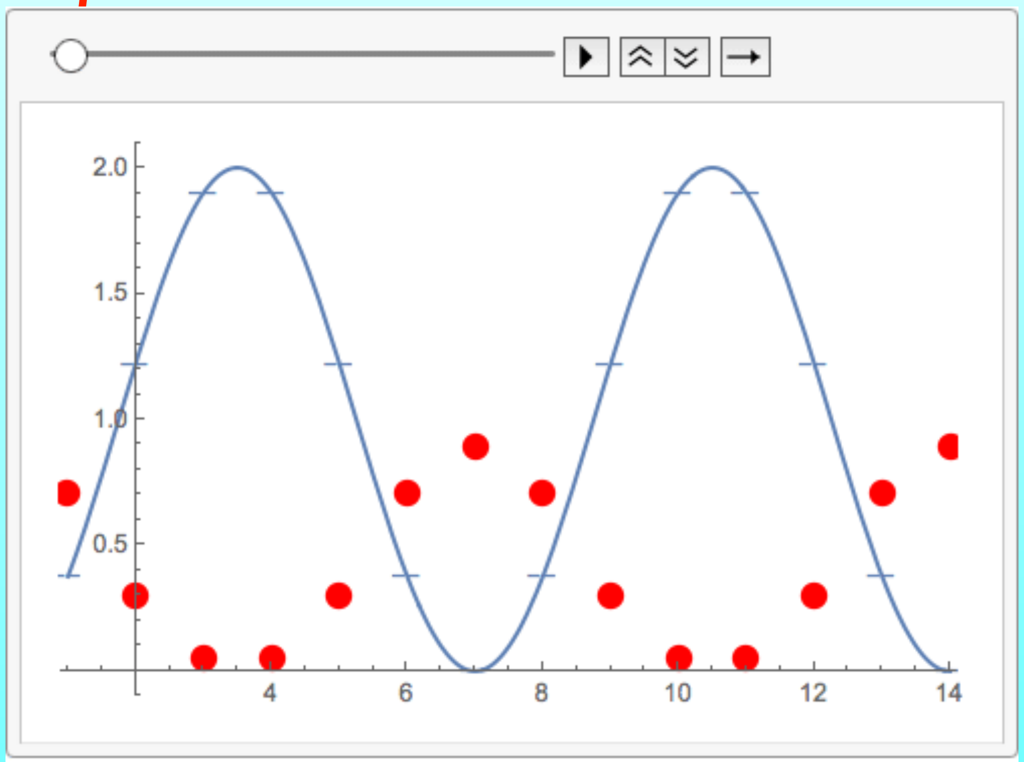
Shift of CM

Pumped charge $\qquad \boxed{\Delta Q = P(T) - P(0)}$

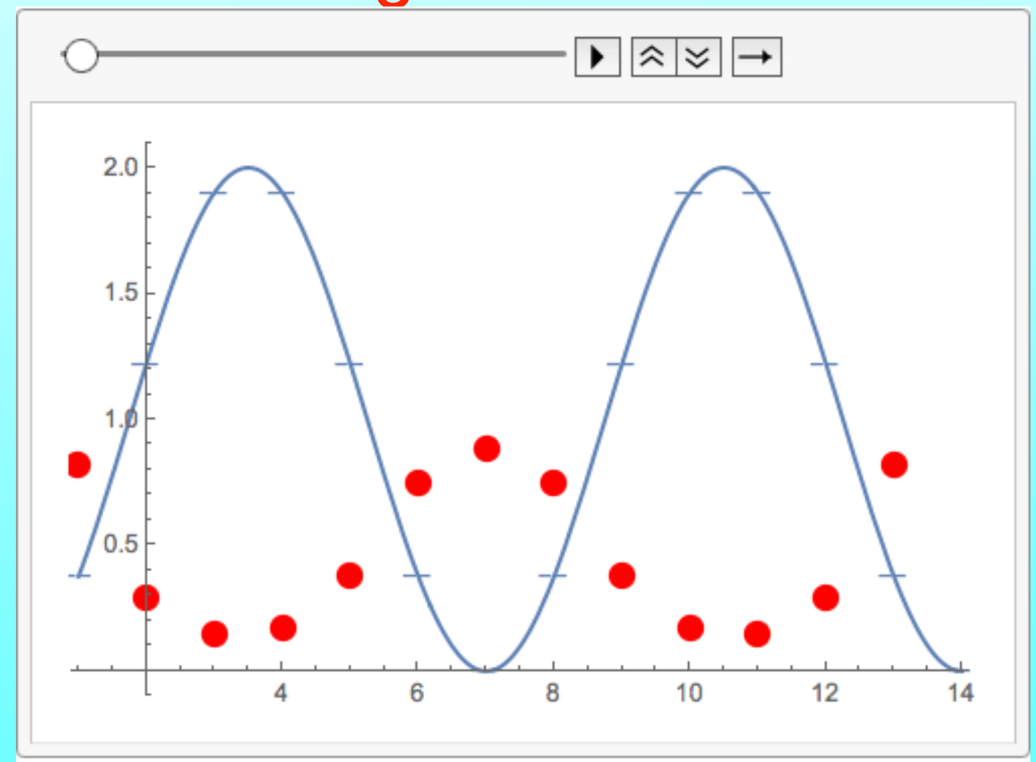
well-defined for open system

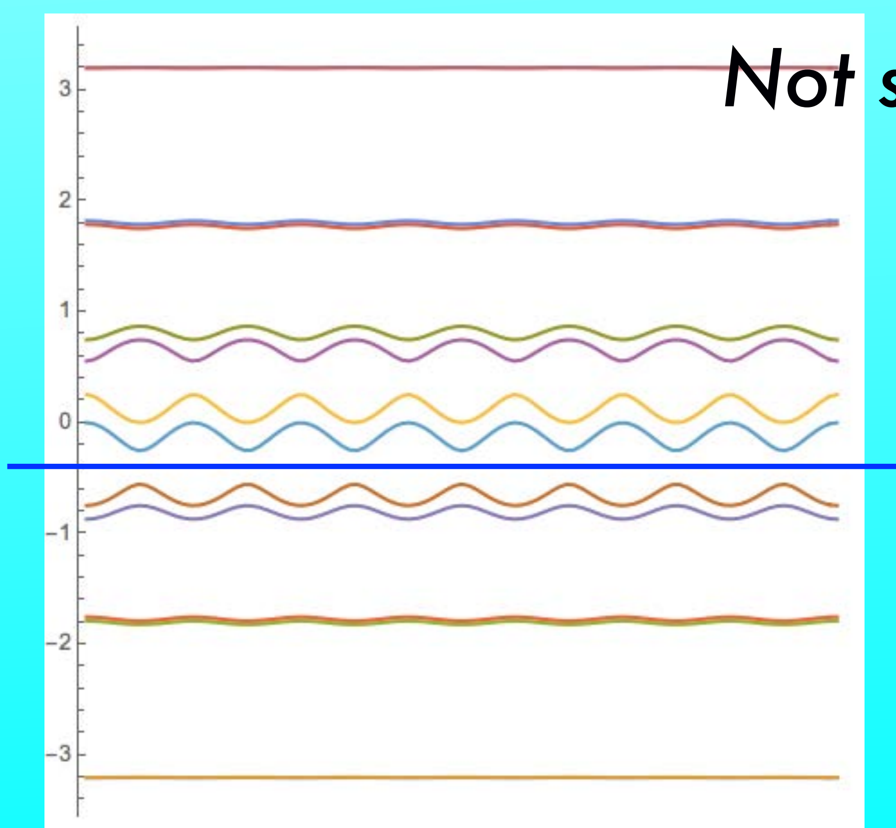
# $\phi = 1/7, \rho = 3/7, C = 3$ With/without edges

periodic

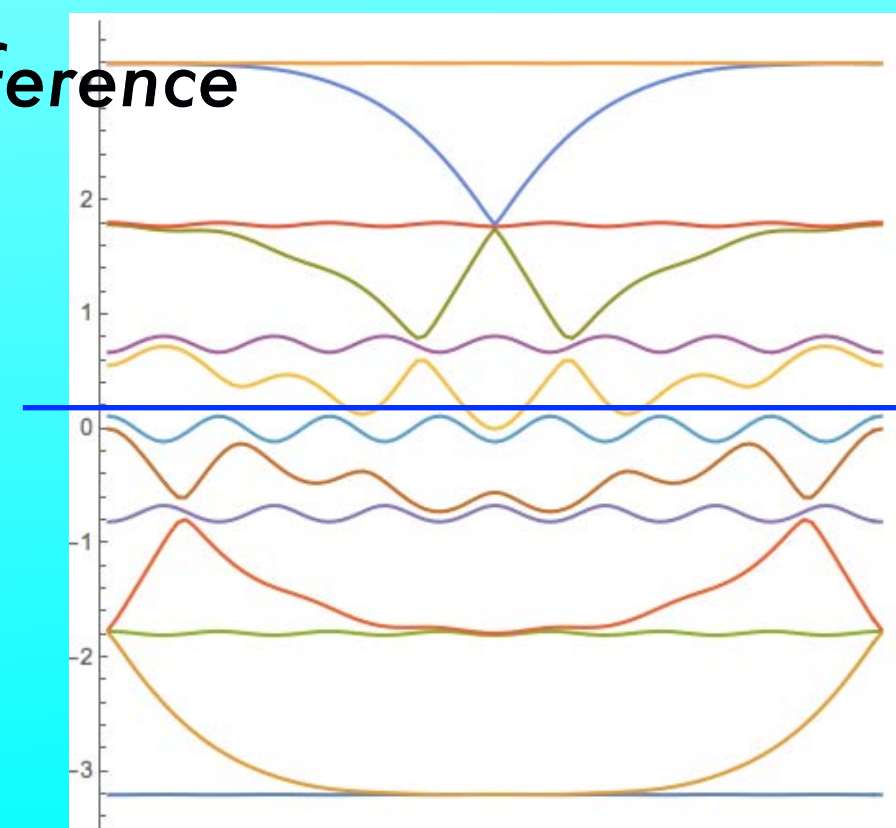



with edges




Not so much difference



# Center of mass with edges

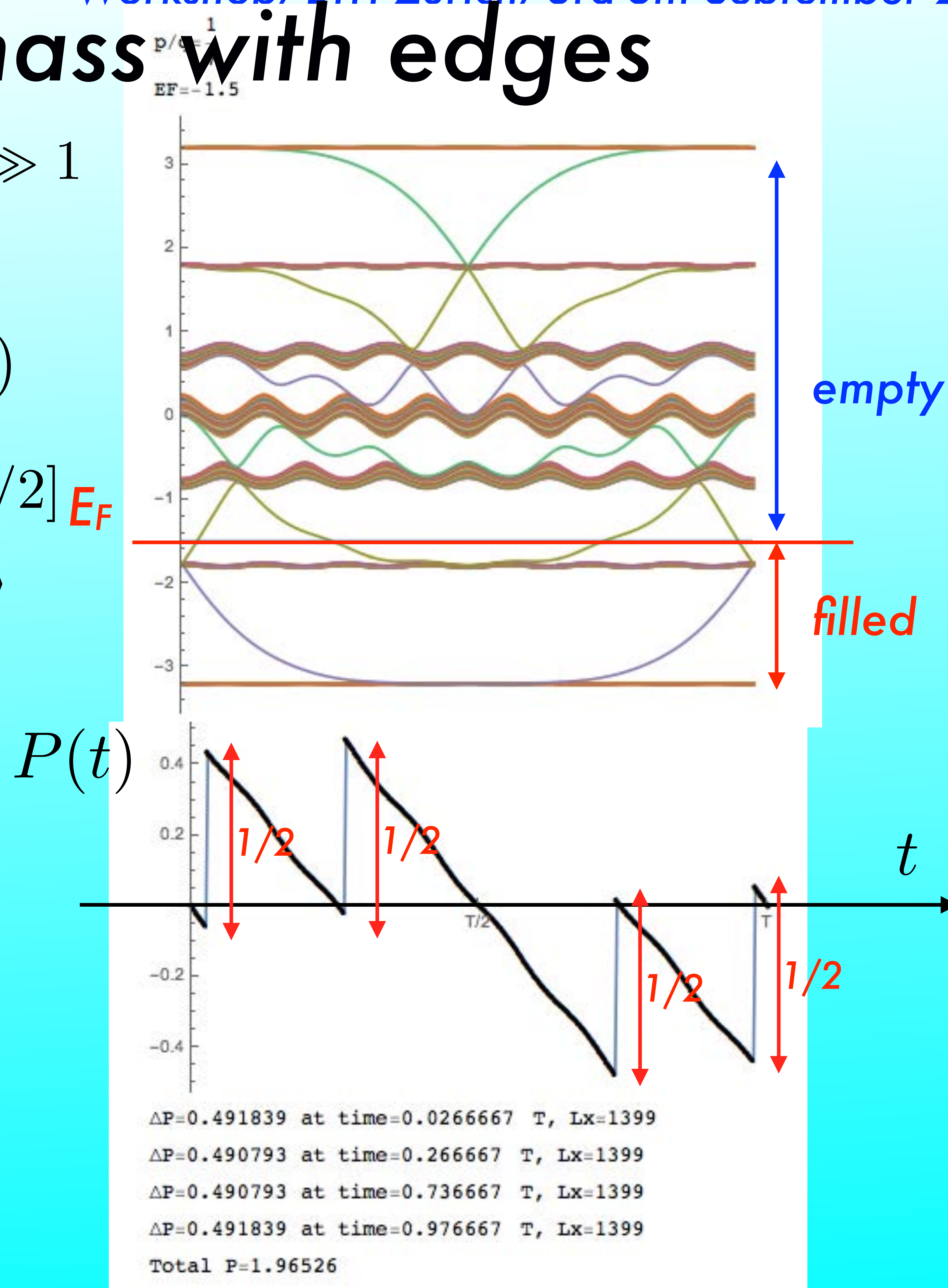
$\phi = 1/7, \qquad N \sim L \gg 1$

$E_F = -1.5, \rho \sim 2/7$

$$P(t) = \sum_j x_j \rho_j \ \sim O(N^0)$$

$$x_j = (j - j_0)/L \in [-1/2, 1/2]$$

$$\rho_j = \rho(x_j) = \langle g(0)|n_j|g(0)\rangle$$




Singular behavior

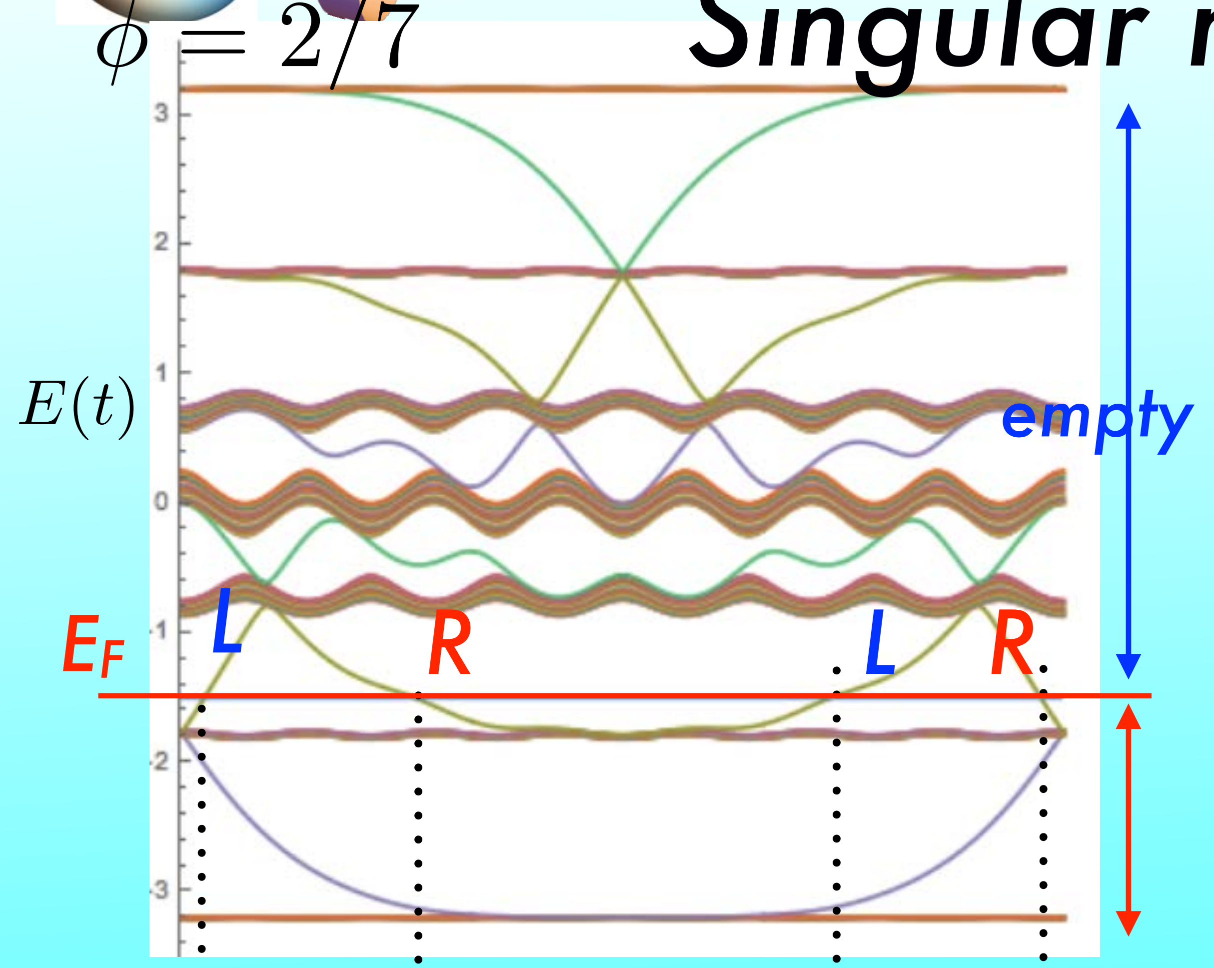
many jumps

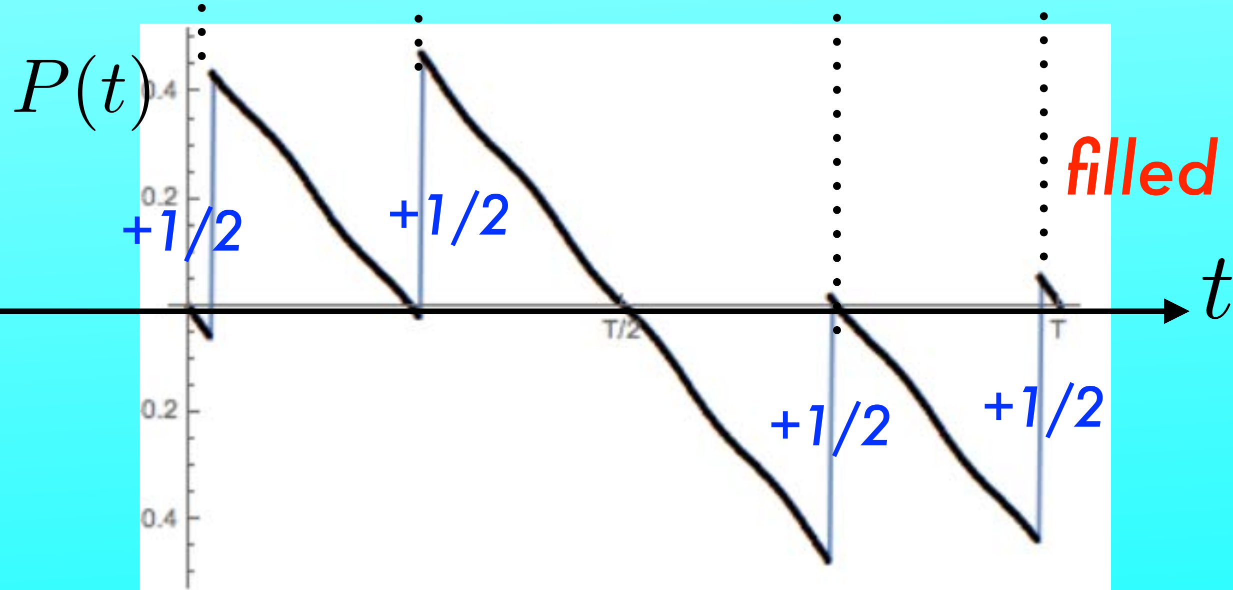
(discontinuities)

Why ?

# Singular motion of CM

## due to edge states




Contribution of the edge state for P is

$$\frac{j-j_0}{L} \to \pm\frac{1}{2}$$

exponentially localized

$j \sim L$

$j \sim 1$

$j_0 = L/2 \quad (L \to \infty)$

$$\Delta P(t_i) = P(t_i^+) - P(t_i^-)$$

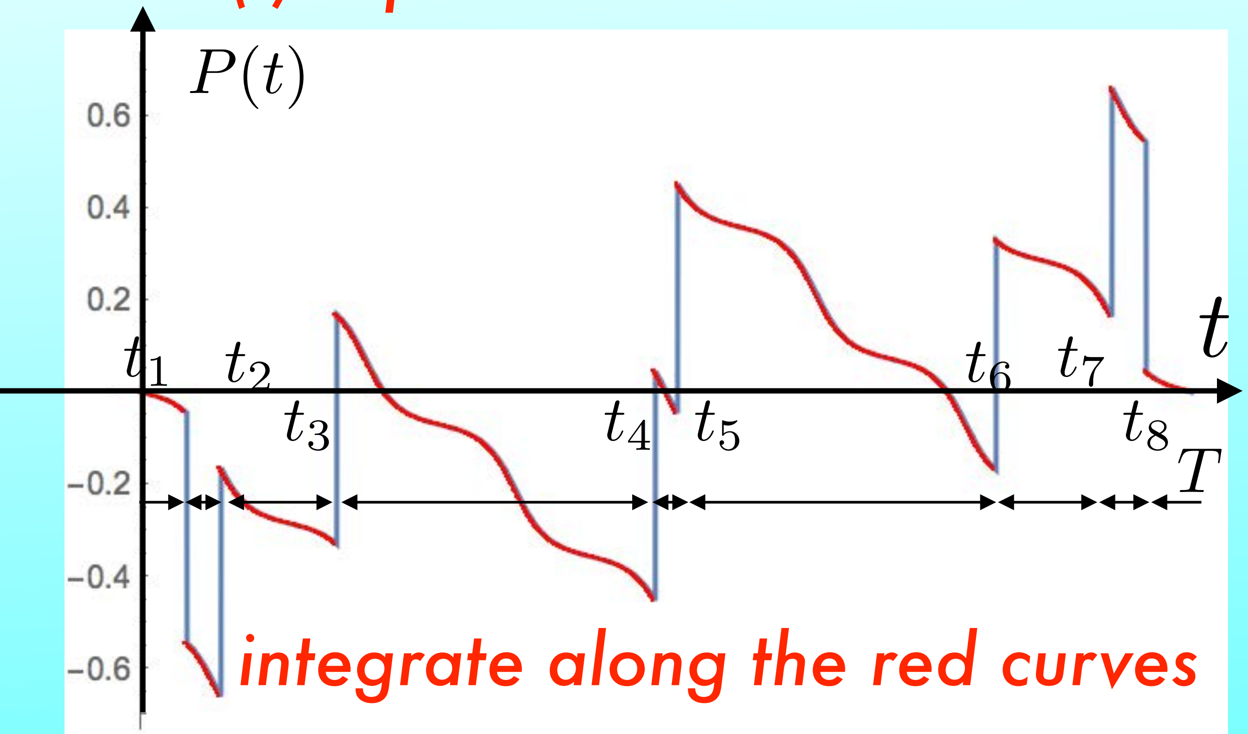
$$= \begin{cases} -1/2\text{: becomes unoccupied at R} \\ +1/2\text{: becomes occupied at R} \\ +1/2\text{: becomes unoccupied at L} \\ -1/2\text{: becomes occupied at L} \end{cases}$$

$$P(t) = \sum_j x_j \rho_j$$

$$x_j = (j - j_0)/L \in [-1/2, 1/2]$$
$$\rho_j = \rho(x_j) = \langle g(0)|n_j|g(0)\rangle$$

# How much pumped?

P(t) is periodic function !




$$\Delta P(t_i) = P(t_i^+) - P(t_i^-)$$

$$= \begin{cases} \text{-1/2: become unoccupied at R} \\ \text{+1/2: become occupied at R} \\ \text{+1/2: become unoccupied at L} \\ \text{-1/2: become occupied at L} \end{cases}$$

jumps are only for adiabatic limit

skipping the jumps due to gapless edge states

Pump by bulk (sudden approximation is justified for finite speed pump near $t_j$ )

$$\Delta Q = \sum_i \int_{t_i^+}^{t_{i+1}^-} dt\, \partial_t P(t) = \sum_i \left[P(t_{i+1}^-) - P(t_i^+)\right]$$

$$= -\sum_i \left[P(t_i^+) - P(t_i^-)\right] = -\boxed{\sum_i \Delta P(t_i)}$$

periodicity in time

patch work in time domain

sum of the discontinuities

Bulk-edge correspondence in time domain → due to edge states

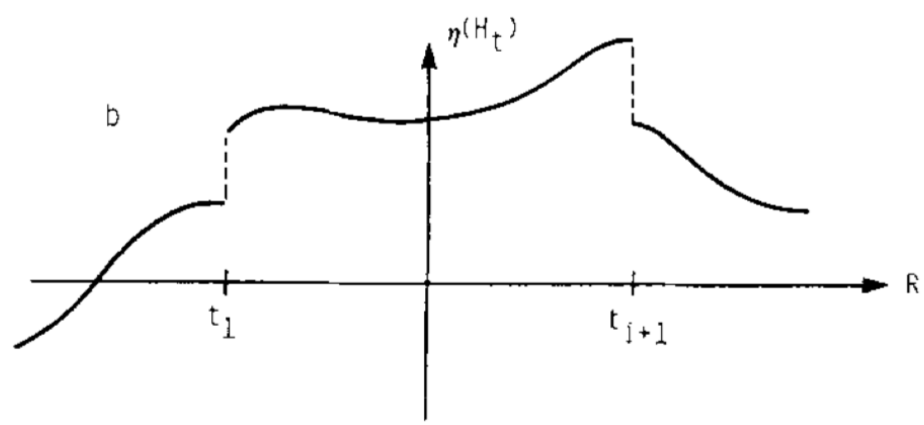
ANNALS OF PHYSICS **163**, 288–317 (1985)

Remark : Similarity !

# Anomalies and Odd Dimensions*

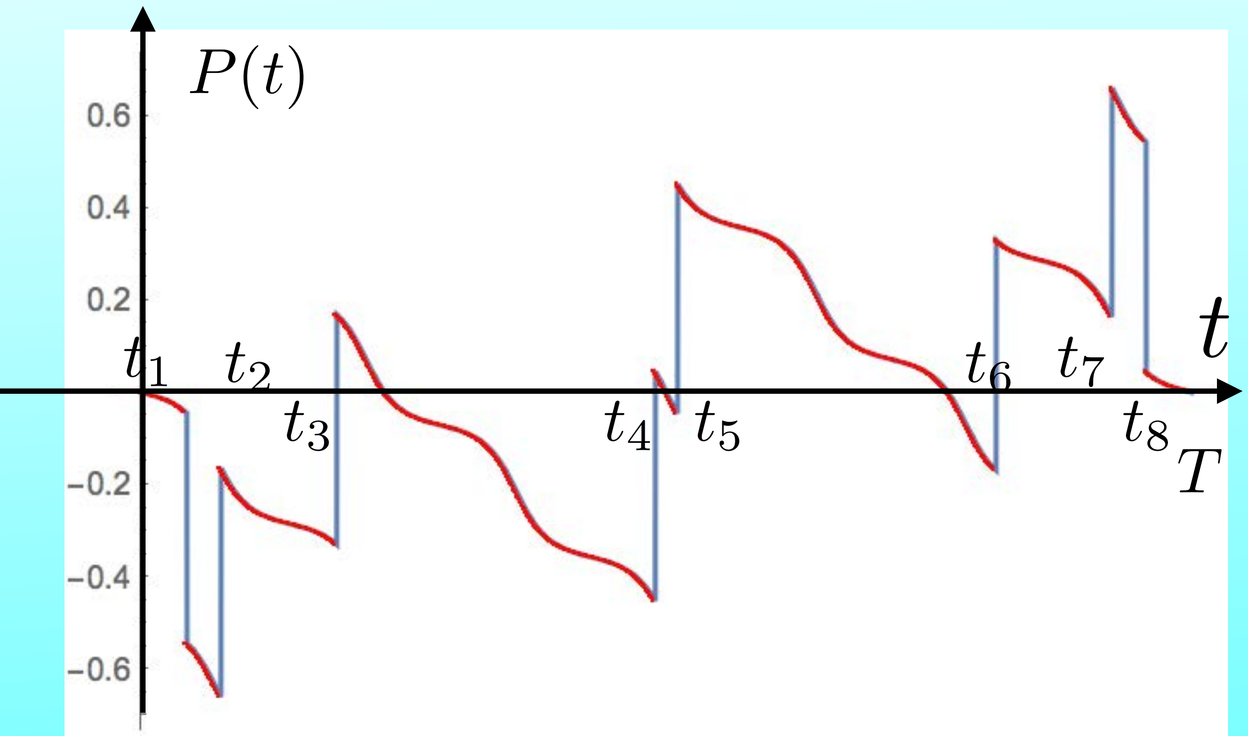
L. ALVAREZ-GAUMÉ, S. DELLA PIETRA,† AND G. MOORE†

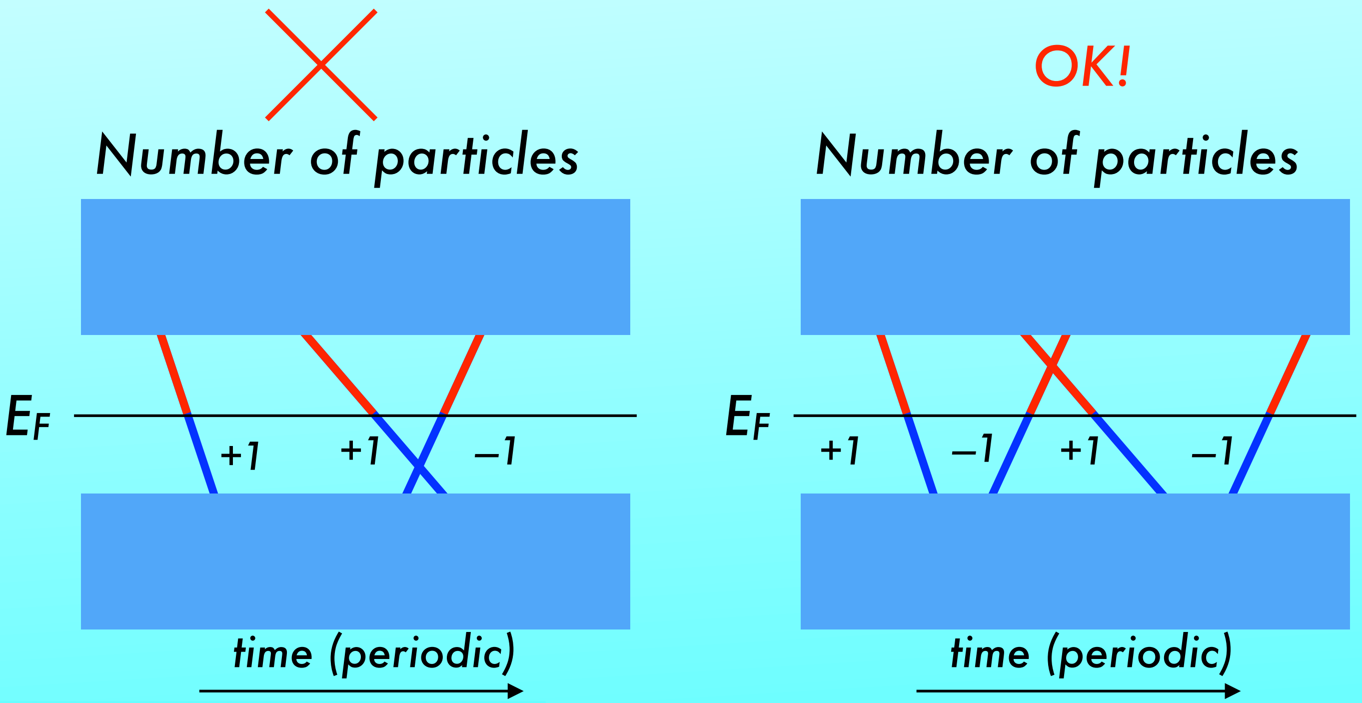
## APPENDIX: THE APS INDEX THEOREM




$$
\begin{aligned}
-\operatorname{ind} D\!\!\!/^{2n+2} &= \sum_{t_i} \pm 1 \\
&= \frac{1}{2} \sum_{i=1}^{k} \left(\eta(H_{t_i^+}) - \eta(H_{t_i^-})\right). \qquad \text{(A.5)}
\end{aligned}
$$

# Quantization due to charge conservation




$$\boxed{\Delta Q = -\sum_i \Delta P(t_i)}$$

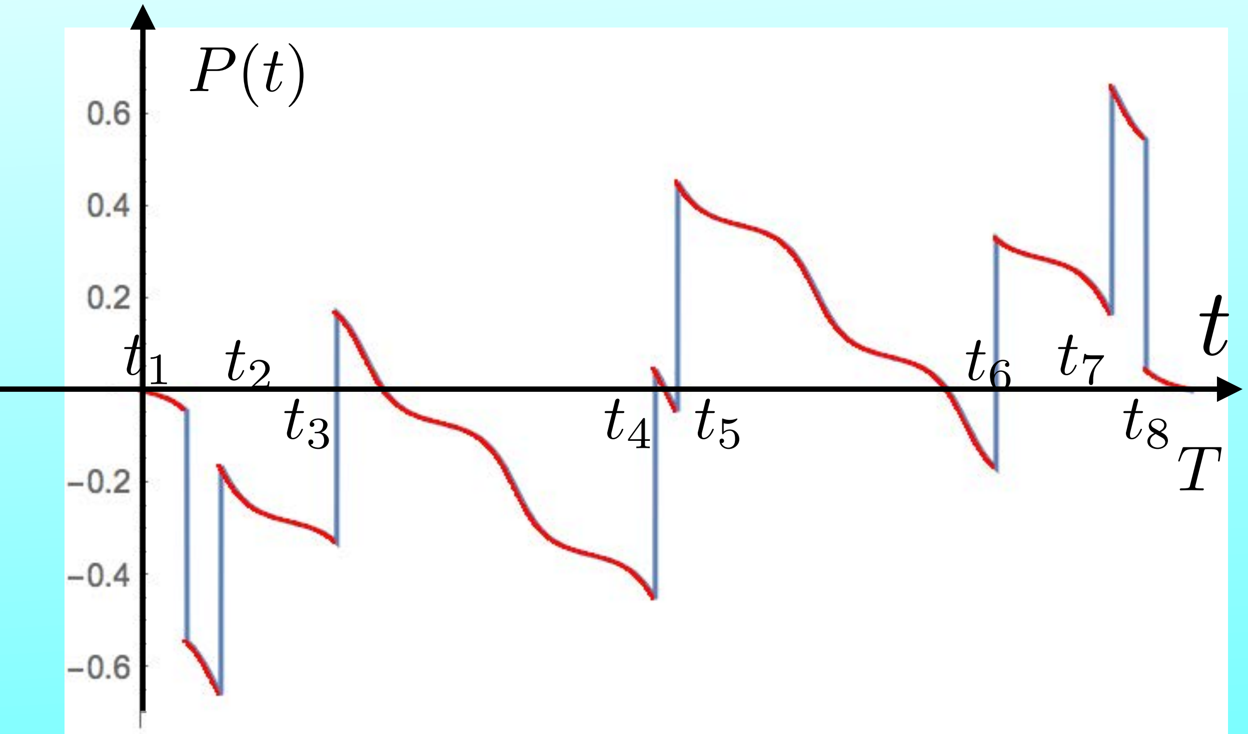
*Number of the discontinuities (SUM) are EVEN !*

*Conservation of charge & periodicity in time*

*become occupied* ⟷ *become unoccupied*

*paired*

# Quantization & conservation law




$$\Delta P(t_i) = P(t_i^+) - P(t_i^-)$$

$=$

- -1/2: become unoccupied at R
- +1/2: become occupied at R
- +1/2: become unoccupied at L
- -1/2: become occupied at L

contribution from "edge"

$$\boxed{\Delta Q = -\sum_i \Delta P(t_i) = -\sum_i \left( \pm \frac{1}{2} \right) = \text{integer } I}$$

*Number of the discontinuities (SUM) are EVEN !*

Conservation of charge & periodicity in time

become occupied ⟷ become unoccupied

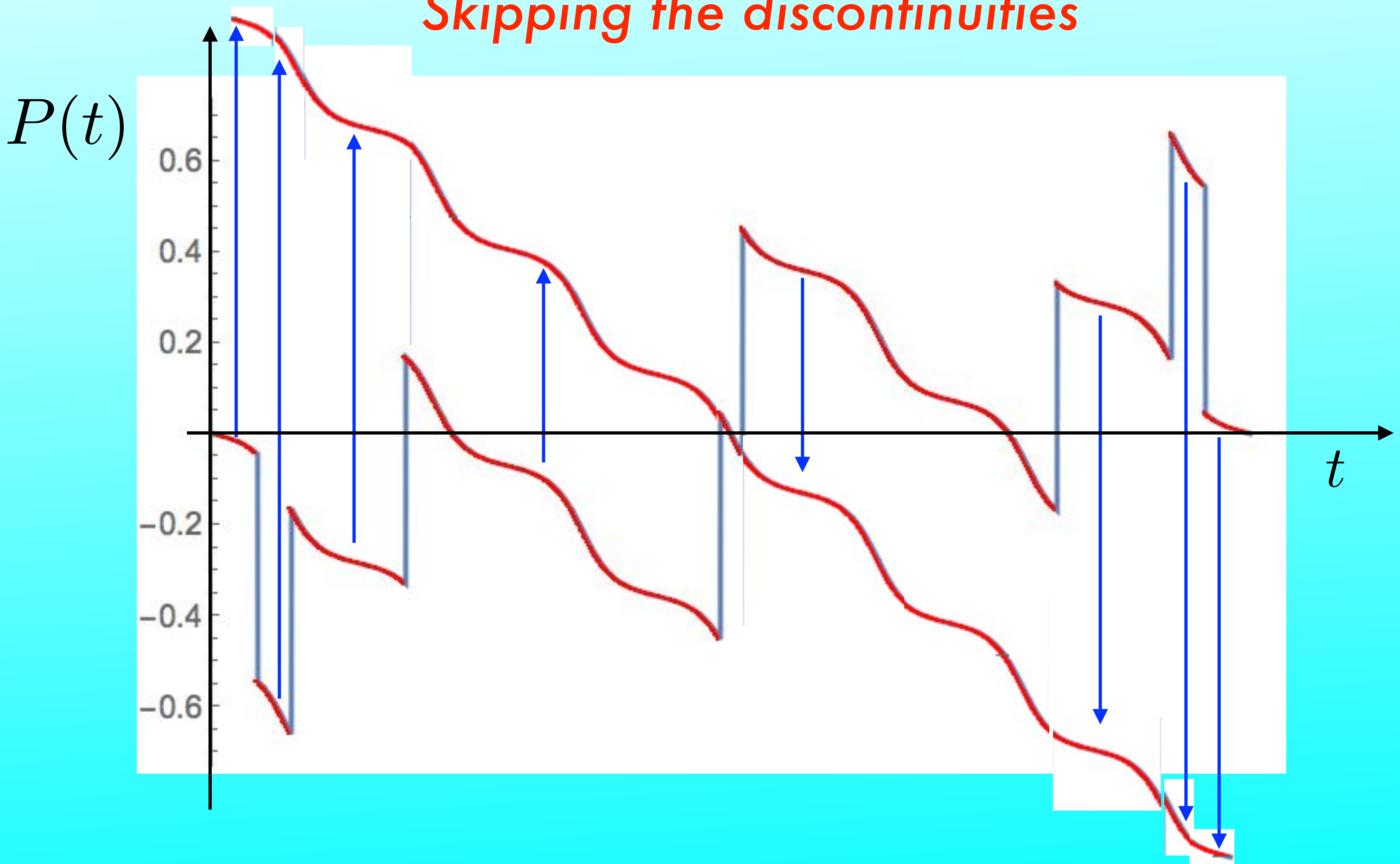
paired

# Pumped charge as a Chern number

Skipping the discontinuities

# Pumped charge as a Chern number

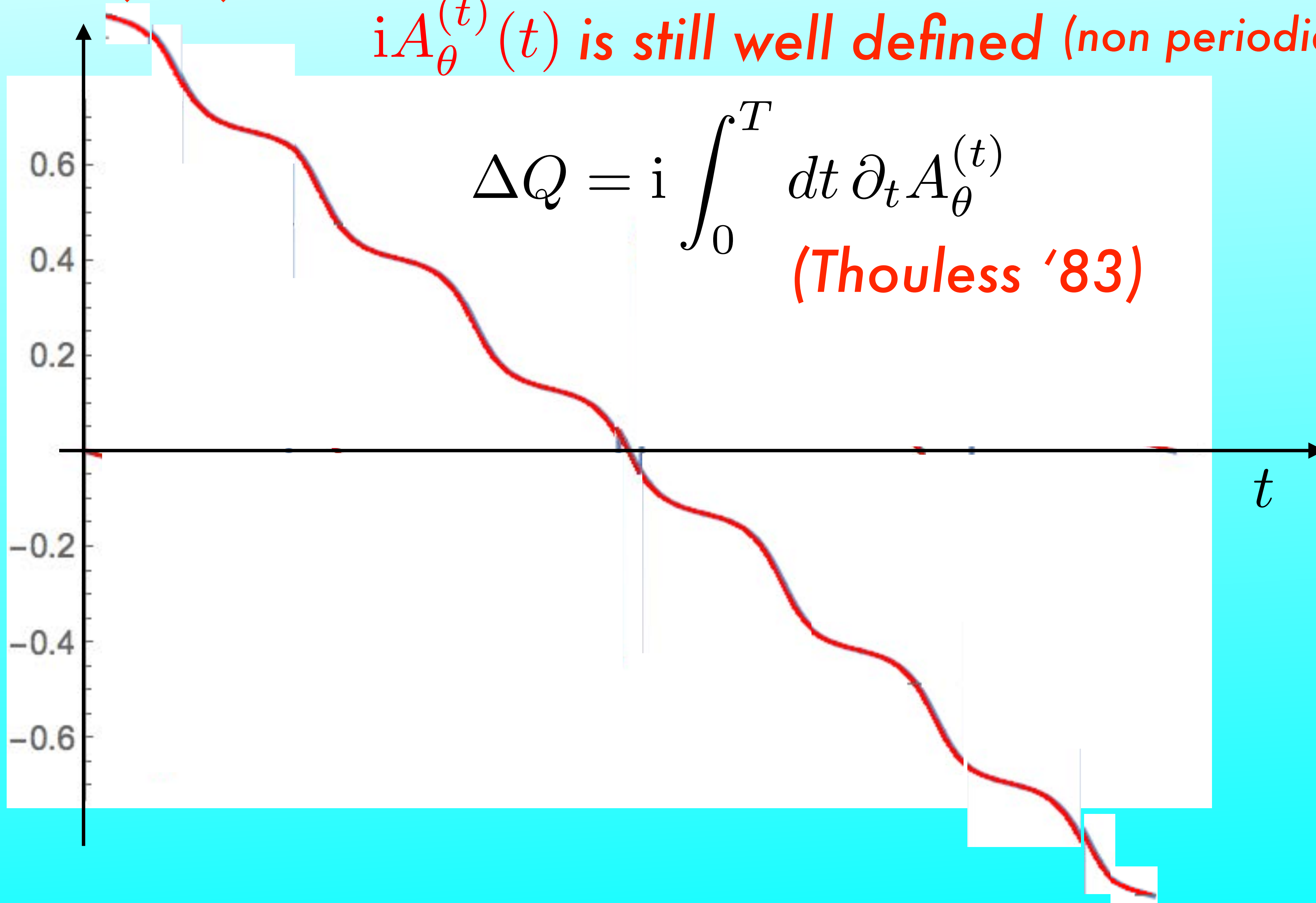
Skipping the discontinuities



# Pumped charge as a Chern number

$\mathrm{i}A_\theta^{(t)}(t) = \cancel{P}(t)$ CM is not well defined for the bulk (Bloch state)

$\mathrm{i}A_\theta^{(t)}(t)$ is still well defined (non periodic in time)




$$\Delta Q = \mathrm{i}\int_0^T dt\, \partial_t A_\theta^{(t)}$$

(Thouless '83)

# Pumped charge as a Chern number

Thouless '83

twist $k_x \sim \theta / L_x$

Chern number in temporal gauge

$$\Delta Q = \mathrm{i} \int_0^T dt\, \partial_t A_\theta^{(t)} = \frac{1}{2\pi \mathrm{i}} \int_0^T dt \int_0^{\Delta k} dk_x\, b(k_x, t) \equiv C$$

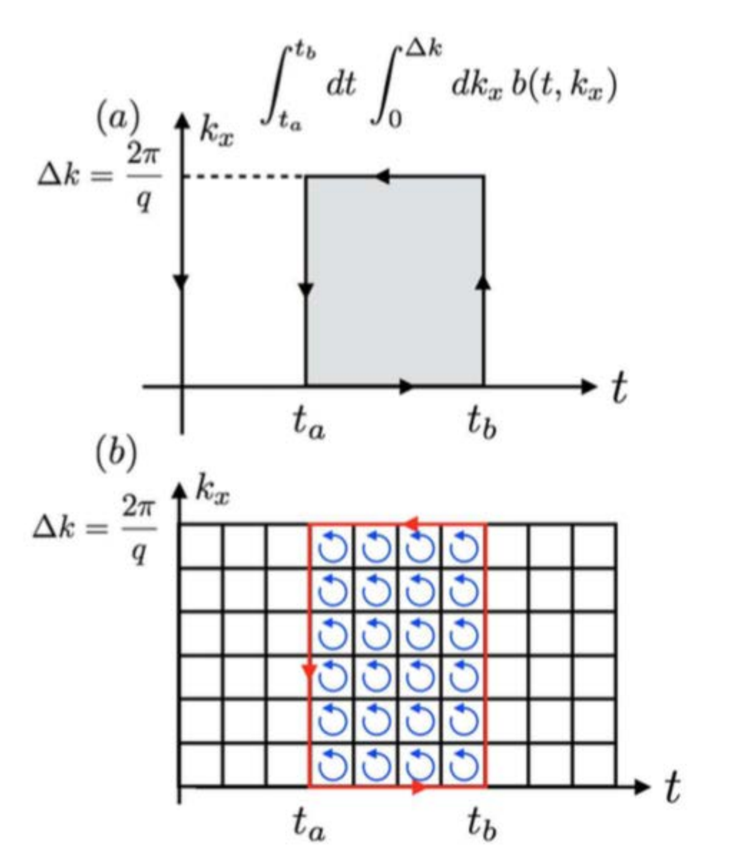
$$b = \partial_{k_x} a_t - \partial_t a_{k_x}$$

$$a_{k_x}^{(t)} = \mathrm{Tr}_M \mathcal{A}_{k_x}^{(t)}$$

$$\mathcal{A}_{k_x}^{(t)} = u^\dagger \partial_{k_x} u$$

$$u = (\boldsymbol{u}_1, \cdots, \boldsymbol{u}_M),$$

$\boldsymbol{u}_\ell(k_x, t)$ Bloch state of the $\ell$-th band



non periodic gauge fixing

YH-Fukui '16 $\boxed{I(\mathrm{edge}) = C(\mathrm{bulk})}$

discontinuities Chern number

Bulk-edge correspondence in topological pumping

# Note !

*Pumped charge is carried by bulk*
*but is described*
*by the discontinuity due to edge states*

*bulk-edge correspondence*

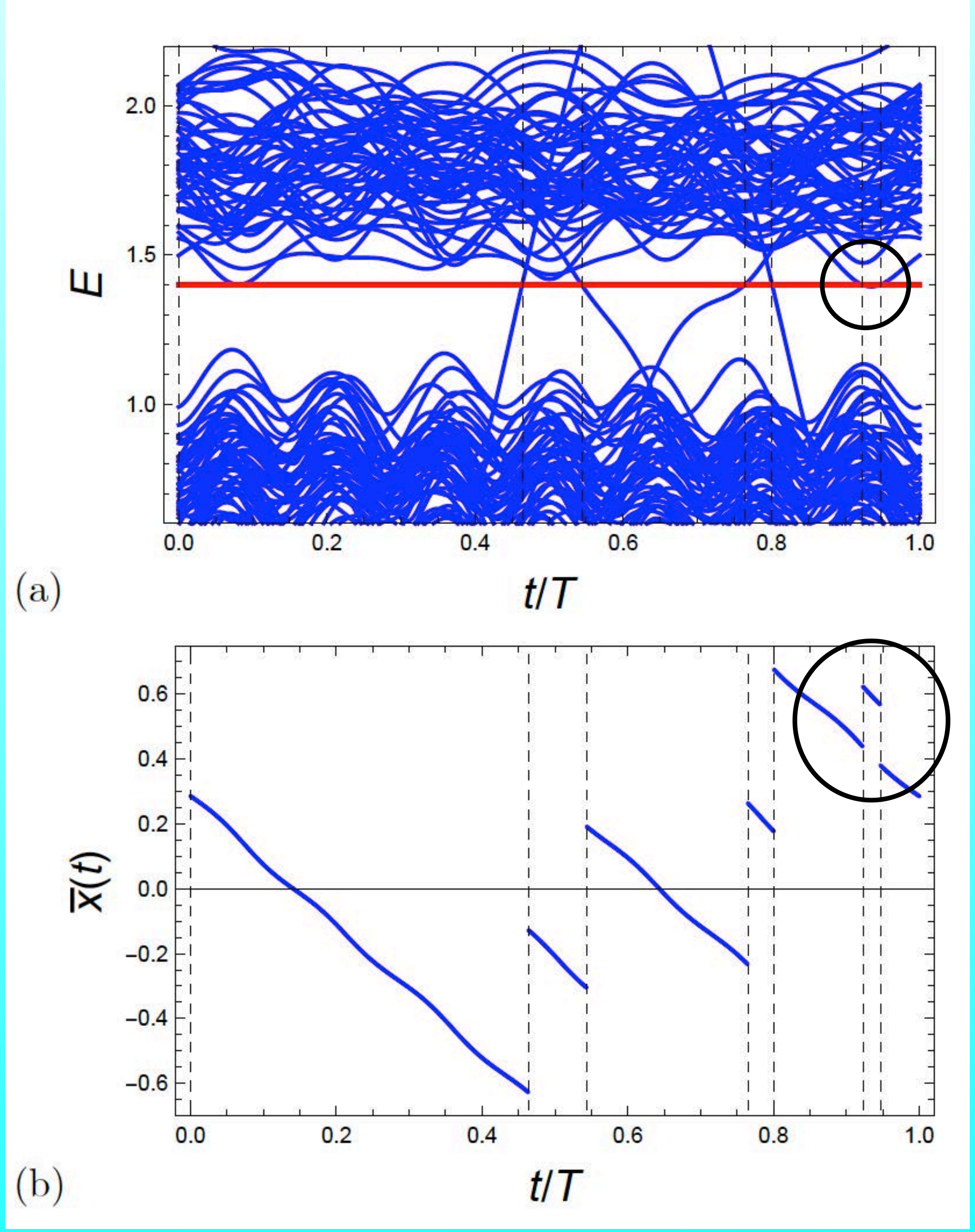
*Discontinuity: breakdown of the adiabaticity*
*due to gapless edge states, then it is never observed in real experiments of finite speed pump !*

*"Bulk" is the physical observable, "edge" is NOT*

# Effects of randomness

Anderson localization

More localize states $\Leftrightarrow$ More discontinuities




localize state at $x = x_r = (j_r - j_0)/L$

induces non quantized jump $\Delta P \neq \pm\frac{1}{2}$

$$\Delta P(t_i) = \begin{cases} +x_r & \psi_r \text{ becomes occupied} \\ -x_r & \psi_r \text{ becomes unoccupied} \end{cases}$$

: always paired and canceled

$$\Delta Q = -\sum_i \Delta P(t_i)$$

unchanged

<u>Topological stability of the pumping</u>

K.-I. Imura, Y. Yoshimura, T. Fukui and Y. Hatsugai, J. Phys. Conf. Ser. 969, 012133 ('18).

# Summary

## Bulk-edge correspondence

$$\boxed{I(\text{edge}) = C(\text{bulk})}$$

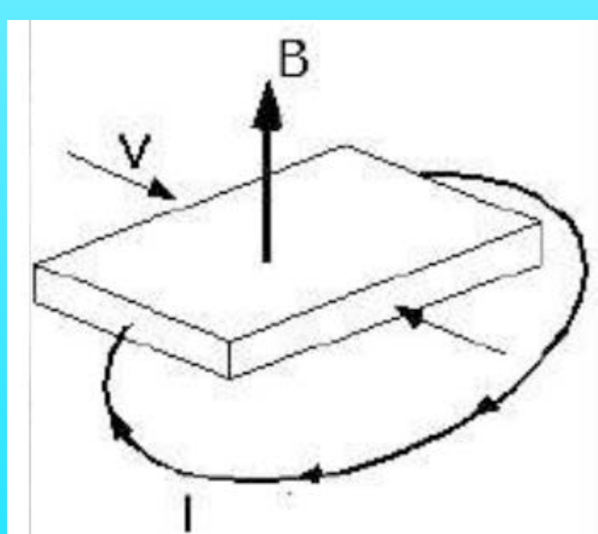
### Quantum Hall Effect

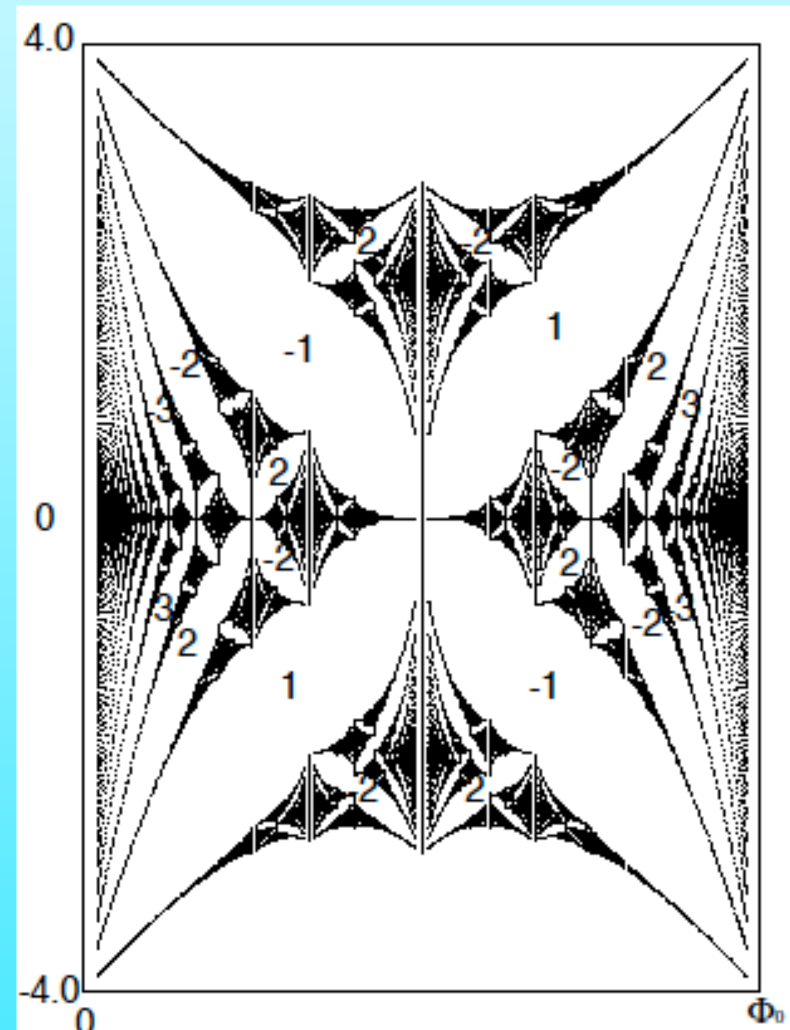
bulk

$C$ : Chern number
(Hall conductance)

edge

$I$ : Winding number




The same model
but
different physics

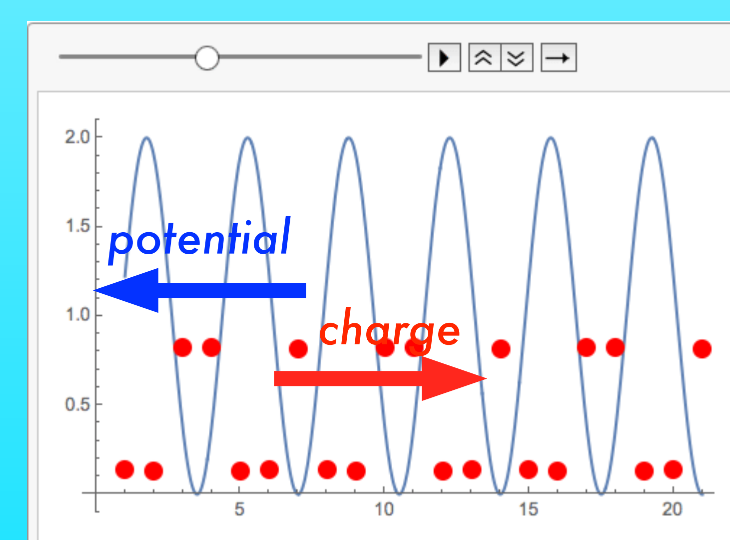
### Topological pump 1D

bulk

$C$ : Chern number
(pumped charge)

edge

$I$ : Sum of discontinuities in center of mass



Thank you